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Graphs that have separator tree representations

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Abstract

Separator trees are defined much like the clique trees that underlie chordal graph theory, except that separator trees are formed from minimal vertex separators instead of maximal complete subgraphs. The graphs G that have separator trees can be characterized using the maximal complete subgraphs of an auxiliary graph that is formed by those edges of G that are unique chords of cycles of G together with new edges that correspond to nonadjacent vertices of induced 4-cycles of G. Moreover, the chordal graphs that are strongly chordal (which have been characterized using weighted edges and strong clique trees) are precisely the chordal graphs that have (similarly defined) strong separator trees; thus, strongly chordal graphs can also be characterized using the maximal complete subgraphs of that auxiliary graph.

1 Introduction to separator trees

A set $S \subset V(G)$ is a *u,v-separator* of a graph G if $u, v \notin S$ are vertices in different components of $G-S$, and a *vertex separator* is a u,v-separator for some $u, v \in V(G)$. A *minimal u,v-separator* (abbreviated as *minimal separator*) is an inclusion-minimal u,v-separator for some pair $u, v \in V(G)$. Thus, a minimal separator can be properly contained inside of another minimal separator; see §12.5 of [3] for other standard properties of minimal separators that are used below. Let $N(v)$ denote the open neighborhood of v, and call a set S or graph G *nontrivial* if $|S| \geq 2$ or $|V(G)| \geq 2$.

Minimal separators are widely studied [2], especially in connection to *chordal graphs*—the graphs that have no induced cycles larger than triangles [3, 11]. Among the many characterizations of being chordal is that every minimal separator induces a complete subgraph. (Further connections between minimal separators and chordal graphs will occur below, largely reflecting the point of view used in Chapter 2 of [11].) Every vertex v of every graph G is in at least one inclusion-maximal complete subgraph of G (called a *maxclique* of G), and v is called a *simplicial vertex* of G if v is

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in exactly one maxclique of G. Another common characterization of a graph being chordal is that every induced subgraph has a simplicial vertex.

Lemma 1.1 *A vertex is in some minimal separator if and only if it is not a simplicial vertex.*

Proof. First, suppose a simplicial vertex x of G is in a minimal u, v-separator S (arguing by contradiction). By the minimality of S, there are neighbors u' and v' of x that are in the components of $G - S$ that contain, respectively, u and v [3]. But then $u', v' \in N(x)$ would be nonadjacent, contradicting x being simplicial.

Conversely, suppose x is not simplicial, say with two nonadjacent neighbors u and v. Then $N(u)$ contains a minimal u, v-separator of G that also contains x. □

For any graph G and any family $\mathcal{F} = \mathcal{F}(G)$ of subgraphs of G, let $K_{\mathcal{F}(G)}$ denote the complete graph that has *node set* $\mathcal{F}$ (calling the members of $\mathcal{F}$ the *nodes* of $K_{\mathcal{F}(G)}$ to avoid confusing them with the vertices of G). For convenience, the nodes of $K_{\mathcal{F}(G)}$ will also be identified with the subsets of $V(G)$ that induce the subgraphs of $\mathcal{F}(G)$. Call a spanning tree T of $K_{\mathcal{F}(G)}$ an *$\mathcal{F}$-tree* of G [9] if, for each $v \in V(G)$, all the nodes of $K_{\mathcal{F}(G)}$ that contain v induce a connected subgraph, denoted T_v, of T (in other words, if each T_v is a subtree of T).

A *clique tree* of a graph G is a $\mathcal{C}$-tree, where $\mathcal{C}(G)$ denotes the family of all the maxcliques of G. A graph is chordal if and only if it has a clique tree [3, 11]; moreover, if a chordal graph has a clique tree T, then the intersections of adjacent nodes of T are the minimal separators of T [1, 4]. In spite of an abundance of characterizations of chordal graphs in [3, 11], and although several species of $\mathcal{F}$-trees with $\mathcal{F}(G) \neq \mathcal{C}(G)$ are described in [8, 9], the graphs that have $\mathcal{F}$-trees have been neglected when $\mathcal{F}(G)$ is the set of all the minimal separators of G. Letting $\mathcal{S}(G)$ denote the family of all minimal separators of G and calling such $\mathcal{S}$-trees *separator trees*, the remainder of this paper will focus on the graphs that have separator trees. (An example of a graph that has an $\mathcal{S}$-tree with an induced subgraph that does not will be mentioned right before the statement of Theorem 3.1, showing that having a separator tree is not a "hereditary" property as in [3, 11].)

Figure 1 shows a graph G_0 that has a unique minimal separator (the minimal u, v-separator $\{x, y, z\}$), and so is the unique node of the separator tree for G_0. The graph G_1 shown has exactly three minimal separators (the minimal u, z-separator $\{x, y\}$, along with $\{x, z\}$ and $\{y, z\}$); but these cannot be three nodes of a separator tree T for G_1 (since every two of these nodes would contain a vertex not in the third, requiring a path between them but not the third, and so forcing all three to be in a cycle, contradicting T being a tree.)

For a cycle C of G and $x, y \in V(C)$, define $xy \in E(G)$ to be, as in [7], the *unichord* of C (and a unichord of G) if xy is the unique chord of C; define $xy \notin E(G)$ to be an *antichord* of C (and of G) if C is an induced cycle—in other words, if C is a chordless cycle—of G with nonadjacent vertices x and y. For example, both of the graphs G_i in Figure 1 have exactly three unichords (xy, xz, and yz) and have no antichords, since

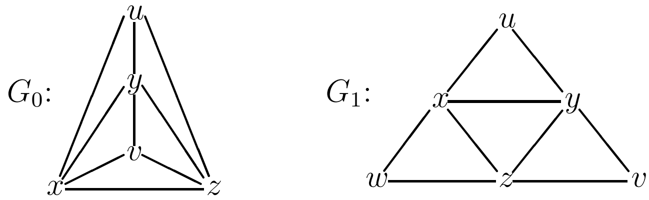

Figure 1: Graph G_0 has a separator tree, but graph G_1 does not.

each of their induced cycles is a triangle and so does not have nonadjacent vertices. (The graphs that have no unichords were named *unichord-free graphs* in [7], after having been introduced in [10, 12].)

Lemma 1.2 *If xy is a unichord or an antichord of a cycle C of a graph G, then $\{x, y\}$ is contained in at least one minimal separator S of G such that $S \cap V(C) = \{x, y\}$.*

Proof. Suppose xy is a unichord or an antichord of a cycle C of G. In either case, $xy \notin E(C)$ and so there exist $u, v \in V(C)$ in separate components of $C - \{x, y\}$. Thus $\{x, y\}$ is contained in each minimal u, v-separator S of G that is contained in the u, v-separator $[V(G) \setminus V(C)] \cup \{x, y\}$ of G, and so some minimal u, v-separator S will have $S \cap V(C) = \{x, y\}$ by [3]. □

A graph is *k-chordal* [5] if no induced cycle has length greater than k; thus, the chordal graphs are precisely the 3-chordal graphs.

Lemma 1.3 *Every graph that has a separator tree is a 4-chordal graph.*

Proof. Suppose G has a separator tree T (so G is chordal) and G has an induced k-cycle C with $k \geq 5$ (arguing by contradiction); say v_1, v_2, v_3, v_4, v_5 is a subpath of C. If $i \in \{1, 2, 3, 4, 5\}$, then each $\{v_i, v_{i+2}\}$ is a subset of a minimal v_{i+1}, v_{i+3}-separator $S_i \in \mathcal{S}(G)$ (calculating subscripts modulo 5). But now the nodes $S_1, S_3, S_5, S_2, S_4, S_1 \in \mathcal{S}$ of T would be—in that order—among the nodes of a cycle of T (contradicting T being a tree). □

Lemma 1.4 *If a graph has a separator tree, then every cycle that has a unichord must have length* 4, 5 *or* 6, *and every cycle that has an antichord must have length* 4.

Proof. This follows from Lemma 1.3. □

Section 2 will show how unichords and antichords play fundamental (and surprisingly parallel) roles in the identification of those graphs G that have separator trees, by means of an auxiliary graph that is denoted $G^{\pm}$; this will involve interplay between the (quite nonparallel) roles of minimal separators and maximal cliques of G and $G^{\pm}$. Section 3 will then assign weights to the edges of $G^{\pm}$ (and so to the unichords and antichords of G) that will be used to characterize the class of graphs that have "strong separator trees," paralleling the well-studied class of "strongly chordal graphs."

2 Separator trees and $G^{\pm}$ graphs

Define an unconventional auxiliary graph $G^{\pm}$ from G by inserting new edges that correspond to all the antichords of cycles of G while simultaneously deleting all the edges from G that are not unichords of cycles of G; thus, $V(G^{\pm}) = V(G)$, and $E(G^{\pm})$ consists of exactly the unichords and antichords of G. (The $G^{\pm}$ notation reflects that $G^{\pm}$ is created from G by both inserting certain new edges and deleting certain old edges.) Several examples of such $G^{\pm}$ graphs will be given below (to illustrate Theorem 2.1), but note for now that, since neither graph G_i in Figure 1 has antichords, both those $G_i^{\pm}$ graphs consist of the triangle xyz and $|V(G_i)\setminus\{x,y,z\}|$ isolated vertices.

Theorem 2.1 *Each minimal separator of each graph G induces a complete subgraph of $G^{\pm}$.*

Proof. Suppose $S \subset V(G)$ is a minimal u,v-separator of a graph G, and let $G^{\pm}_S$ denote the subgraph of $G^{\pm}$ that is induced by S. Let π° denote the internal vertices of any u-to-v path π, so there exist u-to-v paths π_x and π_y of G for each pair $x,y \in S$ such that $x \in \pi^{\circ}_x \setminus \pi^{\circ}_y$ and $y \in \pi^{\circ}_y \setminus \pi^{\circ}_x$. Since such u, v, π_x, and π_y can be chosen [3] so that π_x and π_y are minimal-length, internally-disjoint u-to-v paths, $\pi_x \cup \pi_y$ will be a cycle of G and, moreover, if $xy \in E(G)$, then xy is the unichord of $\pi_x \cup \pi_y$, and if $xy \notin E(G)$, then xy is an antichord of the induced cycle $\pi_x \cup \pi_y$; so either way, $xy \in E(G^{\pm})$. Since $x, y \in S$ were arbitrary, $G^{\pm}_S$ is a complete subgraph of $G^{\pm}$. □

The graph G_2 in Figure 2 has exactly two minimal separators (namely, $\{1,2,5,6\}$ and $\{2,3,4,5\}$), and these become the the nodes of the separator tree T_2 for G_2. Each of these two minimal separators induces a complete subgraph—indeed, a maxclique—of $G^{\pm}_2$. (Note that 25 is a unichord of G_2 because of the cycle induced by $\{2,3,4,5\}$ of G_2, and 16 is an antichord of G_2 because of the cycle induced by $\{1,3,4,6\}$ in G_2.)

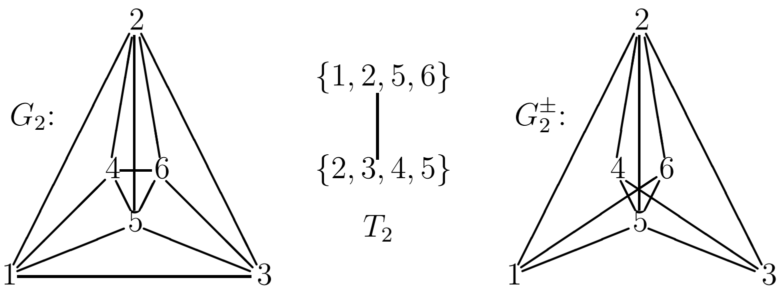

Figure 2: A graph G_2, its unique separator tree T_2, and the graph $G^{\pm}_2$ (of which all the edges *except* 16 and 34 come from unichords of G_2).

Deleting the edge 25 from G_2 would produce the octahedral graph $G'_2 \cong K_{2,2,2}$ (not shown), which has a third minimal separator, $\{1,3,4,6\}$; but G'_2 (much like G_1

of Figure 1) has no separator tree. Thus $G_2'^{\pm}$ would have all twelve edges from G_2' as unichords, along with the three antichords 16, 25, and 34, and so $G_2'^{\pm} \cong K_6$. Each of the three minimal separators of G_2' would induce a complete subgraph—but not a maxclique—of $G_2'^{\pm}$.

The graph G_3 in Figure 3 has five minimal separators ($\{1,3,7\}$, $\{2,4,8\}$, $\{4,6\}$, $\{4,7\}$, and $\{5,7\}$, corresponding to the two order-3 and three order-2 maxcliques of $G_3^{\pm}$), along with two additional minimal separators ($\{1,7\}$ and $\{2,4\}$, neither of which corresponds to a maxclique of $G_3^{\pm}$).

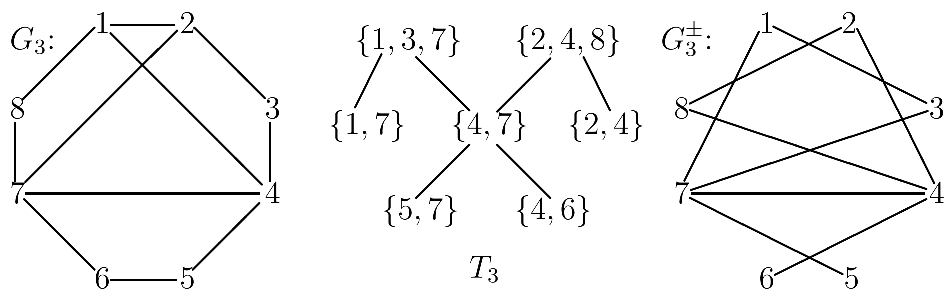

Figure 3: A graph G_3, one possible separator tree T_3, and the graph $G_3^{\pm}$ (of which all the edges *except* 47 come from antichords of G_3).

Theorem 2.2 *A graph G has a separator tree if and only if each nontrivial set that induces a maxclique of $G^{\pm}$ is a minimal separator of G and $G^{\pm}$ is chordal.*

Proof. To prove necessity, suppose T is a separator tree of G and $S \subseteq V(G)$ is nontrivial. Let $G_S^{\pm}$ denote the subgraph of $G^{\pm}$ that is induced by S.

First, suppose $G_S^{\pm} \cong K_n$ is a maxclique of $G^{\pm}$ (toward showing S is a minimal separator of G). Since S is nontrivial, $n \geq 2$. If $n = 2$, then $S = \{x, y\}$ where xy is either a unichord or an antichord of of some cycle of G, so (by Lemma 1.2) S is contained in a minimal separator (call it $S_{x,y}$) of G that induces a complete subgraph of $G^{\pm}$ (by Theorem 2.1); thus $S = S_{x,y}$ (since $G_S^{\pm}$ is a maxclique of $G^{\pm}$), and so S is a minimal separator of G. Hence, assume $n \geq 3$ and C is any n-cycle of $G_S^{\pm}$ (so $V(C) = S$). If S is not a minimal separator of G, then by Lemma 1.2 the n edges $x_i y_i \in E(C)$ would each be in a minimal separator S_{x_i,y_i} of G with $S_{x_i,y_i} \cap V(C) = \{x_i, y_i\}$, which would produce a cycle of distinct nodes S_{x_i,y_i} in the separator tree T. Therefore, S must again be a minimal separator of G.

Next, suppose S is a minimal separator of G (toward showing $G^{\pm}$ has a clique tree and so is chordal). Thus, S is a node of the separator tree T of G, so S induces a complete subgraph of $G^{\pm}$ by Theorem 2.1, and so S is contained in the vertex set S^+ of some nontrivial maxclique of $G^{\pm}$. By the argument in the preceding paragraph, S^+ is also a minimal separator of G, and so S^+ is also a node of T. If $S_1 S_2$ is any edge of T such that the distinct minimal separators S_1 and S_2 of G have $S_1 \subset S_2$ (so $S_1 \cap S_2 = S_1$) with S_2 a maxclique of G, then let T' be the new tree constructed from T by contracting the edge S_1S_2 (replacing node S_1 with S_2). Repeat this to construct new trees $T', T'', \ldots$, eventually ending with a tree (call it T^*) in which each node is a minimal separator of G that is a maxclique of G. The node set of T^* is then the set $\mathcal{C}(G)$ of maxcliques of G as in [1, 4], and so T^* is a clique tree for $G^\pm$. Therefore, as in [3, 11], $G^\pm$ is a chordal graph.

To prove sufficiency, suppose G is any graph such that each nontrivial maxclique Q of $G^\pm$ is induced by a minimal separator $V(Q)$ of G and $G^\pm$ is chordal (toward showing G has a separator tree). Suppose T is a clique tree of the chordal graph $G^\pm$, and construct a new tree T^* from T as follows: For each minimal separator S of G (so S induces a complete subgraph of $G^\pm$ by Theorem 2.1) that does not induce a maxclique of $G^\pm$, insert a new node S into T^* such that S is adjacent to an existing node $V(Q)$ of T for which Q is a maxclique of $G^\pm$ that has $S \subset V(Q)$. The tree T^* is then a separator tree for the chordal graph G by [1, 4]. □

To help illustrate Theorem 2.2, observe that, in Figure 2, the graph G_2 has the separator tree T_2, and the chordal graph $G_2^\pm$ has exactly two maxcliques, induced by the minimal separators $\{1, 2, 5, 6\}$ and $\{2, 3, 4, 5\}$ of G_2. (In contrast, if $G_2' \cong K_{2,2,2}$ results from deleting the edge 25 from G_2, then G_2' would not have a separator tree, and the unique maxclique of the chordal graph $G_2'^\pm \cong K_6$ would not be induced by a minimal separator of G_2'.)

The chordal graph $G_3^\pm$ in Figure 3 has five maxcliques (two of order 3, and three of order 2) that are induced by five of the seven minimal separators of G_3 (so five of the nodes of its separator tree T_3); the other two minimal separators are $\{1, 7\}$ and $\{2, 4\}$, which do not induce maxcliques of $G_3^\pm$.

3 Strong separator trees and weighted $G^\pm$ graphs

Recall from Section 1 that, for any family $\mathcal{F}(G)$ of subgraphs of a graph G, an $\mathcal{F}$-tree of G is a spanning tree $T(G)$ of the complete graph $K_{\mathcal{F}(G)}$, where the nodes of $T(G)$ are the vertex sets of the members of $\mathcal{F}(G)$, such that, for each $v \in V(G)$, the nodes of $K_{\mathcal{F}(G)}$ that contain v always induce a subtree of $T(G)$. Starting from any particular $\mathcal{F}$-tree $T(G)$ of G, define families $\mathcal{F}^{(i)}(G)$ and trees $T^{(i)}(G)$ recursively for $i \geq 0$, beginning with $\mathcal{F}^{(0)}(G) = \mathcal{F}(G)$ and $T^{(0)}(G) = T(G)$. For $i \geq 1$, define $\mathcal{F}^{(i)}(G) = \{S \cap S' : SS' \in E(T^{(i-1)}(G))\}$, and define $T^{(i)}(G)$ to have, for each node S of degree $d \geq 2$ of $T^{(i-1)}(G)$, neighbors $S_1, \ldots, S_{d'}$ in $T^{(i-1)}(G)$ that are the $d' \leq d$ distinct members of $\{S \cap S_1, \ldots, S \cap S_d\}$ in $\mathcal{F}^{(i)}(G)$; thus, each such node S will have degree d' in the $\mathcal{F}^{(i)}(G)$-tree $T^{(i)}(G)$. This makes $\mathcal{F}^{(i)}(G)$ the set of nodes of $T^{(i)}(G)$, with $T^{(i)}(G) \cong K_1$ when $|\mathcal{F}^{(i)}(G)| = 1$ and with $\mathcal{F}^{(i)}(G)$ and $T^{(i)}(G)$ undefined when $\mathcal{F}^{(i)}(G) = \emptyset$.

Suppose G_4 is the order-8 graph (not shown) that has $V(G_4) = \{1, 2, 3, 4, 5, 6, 7, 8\}$ with exactly the five maxcliques $\{1, 2, 4\}$, $\{2, 3, 4, 5\}$, $\{2, 3, 5, 6\}$, $\{2, 3, 7\}$, and $\{2, 7, 8\}$, and suppose $\mathcal{F}(G_4)$ is the family consisting of these five maxcliques. Figure 4 shows one possible $\mathcal{F}(G_4)$-tree $T(G_4) = T^{(0)}(G_4)$. (The only other possibility for $T(G_4)$ would have node $\{2, 3, 7\}$ adjacent to node $\{2, 3, 5, 6\}$ instead of

$\{2,3,4,5\}$, since for each $v \in V(G_4)$, the nodes of $K_{\mathcal{F}(G_4)}$ that contain v need to induce a subtree of $T(G_4)$.) Figure 4 then illustrates how $T^{(i)}(G_4)$ is constructed from $T^{(i-1)}(G_4)$ (with $T^{(3)}(G_4)$ consisting of the unique node $\{2\}$, and $T^{(i)}(G_4)$ undefined for $i \geq 4$).

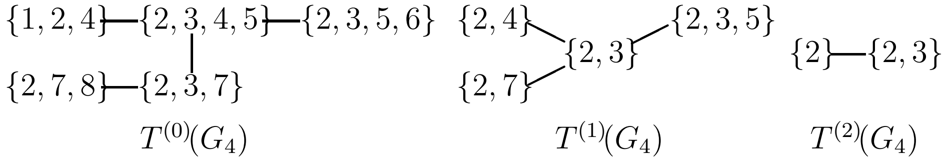

Figure 4: Constructing the $\mathcal{F}^{(i)}(G_4)$-tree $T^{(i)}(G_4)$ from $T^{(i-1)}(G_4)$.

Since each S contains each $S \cap S_j$ in the preceding paragraph, the union of the nontrivial set of nodes of each $T^{(i)}(G)$ that contain v induces a subtree of $T^{(i)}(G)$ for each $v \in V(G)$ (with the order of each tree $T^{(i)}(G)$ less than the order of $T^{(i-1)}(G)$). Every two adjacent nodes S_j and S_k of $T^{(i)}(G)$ will have $S_j \neq S_k$ (but $S_j \subset S_k$ is possible, in which case the node $S_j \cap S_k$ of $T^{(i+1)}(G)$ equals the node S_j of $T^{(i)}(G)$.) Call an $\mathcal{F}$-tree T of G a *strong $\mathcal{F}$-tree* of G if a tree $T^{(i+1)}(G)$ exists whenever $\mathcal{F}^{(i)}(G)$ is nontrivial.

Recall that the clique tree T of a chordal graph G is a $\mathcal{C}$-tree where $\mathcal{C}(G)$ denotes the set of all the maxcliques of G. A graph G is defined to be *strongly chordal* if G has a strong $\mathcal{C}$-tree, called a *strong clique tree*. (In Figure 4, $T^{(0)}(G_4)$ is a strong clique tree for the strongly chordal graph G_4.) See [3, 9, 11] for many additional characterizations of strongly chordal graphs (and Corollary 3.3 will give a new one).

Reference [8] discusses strong $\mathcal{F}$-trees for graph classes other than maxcliques. For instance, if $\mathcal{F}_o(G)$ is the family of all open neighborhoods of vertices of G, then the graphs that have strong $\mathcal{F}_o$-trees turn out to be precisely the "chordal bipartite graphs" (defined in [2, 11]); if $\mathcal{F}_c(G)$ is the family of all closed neighborhoods of vertices of G, then the graphs that have strong $\mathcal{F}_c$-trees turn out to be precisely the strongly chordal graphs. (The graphs that have $\mathcal{F}_o$-trees and $\mathcal{F}_c$-trees are also discussed in [8], but seem less interesting than those that have strong $\mathcal{F}_o$-trees or strong $\mathcal{F}_c$-trees.)

The remainder of this section will characterize the graphs G that have *strong separator trees*, meaning strong $\mathcal{F}$-trees when $\mathcal{F}(G) = \mathcal{S}(G)$ (the family of all nontrivial minimal separators of G). Therefore, whenever G has a strong separator tree T and $i \geq 1$, the sets in $\mathcal{S}^{(i)}(G)$ are precisely the nodes of $T^{(i)}(G)$. Also, the nodes of $T^{(0)}(G)$ are the nontrivial minimal separators of G and, when $i \geq 1$, the nodes of $T^{(i)}(G)$ are the intersections of adjacent nodes of $T^{(i-1)}(G)$, and so every node of every $T^{(i)}(G)$ induces a complete subgraph of $G^{\pm}$ by Theorem 2.1. For instance, both the graphs G_2 and G_3 in Figures 2 and 3, respectively, have strong separator trees $T^{(0)}(G_i)$, where $T^{(1)}(G_2)$ has the unique node $\{2,5\}$ and $T^{(1)}(G_3)$ has the two nodes $\{1,7\}$ and $\{2,4\}$ (the other adjacent nodes have trivial intersections).

Given a separator tree $T = T^{(0)}(G)$ for a graph G, define the *weight of an edge* xy

of $G^{\pm}$, denoted $\text{wgt}_{G^{\pm}}(xy)$, to be the number of nodes of T (the number of minimal separators of G) that contain $\{x, y\}$; also, define the *weight of a cycle* C of $G^{\pm}$, denoted by $\text{wgt}_{G^{\pm}}(C)$, to be the number of nodes of T that contain $V(C)$. Thus, each cycle C satisfies $\text{wgt}_{G^{\pm}}(C) \leq \text{wgt}_{G^{\pm}}(e)$ for all $e \in E(C)$. Note that, whenever G has a separator tree, each edge e of $G^{\pm}$ has $\text{wgt}_{G^{\pm}}(e) \neq 0$ (by Theorem 2.2), while a cycle C of $G^{\pm}$ can have $\text{wgt}_{G^{\pm}}(C) = 0$ (for instance, the cycle induced by $\{2, 4, 5, 6\}$ of $G_2^{\pm}$ in Figure 2).

For instance, Figure 5 shows a graph G_5 that has a separator tree $T_5 = T^{(0)}(G_5)$ but (as shown in the next paragraph) has no strong separator tree. Three of the nine edges (24, 26, and 46) of $G_5^{\pm}$ have weight 2, and the other six have weight 1. (The simplicial vertices 1 and 3 of G_5 become isolated vertices of $G_5^{\pm}$, as in Lemma 1.1.) Only one possible separator tree T_5 of G_5 is shown, but the node $\{2, 5, 7, 8\}$ could be adjacent to any one of the three nodes of T_5 that contain vertex 2 and still have, for every v, the nodes that contain v forming a subtree. (It does not matter which separator tree is chosen.)

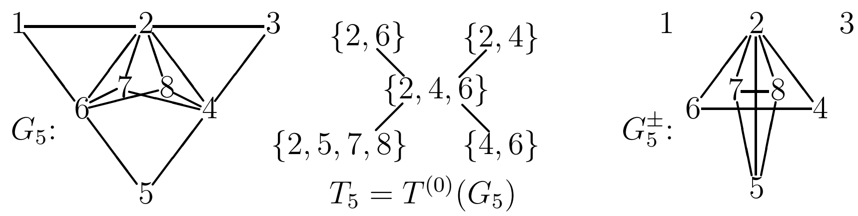

Figure 5: A graph G_5, a (non-strong!) separator tree T_5, and the graph $G_5^{\pm}$ (of which all the edges *except* $24, 26, 27$ and 28 come from antichords of G_5).

There are five minimal separators of G_5 (the five nodes of T_5), where $\{2, 5, 7, 8\}$ is a minimal $4, 6$-separator of G_5, and $\{2, 4, 6\}$ is a minimal $7, 8$-separator. The separator tree T_5 cannot be a strong separator tree, since otherwise, $T^{(1)}(G_5)$ would have the nodes $\{2\}$, $\{2, 4\}$, $\{2, 6\}$, and $\{4, 6\}$ (from the four edges of T_5) with the last three of these forming a triangle of $T^{(1)}(G_5)$ (in order to have the subgraphs corresponding to each $v \in \{2, 4, 6\}$ be subtrees of $T^{(1)}(G_5)$); but this triangle would contradict $T^{(1)}(G_5)$ being a tree. Note that the three minimal separators $\{2, 4\}$, $\{2, 6\}$, and $\{4, 6\}$ of G_5 are the edges of a triangle C of $G_5^{\pm}$ that has $\text{wgt}_{G_5^{\pm}}(C) = 1 < \text{wgt}_{G_5^{\pm}}(e) = 2$ for each $e \in E(C)$. (Also, Theorem 2.2 shows that the graph obtained by deleting vertex 8 from G_5 has no separator tree.)

Theorem 3.1 *A separator tree for a graph G is a strong separator tree for G if and only if each cycle C of $G^{\pm}$ has an edge e with* $\text{wgt}_{G^{\pm}}(C) = \text{wgt}_{G^{\pm}}(e)$.

Proof. First, suppose T is a strong separator tree for G, and C is a cycle of $G^{\pm}$ that has $\text{wgt}_{G^{\pm}}(C) < \min\{\text{wgt}_{G^{\pm}}(e) : e \in E(C)\}$ (arguing by contradiction). Let $\mathcal{S}(C)$ denote the set of the minimal separators of G (in other words, the nodes of T) that are in $V(C)$. For each $e \in E(C)$, let S_e be a minimal separator of G that has $V(e) \subseteq S_e \notin \mathcal{S}(C)$ (where S_e exists since $\text{wgt}_{G^{\pm}}(C) < \text{wgt}_{G^{\pm}}(e)$). Thus, there are edges $e_1, e_2, \ldots, e_n \in E(C)$ with $3 \leq n \leq |E(C)|$ such that $S_{e_1}, S_{e_2}, \ldots, S_{e_n}$ are pairwise distinct with $S_{e_i} \cap S_{e_{i+1}} \neq \emptyset$ when $1 \leq i < n$ and $S_{e_n} \cap S_{e_1} \neq \emptyset$. But now, these n minimal separators would be the nodes of an n-cycle of $T^{(1)}$(G), contradicting $T^{(1)}(G)$ being a tree.

Conversely, suppose G has a separator tree T that is not a strong separator tree. Hence, some $T^{(k)}(G)$ with $k \geq 1$ has a cycle of length n_k, and so G has minimal separators $S_1^{(k)}, S_2^{(k)}, \ldots, S_{n_k}^{(k)}$ with $S_i^{(k)} \cap S_{i+1}^{(k)} \neq \emptyset$ when $1 \leq i < n_k$ and $S_{n_k}^{(k)} \cap S_1^{(k)} \neq \emptyset$. This leads to $T^{(k-1)}(G)$ having a cycle of length $n_{k-1} \geq n_k$, and so to G having minimal separators $S_1^{(k-1)}, S_2^{(k-1)}, \ldots, S_{n_{k-1}}^{(k-1)}$ with $S_i^{(k-1)} \cap S_{i+1}^{(k-1)} \neq \emptyset$ when $1 \leq i < n_{k-1}$ and $S_{n_{k-1}}^{(k-1)} \cap S_1^{(k-1)} \neq \emptyset$. Repeating this ends with $T^{(0)}(G) = T(G)$ having a cycle of length n, and so to G having minimal separators $S_1, S_2, \ldots, S_n$ with $S_i \cap S_{i+1} \neq \emptyset$ whenever $1 \leq i < n$ and $S_n \cap S_1 \neq \emptyset$. Therefore, $G^{\pm}$ would have a cycle C with $E(C) = \{e_1, e_2, \ldots, e_n\}$ (each S_i containing the endpoints of e_i) and $\mathrm{wgt}_{G^{\pm}}(C) < \mathrm{wgt}_{G^{\pm}}(e)$ for every edge e of C. □

Lemma 3.2 will characterize strongly chordal graphs in terms of strong separator trees (and is comparable to the characterization in [6]). Based on Theorem 3.1 and Lemma 3.2, Corollary 3.3 will then be a new characterization of a graph G being strongly chordal in terms of the auxiliary graph $G^{\pm}$.

Lemma 3.2 *A chordal graph is strongly chordal if and only if it has a strong separator tree.*

Proof. Suppose G is a chordal graph. Recalling that $\mathcal{C}^{(0)}(G) = \mathcal{C}(G)$ is the family of all the maxcliques of G, [1, 4] show that G has a clique tree T such that $\mathcal{C}^{(1)}(G) = \{S \cap S' : S, S' \in \mathcal{C}^{(0)}(G)$ are adjacent nodes in $T\}$, and so that $\mathcal{C}^{(1)}(G) = \mathcal{S}(G)$. Therefore, G is strongly chordal if and only if G has a strong separator tree. □

Corollary 3.3 *A chordal graph G is strongly chordal if and only if each cycle C of $G^{\pm}$ has an edge e with $\mathrm{wgt}_{G^{\pm}}(C) = \mathrm{wgt}_{G^{\pm}}(e)$.*

Proof. This follows directly from Theorem 3.1 and Lemma 3.2. □

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