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Partitioning the vertices of a graph or its complement into a total dominating set and an independent dominating set

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Abstract

A graph G whose vertex set can be partitioned into a total dominating set and an independent dominating set is called a TI-graph. There exist infinite families of graphs that are not TI-graphs. We show that, with a few exceptions, every graph or its complement is a TI-graph. From this result, it follows that with the exception of the cycle on five vertices, every nontrivial, self-complementary graph is a TI-graph. We also characterize the complementary prisms which are TI-graphs and explore such partitions in prisms.

1 Introduction

We study graphs whose vertex set can be partitioned into a total dominating set and an independent dominating set. We begin with some basic definitions. Let G be a graph with vertex set $V = V(G)$, edge set $E = E(G)$, and *order* $n = |V|$. Let $\overline{G}$ denote the complement of G. The *open neighborhood* $N_G(v)$ of a vertex $v \in V$ is the

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set of vertices adjacent to v, and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ is $N_G(S) = \bigcup_{v \in S} N_G(v)$, while the *closed neighborhood of a set* $S \subseteq V$ is the set $N_G[S] = \bigcup_{v \in S} N_G[v]$. Two vertices are *neighbors* if they are adjacent. The *degree* of a vertex v is $\deg_G(v) = |N_G(v)|$. The minimum and maximum degrees of a vertex in a graph G are denoted $\delta(G)$ and $\Delta(G)$, respectively. An *isolated vertex* in G is a vertex of degree 0 in G. An *isolate-free graph* is a graph which contains no isolated vertex. A *trivial graph* is the graph of order 1, and a *nontrivial graph* has order at least 2. A *self-complementary graph* is a graph which is isomorphic to its complement. If G is clear from the context, then we will use $N(v)$, $N[v]$, $N[S]$, $N(S)$ and $\deg(v)$ in place of $N_G(v)$, $N_G[v]$, $N_G[S]$, $N_G(S)$ and $\deg_G(v)$, respectively.

The subgraph of G induced by a set $S \subseteq V$ is denoted by $G[S]$. A set S is a *dominating set* of a graph G if $N[S] = V$, that is, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set in a graph G is the *domination number* of G and is denoted by $\gamma(G)$. A *universal vertex* in a graph G of order n, also called a *dominating vertex* in the literature, is a vertex v adjacent to every other vertex of G, and so $\deg(v) = n - 1$. A dominating set S is a *total dominating set*, abbreviated TD-set, of an isolate-free graph G if $G[S]$ has no isolated vertices, that is, $N(S) = V$. If X and Y are sets of vertices in G, where possibly $X = Y$, then the set X *totally dominates* the set Y if every vertex in Y has a neighbor in X. A dominating set S is an *independent dominating set*, abbreviated ID-set, of G if S is an independent set in G, that is, $G[S]$ consists of isolated vertices. The *independent domination number* $i(G)$ is the minimum cardinality of a ID-set of G and an ID-set of cardinality $i(G)$ is called an *i-set* of G. For other graph theory terminology not defined herein, the reader is referred to [12], and for other recent books on domination in graphs, we refer the reader to [10, 11, 17].

2 Motivation and known results

The following classic 1962 result by Ore showed that for any isolate-free graph, its vertex set can be partitioned into two dominating sets.

Theorem 2.1 ([20]) *If G is an isolate-free graph and S is a minimal dominating set of G, then $V \setminus S$ is a dominating set.*

A natural question is which graphs can be partitioned into two types of dominating sets. Let C_n denote the cycle on n vertices. Note that the cycle C_5 cannot be partitioned into a dominating set and a TD-set. Graphs having such a partition were studied in [15, 16]. Henning and Southey [15] established the following sufficient condition for graphs whose vertex set can be partitioned into a dominating set and a TD-set. An *F-component* of a graph G is a component of G that is isomorphic to F.

Theorem 2.2 ([15]) *If G is a graph with $\delta(G) \geq 2$ and no C_5-component, then the vertices of G can be partitioned into a dominating set and a total dominating set.*

In 2019 Delgado, Desormeaux, and Haynes [7] studied graphs whose vertex set can be partitioned into a TD-set and an ID-set, which is the problem we consider in this paper. We refer to such a partition of the vertices of a graph G as a *TDID-partition* of G. If G has a TDID-partition, then we say that G is a *TI-graph*. We remark that if a graph G is a TI-graph, then every TD-set of G contains at least two vertices from every component of G and every ID-set of G contains at least one vertex from every component of G, implying that every component of G has order at least 3. In particular, if G is connected, then G has order at least 3. Not all graphs are TI-graphs as can be easily seen with the cycle C_5 and the path P_5. The paths and cycles having a TDID-partition were determined in [7].

Proposition 2.1 ([7]) *The following hold.*

(a) *A cycle C_n is a TI-graph if and only if $n \equiv 0 \pmod 3$.*
(b) *A nontrivial path P_n is a TI-graph if and only if $n \equiv 1 \pmod 3$.*

A constructive characterization of TI-trees is given in [7], as well as a characterization of the TI-graphs of diameter 2.

Proposition 2.2 ([7]) *A graph G of diameter 2 is a TI-graph if and only if G has a maximal independent set that is not the open neighborhood of some vertex.*

Several sufficient conditions for a graph to be a TI-graph were also given in [7].

Theorem 2.3 ([7]) *Let G be a graph of order n and minimum degree $\delta(G)$, and let $\overline{G}$ be the complement of G. If any of the following conditions holds, then G is a TI-graph:*

(a) $\delta(G) > \frac{1}{2}n$.
(b) $\gamma(\overline{G}) \ge 3$.
(c) *G is claw-free and $\delta(G) \ge 3$.*
(d) $i(G) < \delta(G)$.

In general, characterizing TI-graphs seems to be a challenging problem. Delgado et al. [7] claimed that with the exception of a few graphs, every graph or its complement is a TI-graph. Unfortunately, their result is missing a case and thus is incorrect. One main aim of this paper is to correct this and to characterize the graphs G for which at least one of G or $\overline{G}$ is a TI-graph. Other goals are to characterize the complementary prisms which are TI-graphs and investigate TDID-partitions in prisms.

3 Graphs and their complements

In this section we characterize the graphs G such that at least one of G and its complement $\overline{G}$ is a TI-graph. We show that unless a graph G or its complement $\overline{G}$ is in a given family of graphs, then G or $\overline{G}$ has a TDID-partition.

Definition 3.1 *Let $\mathcal{A}$ be the family that consist of the following graphs:*

- *the trivial graph K_1,*
- *the cycle C_5,*
- *the complete bipartite graphs $K_{r,s}$ for $r \in \{1,2\}$ and $r \le s$, and*
- *the disjoint union $K_r \cup K_s$ for $r \in \{1,2\}$ and $r \le s$.*

We first show that no graph in the family $\mathcal{A}$ is a TI-graph.

Proposition 3.1 *If $G \in \mathcal{A}$, then neither G nor its complement $\overline{G}$ is a TI-graph.*

Proof. Let $G \in \mathcal{A}$. By our earlier observations, every TI-graph has order at least 3. Thus, if $G = K_1$, then G is not a TI-graph. If $G = C_5$, then every TD-set in G contains at least three consecutive vertices of the cycle, and therefore its vertex set cannot be partitioned into a TD-set and an ID-set. Since the 5-cycle is self-complementary, we have $G \cong \overline{G} = C_5$ and therefore infer that $\overline{G}$ has no TDID-partition. Further, since the only ID-set of a bipartite graph $K_{r,s}$, for $1 \le r \le s$, is one of its partite sets and the remaining partite set is not a TD-set, the graph $K_{r,s}$ is not a TI-graph. Moreover the complement of $K_{r,s}$ is the graph $K_r \cup K_s$. As observed earlier, every component of a TI-graph has order at least 3. Hence if $r \in \{1,2\}$, then $K_r \cup K_s$ has a component of order at most 2, and is therefore not a TI-graph. □

We next present a characterization of the graphs G such that at least one of G and its complement $\overline{G}$ is a TI-graph.

Theorem 3.1 *At least one of a graph G and its complement $\overline{G}$ is a TI-graph if and only if $G \notin \mathcal{A}$.*

Proof. By Proposition 3.1, if $G \in \mathcal{A}$, then neither G nor $\overline{G}$ is a TI-graph. Thus, it remains to show that if $G \notin \mathcal{A}$, then G or $\overline{G}$ is a TI-graph. Assume that G is a graph of order n and $G \notin \mathcal{A}$. Thus, $\overline{G} \notin \mathcal{A}$ since K_1 and C_5 are self-complementary, and $K_{r,s}$ and $K_r \cup K_s$ are complements. Since $G \notin \mathcal{A}$ and $\overline{G} \notin \mathcal{A}$, it follows that G has order $n \ge 3$. We proceed by a series of claims.

Claim 1 *If G has a complete component, then G or $\overline{G}$ is a TI-graph.*

Proof. Assume that G has a complete component $F \cong K_k$ for some $k \ge 1$. If G is the complete graph, that is, $G = F$, then since $n \ge 3$, the sets $I = \{x\}$ for any vertex x of G and $T = V(G) \setminus I$ form a TDID-partition of G. Thus, G is a TI-graph. Hence, we may assume that G is not complete.

Let X be the set of vertices in the complete component F, and so $F = G[X]$. Let $Y = V(G) \setminus X$. Since G is not a complete graph, the set $Y \neq \emptyset$. In the complement $\overline{G}$, we note that X is an independent set and $[X, Y]$ is full, that is, every vertex in X is adjacent to every vertex Y in $\overline{G}$. Since $\overline{G} \notin \mathcal{A}$, we have that $\overline{G} \neq K_{r,s}$ for $r \leq s$ and $r \in \{1, 2\}$. Hence, either $\overline{G}$ is the graph $K_{r,s}$ for $3 \leq r \leq s$, or the subgraph $\overline{G}[Y]$ of $\overline{G}$ induced by the set Y has at least one edge. If $\overline{G} = K_{r,s}$ for $3 \leq r \leq s$, then $G = K_r \cup K_s$ for $3 \leq r \leq s$ and G is a TI-graph.

Thus, we may assume that the subgraph $\overline{G}[Y]$ has at least one edge. Let I be a maximal independent set of $\overline{G}[Y]$, and let $T = V(\overline{G}) \setminus I$. Note that $I \subset Y$, $|I| < |Y|$, and so $|T| \geq 2$. By our previous observations, every vertex in $V(\overline{G}) \setminus X$ is adjacent to a vertex in X in $\overline{G}$. Hence, $\overline{G}[T]$ is isolate-free. Since every superset of a dominating set is a dominating set, and since $X \subset T$, we therefore infer that T is a TD-set of $\overline{G}$. Thus, T and I is a partition of the vertex set of $\overline{G}$ into a TD-set and an ID-set, respectively. Hence, $\overline{G}$ is a TI-graph. (□)

By Claim 1, we may assume that G has no complete component, for otherwise the result holds. In particular, we may assume that G has no isolated vertex, that is, $\delta(G) \geq 1$. Similarly, $\delta(\overline{G}) \geq 1$. Since neither G nor $\overline{G}$ has an isolated vertex, it follows that $n \geq 4$. Further, since G has no isolated vertex, the graph $\overline{G}$ has no universal vertex and so $i(\overline{G}) \geq \gamma(\overline{G}) \geq 2$. Similarly, $i(G) \geq \gamma(G) \geq 2$.

Claim 2 *If $i(\overline{G}) \geq 3$, then G is a TI-graph.*

Proof. Assume that $i(\overline{G}) \geq 3$. Consider the graph G, and let I be any i-set of G. Since $\delta(G) \geq 1$, every vertex in I has a neighbor in $V \setminus I$, that is, $V \setminus I$ is a dominating set of G. Suppose that $V \setminus I$ is not a TD-set of G. In this case, there exists a vertex $v \in V \setminus I$ such that v is an isolated vertex in $G[V \setminus I]$, that is, $N_G(v) \subseteq I$. But then the set $\{u, v\}$, where $u \in N_G(v) \cap I$, is an ID-set of $\overline{G}$, and so, $i(\overline{G}) \leq |\{u, v\}| = 2$, contradicting our supposition that $i(\overline{G}) \geq 3$. Hence, $V \setminus I$ is a TD-set of G, and so $V \setminus I$ and I is a partition of the vertex set of G into a TD-set and an ID-set, respectively. Thus, G is a TI-graph. (□)

By Claim 2, we may assume that $i(\overline{G}) \leq 2$, for otherwise the desired result holds. Thus, Claim 1 implies that $i(\overline{G}) = 2$. Similarly, $i(G) = 2$.

Claim 3 *If $\delta(G) \geq 3$, then G is a TI-graph.*

Proof. Assume that $\delta(G) \geq 3$, and let I be any i-set of G. Thus, $|I| = i(G) = 2$. Since $\delta(G) \geq 3$, every vertex in G has a neighbor in $V \setminus I$, implying that $V \setminus I$ is a TD-set of G, and so G is a TI-graph. (□)

By Claims 1 and 3, we may assume that $\delta(G) \in \{1, 2\}$ and $\delta(\overline{G}) \in \{1, 2\}$, for otherwise the result holds. Let $\{a, b\}$ be an i-set of G. Since $\delta(G) \geq 1$, the set

$V \setminus \{a, b\}$ is a dominating set of G. If $G' = G - \{a, b\}$ is an isolate-free graph, then $V \setminus \{a, b\}$ is a TD-set of G, and so G is a TI-graph. Thus, we may assume further that there exists a vertex $x \in V \setminus \{a, b\}$ such that $N_G(x) \subseteq \{a, b\}$. We use this terminology as we continue proving three more claims.

Claim 4 *If $\delta(G) = \delta(\overline{G}) = 2$, then G or $\overline{G}$ is a TI-graph.*

Proof. Assume that $\delta(G) = \delta(\overline{G}) = 2$. Since $\delta(G) = 2$ and $N_G(x) \subseteq \{a, b\}$, we have $N_G(x) = \{a, b\}$. Each of $\{a, x\}$ and $\{b, x\}$ is therefore an ID-set in the graph $\overline{G}$. If $V \setminus \{a, x\}$ or $V \setminus \{b, x\}$ is a TD-set of $\overline{G}$, then $\overline{G}$ is a TI-graph and the result holds. Hence, we may assume that there is a vertex $w \in V \setminus \{a, x\}$ such that $N_{\overline{G}}(w) \subseteq \{a, x\}$ and a vertex $y \in V \setminus \{b, x\}$ such that $N_{\overline{G}}(y) \subseteq \{b, x\}$. Since $\delta(G) = \delta(\overline{G}) = 2$, it follows that $N_{\overline{G}}(w) = \{a, x\}$ and $N_{\overline{G}}(y) = \{b, x\}$, and so $w \neq y$. Thus, $abyxwa$ is an induced 5-cycle in $\overline{G}$.

If $n = 5$, then $\overline{G} = G = C_5 \in \mathcal{A}$, a contradiction. Thus, $n \geq 6$. Since the vertex w is adjacent in G to every vertex except for the vertices a and x, and since the vertex a is adjacent in G to x, the set $I = \{a, w\}$ is an ID-set of G. Analogously, the set $\{b, y\}$ is an ID-set of G. In particular, $\{b, y\}$ is a dominating set of G. Let $T = V(G) \setminus I$. Since every superset of a dominating set is a dominating set and since $\{b, y\} \subset T$, the set T is a dominating set of G. Moreover, since the vertex y is adjacent in G to every vertex except for the vertices b and x, and since the vertex b is adjacent in G to x, the subgraph of G induced by the set T is isolate-free. Hence, the set T is a TD-set of G. Therefore, T and I is a partition of the vertex set of G into a TD-set and an ID-set, respectively. Thus, G is a TI-graph. (□)

By Claim 4, we may assume that $\delta(G) = 1$ or $\delta(\overline{G}) = 1$. Without loss of generality, assume that $\delta(G) = 1$ and $\delta(\overline{G}) \in \{1, 2\}$.

Claim 5 *If $N_G(x) = \{a, b\}$, then G or $\overline{G}$ is a TI-graph.*

Proof. Let $N_G(x) = \{a, b\}$. Since $\delta(G) = 1$, there exist a vertex $w \in V \setminus \{x\}$ such that $\deg_G(w) = 1$. First suppose that $w = a$. Then since $\{a, b\}$ is an ID-set of G, the vertex b dominates $V \setminus \{a\}$. Recall by our earlier assumptions that $n \geq 4$, and so $V \setminus \{a, b, x\} \neq \emptyset$. Let $G_b = G - \{a, b, x\}$ and let I_b be an ID-set of G_b. In particular, we note that $b \notin I_b$. The set $I_b \cup \{a\}$ is an ID-set of G and $V(G) \setminus (I_b \cup \{a\})$ is a TD-set of G (that contains both vertices b and x). Thus, G is a TI-graph. Hence, we may assume that $w \neq a$ and similarly, $w \neq b$, for otherwise the result holds. Thus, $\deg_G(a) \geq 2$ and $\deg_G(b) \geq 2$. Therefore, $w \in V \setminus \{a, b, x\}$ and since $\{a, b\}$ is an ID-set, the neighbor of w is either a or b, say a. But now $\{a, x\}$ is an ID-set of $\overline{G}$ and $V(\overline{G}) \setminus \{a, x\}$ is a TD-set of $\overline{G}$. Hence, $\overline{G}$ is a TI-graph. (□)

By Claim 5, we may assume that x is adjacent to exactly one of a and b in G, and so, $\deg_G(x) = 1$. Without loss of generality, let x be adjacent to a. Since $\delta(G) = 1$, vertex b has a neighbor in $V \setminus \{a, b, x\}$. Furthermore, since by Claim 1 the graph G has no complete component, it follows that a has a neighbor in $V \setminus \{a, b, x\}$ and

$\deg_{\overline{G}}(a) \ge 2$. As before, $\{a, x\}$ is an ID-set of $\overline{G}$ and $V \setminus \{a, x\}$ is a dominating set of $\overline{G}$. If $V \setminus \{a, x\}$ is a TD-set of $\overline{G}$, then the result holds. Thus, assume that there is a vertex $y \in V \setminus \{a, x\}$ such that $N_{\overline{G}}(y) \subseteq \{a, x\}$.

Claim 6 *If $y = b$, then G or $\overline{G}$ is a TI-graph.*

Proof. Suppose that $y = b$. This implies that the vertex b is adjacent in G to every vertex different from a and x, that is, $N_G(b) = V \setminus \{a, b, x\}$. By the structure of the graph G, the set $\{b, x\}$ is an ID-set of G and the set $V \setminus \{b, x\}$ is a dominating set of G. If $V \setminus \{b, x\}$ is a TD-set of G, then G is a TI-graph and the result holds. Thus, assume that $V \setminus \{b, x\}$ is not a TD-set of G, that is, there exists a vertex $z \in V \setminus \{b, x\}$ such that $N_G(z) \subseteq \{b, x\}$. Since $\deg_{\overline{G}}(a) \ge 2$ and a is not adjacent to b in G, we note that $a \ne z$, and so $N_G(z) = \{b\}$.

Let a' be a neighbor of a in G different from x. We note that $a' \notin \{b, x, z\}$, implying that G has order at least 5, that is, $n \ge 5$. The set $\{b, z\}$ is an ID-set of $\overline{G}$ and the set $V \setminus \{b, z\}$ is a dominating set of $\overline{G}$. Let $S = V \setminus \{b, z\}$. If S is a TD-set of $\overline{G}$, then again $\overline{G}$ is a TI-graph and the result holds. Hence, we assume that S is not a TD-set of $\overline{G}$. Since x is adjacent to every vertex in $\overline{G}$ except vertex a and since $n \ge 5$, the only possible vertex that is isolated in the subgraph $\overline{G}[S]$ of $\overline{G}$ induced by the set S is the vertex a. But then $\{a, z\}$ is an ID-set of G. Let $W = V \setminus \{a, b, x, z\}$. By our earlier observations, every vertex in W is adjacent in G to both the vertex a and the vertex b. If there is an edge in $G[W]$, then let I_W be an ID-set of $G[W]$. In this case, $I_W \cup \{x, z\}$ is an ID-set of G and $V \setminus (I_W \cup \{x, z\})$ is a TD-set of G, and so G is a TI-graph. If there is no an edge in $G[W]$, then $V \setminus \{a, b\}$ is a clique in $\overline{G}$. In this case, $\{a, a'\}$ is an ID-set of $\overline{G}$ and $V \setminus \{a, a'\}$ is a TD-set of $\overline{G}$. Hence, $\overline{G}$ is a TI-graph and the result holds. (□)

By Claim 6, we may assume that $y \ne b$, for otherwise the desired result follows. Let b' be a neighbor of b in $\overline{G}$ different from a and x. By assumption, $y \ne b'$. We note that y is adjacent to x in $\overline{G}$. However, y may or may not be adjacent to a in $\overline{G}$. The vertex b' is not adjacent in G to the vertex b. Since $\{a, b\}$ is an ID-set of G, vertex b' is therefore adjacent to vertex a. Furthermore, vertex y dominates $V \setminus \{a, x\}$ in G. Thus, $\{x, y\}$ is an ID-set of G, and $V \setminus \{x, y\}$ is a dominating set of G.

If $V \setminus \{x, y\}$ is a TD-set of G, then G is a TI-graph and the result holds. Thus, assume that there exists a vertex $z \in V \setminus \{x, y\}$ such that $N_G(z) \subseteq \{x, y\}$. Since $\{a, b\} \subset V \setminus \{x, y\}$, every vertex in $V \setminus \{a, b\}$ has a neighbor in $\{a, b\}$, and a is adjacent to b', we infer that $z = b$ and $N_G(b) = \{y\}$. Hence, in the graph G, the vertex a dominates $V \setminus \{b, y\}$, the vertex y dominates $V \setminus \{a, x\}$, and possibly a is adjacent to y.

Let $R = \{a, b, x, y\}$ and let $G_R = G - R$. Let I be an ID-set of G_R. If a is adjacent to y in G or if $V \setminus (I \cup R) \ne \emptyset$, then $I \cup \{b, x\}$ is an ID-set of G and $V \setminus (I \cup \{b, x\})$ is a TD-set of G. If a is not adjacent to y in G and $V \setminus (I \cup R) = \emptyset$, then $V \setminus R$ is an independent set of vertices in G, and so $\overline{G}[V \setminus R]$ is a clique. From the structure of the graph $\overline{G}$, we infer that $\{y, b'\}$ is an i-set of $\overline{G}$ and $V \setminus \{y, b'\}$ is a TD-set of $\overline{G}$. Hence, $\overline{G}$ is a TI-graph. This completes the proof of Theorem 3.1. □

Since the trivial graph K_1 and the cycle C_5 are the only self-complementary graphs in $\mathcal{A}$, the following result is an immediate consequence of Theorem 3.1.

Corollary 3.1 *Every nontrivial, self-complementary graph different from the* 5*-cycle is a TI-graph.*

4 Complementary prisms

Let G be a graph and $\overline{G}$ its complement. For every vertex $v \in V(G)$, we denote $\overline{v} \in V(\overline{G})$ as its corresponding vertex, and for a set $X \subseteq V(G)$, let $\overline{X}$ denote the corresponding set of vertices in $V(\overline{G})$. A variation of the prism, called a complementary prism, was introduced in [13] as follows, and is studied, for example, in [2, 3, 5, 14] and elsewhere.

For a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$, the *complementary prism* of G is the graph, denoted by $G\overline{G}$, with vertex set $V(G\overline{G}) = \{v_1, \ldots, v_n\} \cup \{\overline{v}_1, \ldots, \overline{v}_n\}$ and edge set $E(G\overline{G}) = E(G) \cup E(\overline{G}) \cup \{v_1\overline{v}_1, \ldots, v_n\overline{v}_n\}$. Thus, $G\overline{G}$ is constructed from $G \cup \overline{G}$ by adding a perfect matching between the vertices of G and the corresponding vertices of $\overline{G}$.

For example, if G is the 5-cycle given by $v_1v_2v_3v_4v_5v_1$, then the complementary prism $G\overline{G}$ is the Petersen graph $P(5, 2)$ illustrated in Figure 1. We observe that the shaded vertices in Figure 1 form an independent set in G and the white vertices form a TD-set in G. Moreover, these two sets partition the set $V(G)$, thereby forming a TDID-partition of G. Thus, the complementary prism $C_5\overline{C}_5$ of a 5-cycle is the Petersen graph, which is a TI-graph.

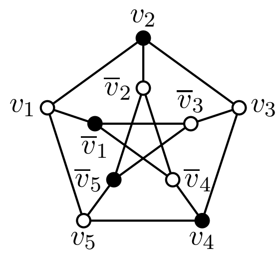
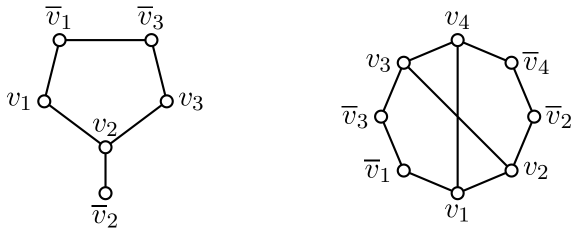


Figure 1: The complementary prism $C_5\overline{C}_5$ of a 5-cycle C_5

Since by Theorem 3.1, if $G \notin \mathcal{A}$, then G or $\overline{G}$ is a TI-graph, it seems logical to next consider for which graphs G is the complementary prism $G\overline{G}$ a TI-graph. Let G_1 be the complementary prism of the path $v_1v_2v_3$, and let G_2 be the complementary prism of the 4-cycle $v_1v_2v_3v_4v_1$. The graphs G_1 and G_2 are illustrated in Figure 2(a) and 2(b), respectively. We shall show that with the exception of these two complementary prisms, G_1 and G_2, every complementary prism of a nontrivial graph is a TI-graph. We show firstly that neither G_1 nor G_2 is a TI-graph.



(a) G_1 (b) G_2

Figure 2: The complementary prisms G_1 and G_2

Proposition 4.1 *The complementary prisms G_1 and G_2 shown in Figure 2 are not TI-graphs.*

Proof. We consider firstly the complementary prism G_1 of the path P_3 given by $v_1v_2v_3$ as shown in Figure 2(a). Suppose, to the contrary, that G_1 contains a TDID-partition $\{I, T\}$ where I is an ID-set of G_1 and T is a TD-set of G_1. In order to totally dominate the vertex $\overline{v}_2$, the TD-set T contains the vertex v_2. Thus, in order to dominate the vertex $\overline{v}_2$, the ID-set I contains the vertex $\overline{v}_2$. In order to totally dominate the vertex v_2, at least one of v_1 and v_3 belongs to the set T. By symmetry, we may assume that $v_1 \in T$. It follows that $\overline{v}_1 \in I$ in order for the ID-set I to dominate the vertex v_1. Since I is an independent set, this in turn implies that $\overline{v}_3 \in T$. In order to totally dominate the vertex $\overline{v}_3$, we infer that $v_3 \in T$. However, v_3 is not dominated by I, a contradiction. Hence, G_1 is not a TI-graph.

Next we consider the complementary prism G_2 of the 4-cycle C_4 given by $v_1v_2v_3v_4v_1$ as shown in Figure 2(b). Suppose, to the contrary, that G_2 contains a TDID-partition $\{I, T\}$ where I is an ID-set of G_2 and T is a TD-set of G_2. Let $X = \{v_1, v_2, v_3, v_4\}$ and let $\overline{X} = \{\overline{v}_1, \overline{v}_2, \overline{v}_3, \overline{v}_4\}$, and so $V(G_2) = X \cup \overline{X}$. If $X \subseteq T$, then in order to dominate the vertices in X, we must have $\overline{X} \subseteq I$. However, the resulting set I is then not an independent set, a contradiction. Hence, at least one vertex of X does not belong to the set T. By symmetry, and renaming vertices if necessary, we may assume that $v_1 \notin T$, and so $v_1 \in I$. Thus, $N(v_1) = \{\overline{v}_1, v_2, v_4\} \subseteq T$. In order to totally dominate the vertex $\overline{v}_1$, we infer that $\overline{v}_3 \in T$. It follows that $v_3 \in I$ in order for the ID-set I to dominate the vertex $\overline{v}_3$. This in turn implies that $\overline{v}_2$ and $\overline{v}_4$ belong to the set T in order for T to totally dominate v_2 and v_4, respectively. But now the ID-set I dominates neither $\overline{v}_2$ nor $\overline{v}_4$, a contradiction. Hence, G_2 is not a TI-graph. $\square$

We proceed further with the following property of TI-graphs.

Lemma 4.1 *If G is a TI-graph and G is not complete, then there exists a TDID-partition $\{T, I\}$ of G with TD-set T and ID-set I such that $|I| \geq 2$.*

Proof. Let G be a TI-graph different from the complete graph. Let $\{T, I\}$ be a TDID-partition of G with TD-set T and ID-set I. If $|I| \geq 2$, then the desired result is immediate. Hence, we may assume that $|I| = 1$, and so $i(G) = 1$. Let $I = \{x\}$, and so the vertex x is a universal vertex of G. Since G is not complete, the set $T = V \setminus \{x\}$ has at least two non-adjacent vertices, say u and v. Let I' be a maximal independent set of G containing the vertices u and v. Since x is adjacent to both u and v, we note that $x \notin I'$. Since T is a TD-set of G, every vertex of G has a neighbor in T, and so $|T \setminus I'| \geq 1$. Hence, I' is an ID-set of G and $T' = V \setminus I'$ is a TD-set of G forming a TDID partition $\{T', I'\}$ such that $|I'| \geq 2$. $\square$

We are now in a position to show that the complementary prism of every nontrivial graph is a TI-graph, unless it is one of the two complementary prisms G_1 and G_2 shown in Figure 2.

Theorem 4.1 *The complementary prism $G\overline{G}$ of a nontrivial graph G is a TI-graph if and only if $G\overline{G}$ is not one of the two graphs G_1 and G_2 shown in Figure 2.*

Proof. Let G be a nontrivial graph. If $G\overline{G}$ is one of the two complementary prisms G_1 and G_2 shown in Figure 2, then, by Proposition 4.1, the complementary prism G is not a TI-graph. Suppose next that $G\overline{G}$ is not one of the two graphs G_1 and G_2 shown in Figure 2. Thus, G has order at least 2 and $G \notin \{P_3, C_4\}$.

Suppose that $G \notin \mathcal{A}$. By Theorem 3.1, at least one of G and $\overline{G}$ is a TI-graph. Without loss of generality, we may assume that G is a TI-graph. In particular, G has order at least 3. If G is a complete graph, then the vertices of G form a TD-set of $G\overline{G}$ and the vertices of $\overline{G}$ form an ID-set of $G\overline{G}$, implying that $G\overline{G}$ is a TI-graph. If G is not a complete graph, then by Lemma 4.1, there exists a TDID-partition $\{T, I\}$ of G, where T is a TD-set of G, I is an ID-set of G, and $|I| \geq 2$. We note that $\overline{I}$ induces a complete graph in $\overline{G}$.

Let $\overline{S}$ be an ID-set of the induced subgraph $\overline{G}[\overline{T}]$. We claim that $I^* = I \cup \overline{S}$ is an ID-set of $G\overline{G}$ and $T^* = V(G\overline{G}) \setminus (I \cup \overline{S})$ is a TD-set of $G\overline{G}$, that is, $G\overline{G}$ is a TI-graph. To see this, we note that I^* is independent, I dominates the vertices of $V(G) \cup \overline{I}$, and $\overline{S}$ dominates the vertices of $\overline{T} = V(\overline{G}) \setminus \overline{I}$. Since T is a TD-set of G, the set T totally dominates the vertices of $V(G)$, and by construction of the complementary prism the set T totally dominates the vertices of $\overline{T}$. Further since $|\overline{I}| \geq 2$ and $\overline{I}$ is a clique, the vertices of $\overline{I}$ are totally dominated by $\overline{I}$. Hence, T^* is a TD-set of $G\overline{G}$. Thus, $G\overline{G}$ is a TI-graph, and therefore has order at least 4.

Next assume that $G \in \mathcal{A}$. Since $G\overline{G}$ has order at least 4, the graph G has order at least 2. In particular, $G \neq K_1$. If $G = C_5$, then $G\overline{G}$ is the Petersen graph, which by our earlier observations is a TI-graph. Suppose, therefore, that G or $\overline{G}$, say G, is a complete bipartite graph $K_{r,s}$ for some $r \in \{1, 2\}$ and $r \leq s$. Suppose firstly that $r = 1$, and so $G = K_{1,s}$ where $s \geq 1$. If $G = K_{1,1}$, then the complementary prism $G\overline{G}$ is a path P_4, which is a TI-graph. Hence, we may assume that $s \geq 2$. If $G = K_{1,2}$, then $G\overline{G}$ is the complementary prism G_1 shown in Figure 2(a), a contradiction. Hence, $s \geq 3$. Let v_1 be the center of the star and label the leaves of G by $v_2, \dots v_{s+1}$. Then, $I = \{\overline{v}_1, v_2, \dots, v_s, \overline{v}_{s+1}\}$ is an ID-set of $G\overline{G}$ and $V(G\overline{G}) \setminus I$ is a TD-set of $G\overline{G}$, implying that $G\overline{G}$ is a TI-graph.

Suppose next that $r = 2$, and so $G = K_{2,s}$ where $s \geq 2$. If $G = K_{2,2} = C_4$, then $G\overline{G}$ is the complementary prism G_2 shown in Figure 2(b), a contradiction. Hence, $s \geq 3$. Let v_1 and v_2 be the vertices in the smaller partite set of G and label the vertices in the other partite set $v_3, v_4, \ldots, v_{s+2}$. Then $I = \{\overline{v}_1, v_3, v_4 \ldots v_{s+1}, \overline{v}_{s+2}\}$ is an ID-set of $G\overline{G}$ and $V(G\overline{G}) \setminus I$ is a TD-set of $G\overline{G}$, implying that $G\overline{G}$ is a TI-graph. $\square$

5 Prisms

The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$ and two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \,\square\, H$ if either $g_1 = g_2$ and h_1h_2 is an edge in H, or $h_1 = h_2$ and g_1g_2 is an edge in G. The *prism* of a graph G is the graph $G \,\square\, K_2$. Thus, it is defined by taking two disjoint copies G_1 and G_2 of G, and adding an edge between each pair of corresponding vertices. The resulting added edges form a perfect matching in the prism. We refer to the vertices joined by such a matching edge as *partners*. If G is a path or a cycle, then we call the prism $G \,\square\, K_2$ a *path prism* and *cycle prism*, respectively. If every vertex of G is contained in a triangle, then we call the prism $G \,\square\, K_2$ a *triangle prism*.

The relationship between domination parameters in the graph and its prism have been studied extensively. See, for example, [1, 4, 6, 8, 9, 14, 18, 19, 22, 21]. Since the complementary prism is a variant of the prism of a graph G where one takes a copy of G and its complement $\overline{G}$ instead of two copies of G, a natural next step is to consider the problem of determining for which graphs G is the prism $G \,\square\, K_2$ a TI-graph. Recall that we showed in Section 4 that the complementary prism of every nontrivial graph is a TI-graph, unless it is one of the two complementary prisms G_1 and G_2 shown in Figure 2. The characterization of prisms that are TI-graphs seems to be a more difficult problem than for complementary prisms. In this section, we characterize the path, cycle, and triangle prisms that are TI-graphs, and provide two infinite families of graphs G for which the prism $G \,\square\, K_2$ is not a TI-graph.

5.1 Path prisms

We show in this section that the path prism $P_n \,\square\, K_2$ is a TI-graph for all $n \geq 3$ except for $n = 4$. We observe that the path prism $P_1 \,\square\, K_2 = K_2$ is not a TI-graph and that $P_2 \,\square\, K_2$ is the 4-cycle, which is not a TI-graph. We show next that the path prism $P_4 \,\square\, K_2$ is not a TI-graph.

Proposition 5.1 *The path prism $P_4 \,\square\, K_2$ is not a TI-graph.*

Proof. Let $G = P_4 \,\square\, K_2$ be the path prism shown in Figure 3. Suppose, to the contrary, that G contains a TDID-partition $\{I, T\}$ where I is an ID-set of G and T is a TD-set of G. If neither u_1 nor v_1 belong to the set I, then $\{u_2, v_2\} \subseteq I$ in order for I to dominate u_1 and v_1, contradicting the independence of the set I. Hence,

u_1 or v_1 belongs to the set I. By symmetry, we may assume that $u_1 \in I$, and so $\{v_1, u_2\} \subseteq T$. Thus, $v_2 \in T$ in order for T to totally dominate v_1, implying that $v_3 \in I$ in order for I to dominate v_2, and so $\{u_3, v_4\} \subseteq T$. Hence, $u_4 \in I$ in order for I to dominate u_4. But then the vertex v_4 is not totally dominated by the set T. $\square$

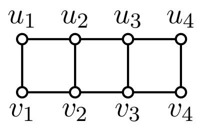


Figure 3: The path prism $P_4 \,\square\, K_2$

Proposition 5.2 *The path prism $P_n \,\square\, K_2$ is a TI-graph for all $n \geq 3$ and $n \neq 4$.*

Proof. For $n \geq 3$ and $n \neq 4$, let G be the path prism $P_n \,\square\, K_2$. Let G_1 and G_2 be the two disjoint copies of the path P_n in the prism G, where G_1 is the path $u_1u_2 \ldots u_n$ and G_2 is the path $v_1v_2 \ldots v_n$. Further, let the vertices u_i and v_i be partners in the path prism G, and so u_iv_i is an edge in G. We consider three cases. In all three cases, we give a TDID-partition $\{I, T\}$ where I is an ID-set of G and T is a TD-set of G.

Case 1. $n \equiv 0 \,(\mathrm{mod}\, 3)$. Thus, $n = 3k$ for some $k \geq 1$. In this case, we let $I = \bigcup_{i=1}^{k} \{u_{3i-2}, v_{3i}\}$ and $T = V(G) \setminus I$. For example, when $n = 9$ (and $k = 3$) the set I is given by the shaded vertices in Figure 4.

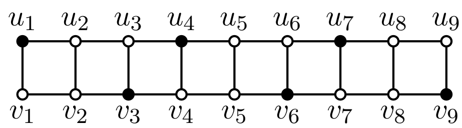


Figure 4: A path prism $P_n \,\square\, K_2$ where $n \equiv 0 \,(\mathrm{mod}\, 3)$

Case 2. $n \equiv 1 \,(\mathrm{mod}\, 3)$ and $n \geq 7$. Thus, $n = 3k + 1$ for some $k \geq 2$. In this case, we let $I = \{u_{n-2}, v_n\} \cup \bigcup_{i=1}^{k-1} \{u_{3i-2}, v_{3i}\}$ and $T = V(G) \setminus I$. For example, when $n = 10$ (and $k = 3$) the set I is given by the shaded vertices in Figure 5.

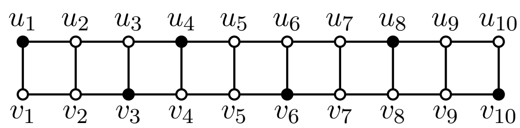


Figure 5: A path prism $P_n \,\square\, K_2$ where $n \equiv 1 \,(\mathrm{mod}\, 3)$ and $n \geq 7$

Case 3. $n \equiv 2 \,(\mathrm{mod}\, 3)$ and $n \geq 5$. Thus, $n = 3k + 2$ for some $k \geq 1$. In this case, we let $I = \{u_n\} \cup \bigcup_{i=1}^{k} \{u_{3i-2}, v_{3i}\}$ and $T = V(G) \setminus I$. For example, when $n = 11$ (and $k = 3$) the set I is given by the shaded vertices in Figure 6.

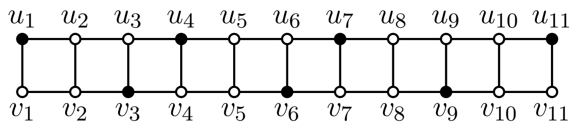


Figure 6: A path prism $P_n \,\square\, K_2$ where $n \equiv 2 \pmod 3$

We deduce from the above three cases that the path prism $G = P_n \,\square\, K_2$ is a TI-graph. $\square$

5.2 Cycle prisms

We show that with the exception of the prism $C_5 \,\square\, K_2$, the cycle prism $C_n \,\square\, K_2$ is a TI-graph for all $n \geq 3$.

Proposition 5.3 *The cycle prism $C_5 \,\square\, K_2$ is not a TI-graph.*

Proof. Let $G = C_5 \,\square\, K_2$ be the cycle prism shown in Figure 7. Suppose, to the contrary, that G contains a TDID-partition $\{I, T\}$ where I is an ID-set of G and T is a TD-set of G. We note that G is vertex-transitive. Renaming the vertices if necessary, we may assume that $v_1 \in I$, implying that $N_G(v_1) = \{v_2, v_5, u_1\} \subseteq T$. Suppose that u_2 or u_5 belongs to the set I. By symmetry, we may assume that $u_2 \in I$. In order for the set T to totally dominate the vertices u_1 and v_2, we infer that $u_5 \in T$ and $v_3 \in T$, respectively. This in turn implies that $u_4 \in I$ in order for the set I to dominate the vertex u_5, and therefore the neighbors u_3 and v_4 of u_4 belong to the set T. But then the set I does not dominate the vertex v_3, a contradiction. Hence, $\{u_2, u_5\} \subset T$. This implies that $\{u_3, u_4\} \subset I$ in order for the set I to dominate the vertices u_2 and u_5. However, the ID-set I then contains two adjacent vertices, namely u_3 and u_4, a contradiction. $\square$

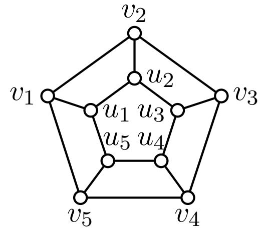


Figure 7: The prism $C_5 \,\square\, K_2$

Proposition 5.4 *The cycle prism $C_n \,\square\, K_2$ is a TI-graph for all $n \geq 3$ and $n \neq 5$.*

Proof. For $n \geq 3$ and $n \neq 5$, let G be the cycle prism $C_n \,\square\, K_2$. Let G_1 and G_2 be the two disjoint copies of the cycle C_n in the prism G, where G_1 is the cycle

$u_1u_2 \dots u_nu_1$ and G_2 is the cycle $v_1v_2 \dots v_nv_1$. Further, let the vertices u_i and v_i be partners in the cycle prism G, and so u_iv_i is an edge in G. We consider three cases. In all three cases, we give a TDID-partition $\{I, T\}$ where I is an ID-set of G and T is a TD-set of G.

Case 1. $n \equiv 0 \pmod 3$. Thus, $n = 3k$ for some $k \geq 1$. In this case, we let $I = \bigcup_{i=1}^{k} \{u_{3i-2}, v_{3i}\}$ and $T = V(G) \setminus I$. For example, when $n = 9$ (and $k = 3$) the set I is given by the shaded vertices in Figure 8.

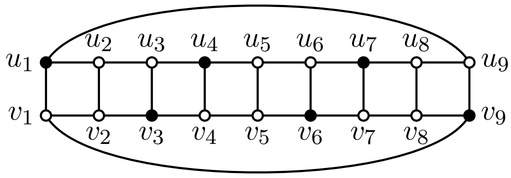


Figure 8: A cycle prism $C_n \,\Box\, K_2$ where $n \equiv 0 \pmod 3$

Case 2. $n \equiv 1 \pmod 3$. Thus, $n = 3k + 1$ for some $k \geq 1$. In this case, we let $I = \bigcup_{i=1}^{k} \{u_{3i-2}, v_{3i}\}$ and $T = V(G) \setminus I$. For example, when $n = 10$ (and $k = 3$) the set I is given by the shaded vertices in Figure 9.

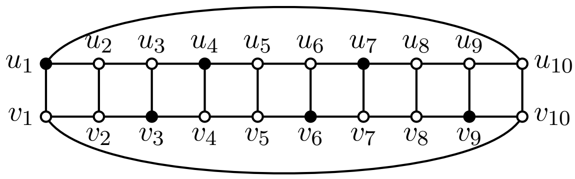


Figure 9: A cycle prism $C_n \,\Box\, K_2$ where $n \equiv 1 \pmod 3$

Case 3. $n \equiv 2 \pmod 3$ *and* $n \geq 8$. Thus, $n = 3k + 2$ for some $k \geq 2$. In this case, we let $I = \{u_{n-3}, v_{n-1}\} \cup \bigcup_{i=1}^{k-1} \{u_{3i-2}, v_{3i}\}$ and $T = V(G) \setminus I$. For example, when $n = 11$ (and $k = 3$) the set I is given by the shaded vertices in Figure 8.

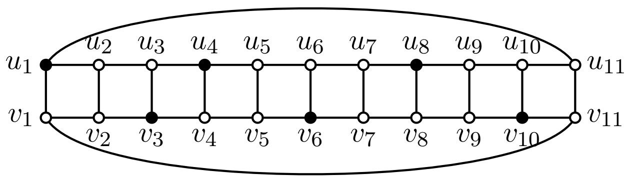


Figure 10: A cycle prism $C_n \,\Box\, K_2$ where $n \equiv 2 \pmod 3$ and $n \geq 8$

We deduce from the above three cases that the graph G is a TI-graph. $\square$

5.3 Triangle prisms

Let G be a graph in which every vertex belongs to a triangle. Let I be an ID-set in G and let $T = V(G) \setminus I$. Let v be an arbitrary vertex in T, and let T_v be a triangle that contains the vertex v. The triangle T_v contains at most one vertex from the independent set I, implying that the vertex v has at least one neighbor in T. Thus, the set T totally dominates the set T. Moreover, since I is an ID-set of G, the set T totally dominates the set I, and so T is a TD-set of G. Hence, $\{T, I\}$ is a TDID-partition of G, and so G is a TI-graph. We state this formally as follows.

Observation 5.1 *If G is a graph in which every vertex belongs to a triangle, then G is a TI-graph.*

As a consequence of Observation 5.1, this yields the following class of graphs G for which the prism $G \,\square\, K_2$ is a TI-graph.

Proposition 5.5 *If G is a graph in which every vertex belongs to a triangle, then the prism $G \,\square\, K_2$ is a TI-graph.*

5.4 Prisms that are not TI-graphs

Next we present several classes of graphs G for which the prism $G \,\square\, K_2$ is not a TI-graph. For notational convenience in this section, we label the two disjoint copies of G used to construct $G \,\square\, K_2$ as G_1 and G_2, where the vertices of G_1 are labeled with subscript 1 and their partners have corresponding labels with subscript 2. Thus, if v is a vertex of G, then v is labeled v_i in G_i for $i \in \{1, 2\}$ and $v_1v_2 \in E(H)$. The *corona* $G \circ K_1$ of a graph G is the graph obtained from G by adding for each vertex $v \in V$ a new vertex v' and the edge vv'. We consider next the prism $(G' \circ K_1) \,\square\, K_2$ of the corona $G' \circ K_1$ of a graph G'.

Proposition 5.6 *If $G = G' \circ K_1$ is the corona of an arbitrary graph G', then the prism $G \,\square\, K_2$ is not a TI-graph.*

Proof. Let G' be an arbitrary graph, $G = G' \circ K_1$, and $H = G \,\square\, K_2$. Using the notation mentioned in our previous comments, we let G_1 and G_2 be the two disjoint copies of $G = G' \circ K_1$ in the prism H.

Suppose, to the contrary, that H contains a TDID-partition $\{I, T\}$ where I is an ID-set of H and T is a TD-set of H. Let v_1 be an arbitrary vertex that belongs to the graph G' in the copy of G_1 and v_2 its partner in G_2. Thus, $v_i \in V(G_i)$ for $i \in \{1, 2\}$ and v_1v_2 is a matching edge in the prism H. Let u_i be the neighbor of v_i of degree 1 in the corona graph G_i for $i \in \{1, 2\}$, and so u_1 and u_2 are partners in H and both have degree 2 in H. As an illustration, when G' is the path P_3 given by vxw and the neighbors of v, x, and w of degree 1 in the corona $G' \circ K_1$ are u, y, and z, respectively, then the vertices in the prism H are as labelled in Figure 11.

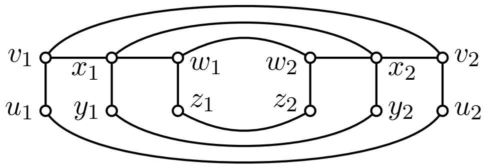


Figure 11: The prism $(P_3 \circ K_1) \,\square\, K_2$

We now consider a vertex u of degree 1 in the corona $G' \circ K_1$, and let v denote the (unique) neighbor of u in $G' \circ K_1$. Thus, $v_1v_2u_2u_1v_1$ is an induced 4-cycle in the prism H. If neither u_1 nor u_2 belongs to the set I, then in order to dominate the vertices u_1 and u_2 the ID-set I contains both v_1 and v_2. However, v_1 and v_2 are adjacent vertices, contradicting the fact that I is an independent set. Hence, exactly one of u_1 and u_2 belongs to the set I. By symmetry, we may assume that $u_1 \in I$, and so $N_H(u_1) = \{v_1, u_2\} \subseteq T$. In order to totally dominate the vertex u_2, the vertex v_2 belongs to the set T. In order to dominate the vertex v_2, a neighbor of v_2, say x_2, belongs to the set I.

Since the partner v_1 of v_2 belongs to the set T, we note that $x_2 \in V(G_2)$ and that its partner x_1 belongs to $V(G_1)$. Let y_i be the neighbor of x_i of degree 1 in the corona graph G_i for $i \in \{1, 2\}$, and so y_1 and y_2 are partners in H and both have degree 2 in H. Moreover, $x_1x_2y_2y_1x_1$ is an induced 4-cycle in the prism H. Since $x_2 \in I$, the neighbors x_1 and y_2 of x_2 belong to the set T. Thus, $N_H(y_1) = \{x_1, y_2\} \subset T$, implying that $y_1 \in I$ in order for the set I to dominate the vertex y_1. But then $N_H(y_2) = \{x_2, y_1\} \subset I$, and so the set T does not totally dominate the vertex y_2, a contradiction. Hence, the prism $H = (G' \circ K_1) \,\square\, K_2$ is not a TI-graph. $\square$

For $k \geq 3$, let H_k be obtained from a complete graph K_k by adding for each vertex v in the complete graph a 5-cycle C_v and adding an edge from v to exactly one vertex in C_v. The graph H_4, for example, is illustrated in Figure 12.

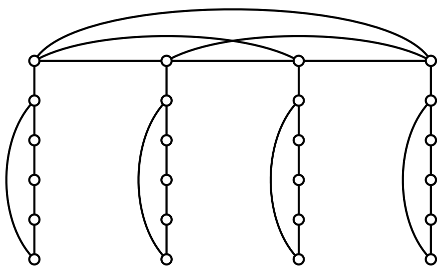


Figure 12: The graph H_4

Proposition 5.7 *For $k \geq 3$, the prism $H_k \,\square\, K_2$ is not a TI-graph.*

Proof. For $k \geq 3$, let $G = H_k$ and H be the prism $G \,\square\, K_2$. Using the labelling notation previously described, let G_1 and G_2 be the two disjoint copies of G in the prism H. Suppose, to the contrary, that H contains a TDID-partition $\{I, T\}$ where I is an ID-set of H and T is a TD-set of H. Let u_1 be an arbitrary vertex in G_1 that belongs to the complete graph K_k in G_1, and let $Q_1\colon v_1w_1x_1y_1z_1v_1$ be the 5-cycle added to the vertex u_1 in the complete graph when constructing G_1, where u_1v_1 is the edge added from u_1 to a vertex of Q_1 in G_1. Then u_2, v_2, w_2, x_2, y_2, and z_2 are the partners of the vertices u_1, v_1, w_1, x_1, y_1, and z_1, respectively, in the prism H. Thus, the graph F shown in Figure 13 is a subgraph of the prism H. We note that if v is a vertex of F different from u_1 and u_2, then the degree of v in F is equal to its degree in H.

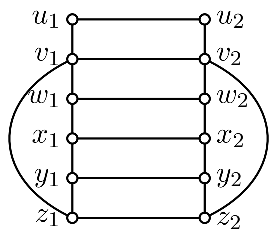


Figure 13: A subgraph F of the prism $H_k \,\Box\, K_2$

We proceed further with the following claim.

Claim 7 *The set I contains one of u_1 and u_2.*

Proof. We show firstly that the ID-set I contains one of z_1 and z_2. Suppose, to the contrary, that neither z_1 nor z_2 belongs to the set I. If neither y_1 nor y_2 belongs to the set I, then in order to dominate the vertices y_1 and y_2, the set I contains both x_1 and x_2, contradicting the fact that I is an independent set. Hence, I contains one of y_1 and y_2. By symmetry, we may assume that $y_1 \in I$, implying that $\{x_1, y_2\} \subseteq T$. In order to dominate the vertex z_2, we have $v_2 \in I$. Thus, $N_H(v_2) \subseteq T$, implying in particular that $\{v_1, w_2\} \subset T$. Thus, $N_H(w_1) = \{v_1, w_2, x_1\} \subset T$ and $N_H(x_2) = \{w_2, x_1, y_2\} \subset T$. In order to dominate the vertices w_1 and x_2, the set I contains both these two vertices. But then $N_H(x_1) = \{w_1, x_2, y_1\} \subset I$, and so the vertex x_1 is not totally dominated by the set T, a contradiction. Hence, the ID-set I contains one of z_1 and z_2.

Suppose that $z_1 \in I$. Hence, $N_H(z_1) = \{v_1, y_1, z_2\} \subseteq T$. If $y_2 \in I$, then the vertex $x_2 \in T$. Moreover, in this case the set T contains the vertex v_2 in order to totally dominate the vertex z_2, and the set T contains the vertex x_1 in order to totally dominate the vertex y_1. This in turn implies that the set I contains the vertex w_1 in order to dominate the vertex x_1. Thus, $w_2 \in T$. Since $\{v_1, v_2, w_2\} \subset T$, the set I contains the vertex u_2 in order to dominate the vertex v_2. If $y_2 \notin I$, then the set I contains the vertex x_2 in order to dominate the vertex y_2. Therefore, the neighbors of x_2 belong to the set T, and so $\{x_1, w_2\} \subset T$. Hence, $N_H(w_1) = \{v_1, w_2, x_1\} \subset T$, and so the set I contains w_1 in order to dominate the vertex w_1. This in turn implies that the set T contains v_2 in order to totally dominate the vertex w_2. Since

$\{v_1, v_2, w_2\} \subset T$, the set I contains the vertex u_2 in order to dominate the vertex v_2. Therefore, we have shown that if $z_1 \in I$, then the set I contains the vertex u_2. By symmetry, if $z_2 \in I$, then the set I contains the vertex u_1. $(\square)$

We now return to the proof of Proposition 5.7. Let X_i be the clique of size k in G_i, and so the set X_i induces a complete graph K_k in G_i for $i \in \{1, 2\}$. Let u_1 be an arbitrary vertex in the clique X_1. Then, u_2 is the partner of u_1, the vertex u_2 belongs to the clique X_2, and u_1u_2 is an edge of H. By Claim 7, the set I contains one of u_1 and u_2. This is true for every vertex that belongs to X_1 and its partner that belongs to X_2. Thus, since $k \geq 3$, the set I contains at least $\lceil k/2 \rceil \geq 2$ vertices that belong the clique K_k in G_1 or the clique K_k in G_2. Hence, the independent set I contains at least two adjacent vertices, a contradiction. Therefore, for $k \geq 3$ and $G = H_k$, the prism $H = G \,\square\, K_2$ is not a TI-graph. $\square$

By Propositions 5.6 and 5.7, there exists an infinite family of connected graphs G with minimum degree $\delta(G) = 1$ and $\delta(G) = 2$, respectively, such that the prism $G \,\square\, K_2$ is not a TI-graph.

6 Concluding remarks and open problems

In this paper we characterize the graphs G such that at least one of G and its complement $\overline{G}$ is a TI-graph. As an application of this characterization, we show that the complementary prism of every nontrivial graph is a TI-graph, unless it is one of the two complementary prisms G_1 and G_2 shown in Figure 2. It remains, however, an open problem to characterize the (connected) graphs G for which the prism $G \,\square\, K_2$ a TI-graph. Among other results, we show that there are infinitely many graphs G for which the prism $G \,\square\, K_2$ is not a TI-graph, and there are infinitely many graphs G for which the prism $G \,\square\, K_2$ is a TI-graph.

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