BRANCHED COVERINGS OF MAPS AND LIFTS OF MAP HOMOMORPHISMS

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ABSTRACT. In this article we generalize both ordinary and permutation voltage constructions to obtain all branched coverings of maps. We approach a map as a set of flags together with three fixed-point-free involutions and relate this approach with other standard representations. We define a lift as a function from these flags into a group. Ordinary voltage and ordinary current assignments are special cases of our lifts. We interpret our construction as an assignment of voltages to the corners of an embedded graph. We describe a simple necessary and sufficient condition for a map homomorphism of base graphs to lift to a homomorphism of covering graphs. As an application we construct centrally-symmetric self-dual spherical polyhedra.

1. INTRODUCTION

There is a rich history of current graphs and their dual concept of voltage graphs. This theory gives a convenient combinatorial method of using embedded graphs to

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describe branched coverings of surfaces (2-manifolds). It is also useful in constructing embeddings of a wide variety of graphs. For example, current graphs were critical in the proof of the "Map Colour Theorem" [**13**]. ARCHDEACON [**1**] has given a common generalization of current and voltage constructions using only voltages on the medial graph.

In this paper we describe a generalization of voltages and currents in the context where maps are represented by three fixed-point-free involutions. The theory turns out to be equivalent to that in [**1**]; however, we extend the theory to permutation voltages (ARCHDEACON dealt only with ordinary voltages). As a result of this extension we obtain all branched coverings between surfaces which are represented by embedded graphs with the branch points at the faces and vertices. Maps related to the given map, such as the dual, also lift nicely. A simple consequence of this point of view is that the dual of the lift is the lift of the dual.

We also provide a necessary and sufficient condition for a graph homomorphism between two maps to have a lift to simultaneous derived graphs. This generalizes a result of GVOZDJAK AND ŠIRÁŇ [**8**] who gave a necessary and sufficient condition for an automorphism of a map to have a lift to a map derived from a voltage construction.

Throughout this work we shall assume some familiarity with standard voltage constructions. A useful reference is Gross and Tucker [**6**].

2. MAPS

We shall mainly be concerned with viewing maps as sets of flags with three involutions on the flags. This form of a map is found, for example, in JAMES AND THORNTON [**9**]; see also COXETER AND MOSER [**5**] for related material.

Definition 1. A *premap* is a finite set $\mathcal{F}$ of *flags* together with three fixed-point-free involutions, R, L, T such that:

(1) for each $x \in \mathcal{F}$, the four flags x, Lx, Tx, LTx are all distinct;
(2) $TL = LT$;
(3) for any flag x, the orbits of x and Tx under RT are disjoint; and
(4) for any flag x, the orbits of x and Lx under RL are disjoint.

Definition 2. A *map* is a premap $(\mathcal{F}, R, L, T)$ such that the group generated by R, L, T acts transitively on $\mathcal{F}$.

Given a map $(\mathcal{F}, R, L, T)$, a *vertex* is an orbit of the flags under the action by $\langle R, T\rangle$, the subgroup generated by R and T. Similarly a *face* is an orbit of the action by $\langle R, L\rangle$, and an *edge* is an orbit of the action by $\langle T, L\rangle$. Observe that each edge contains exactly four flags, x, Lx, Tx, and LTx. These vertices and edges form a graph embedded on the same surface as the map; the faces of the map are the faces of the embedded graph. The orbits $O_{RT}(x)$ and $O_{RT}(Tx)$ give the clockwise and counterclockwise sense around a vertex. Likewise $O_{RL}(x)$ and $O_{RL}(Lx)$ give the two senses around a face. In this graph a flag is interpreted as a pair of directions on an edge, one longitudinal and one transversal. Figure 1 represents the effect on these arrows of the action by T, L, and R.

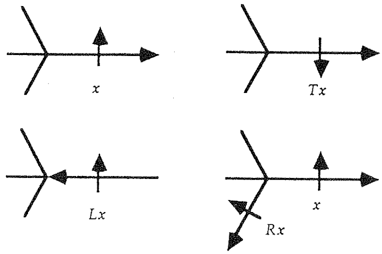
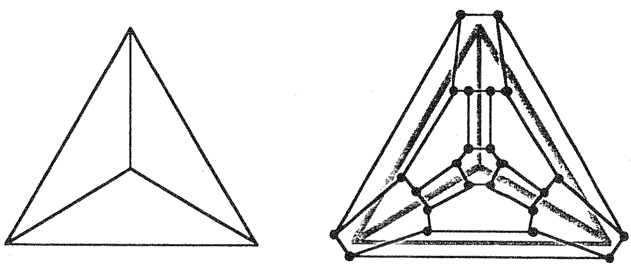


FIGURE 1

The maps here also correspond to TUTTE's [**16**] maps (θ, ϕ, P), where P is the rotation of edges around a vertex — $P = RT$, $\theta = L$ and $\phi = T$. They also quite naturally correspond to the "graph-encoded maps" of LINS [**11**]. The vertices of the "gem" are the flags, and a flag x is adjacent to Rx, Lx, Tx. The latter correspondence will be a useful one to keep in mind for the next section, when we discuss our generalization of voltage and current constructions.

The main point to remember about any of the encodings of an embedded graph is that the encoding is designed to keep track of the corners of the embedding. Another approach was used in [**12**]. The flags use arrows to distinguish a corner; in the gem it is a vertex to pick out a particular corner; and so on.

For example, Figure 2(a) is an embedding of K_4 in the sphere. The gem is obtained by putting two vertices in each corner and joining up the new vertices as shown in Figure 2(b).



(a) (b)

FIGURE 2

The *medial* of an embedded graph G is the 4-regular embedded graph $med(G)$ obtained by inserting a vertex in every edge of G and joining two of these vertices by an edge if the corresponding edges of G are a corner of G. In Figure 3 is the medial of the embedding of K_4 shown in Figure 2(a). Observe that contracting the squares of the gem that represent edges (Figure 2(b)) to vertices turns the gem into the medial (Figure 3).

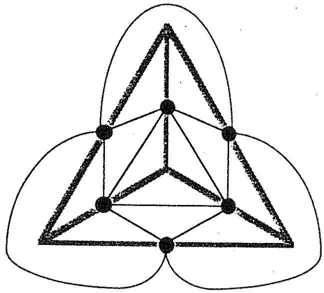


FIGURE 3

There is another relation with the gem and medial graphs. Let G be an embedded graph, and let $gem(G)$ denote the corresponding gem. By contracting all the (x, Rx) edges of $gem(G)$, one gets a 4-regular graph, which is $med(med(G)) = med^2(G)$. We comment that in ARCHDEACON AND RICHTER [4], the graph $med^2(G)$ played a central role, for reasons intimately related to the material discussed here.

3. GENERAL CONSTRUCTION

In this section, we introduce the generalization of the voltage and current graph constructions.

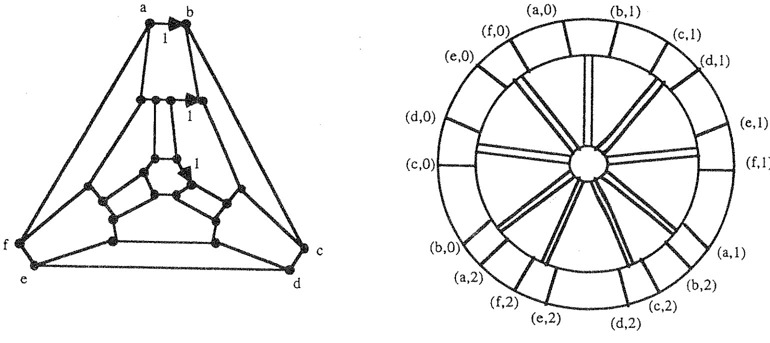
Definition 3. Let $M = (\mathcal{F}, R, L, T)$ be a map and let Γ be a group. A function $\alpha : \mathcal{F} \to \Gamma$ is a *prelift* of M if, for every flag x, $\alpha(Rx) = (\alpha(x))^{-1}$.

If α is a prelift of the map M, then it is easy to show that $\tilde{M} = (\mathcal{F} \times \Gamma, R^*, L^*, T^*)$ is a premap, where $R^*(x, \varepsilon) = (Rx, \varepsilon\alpha(x))$, $L^*(x, \varepsilon) = (Lx, \varepsilon)$ and $T^*(x, \varepsilon) = (Tx, \varepsilon)$.

Definition 4. The prelift α is a *lift* of M if $\tilde{M}$ is a map.

The general lifting described above corresponds to a usual voltage construction on the corresponding gem, where the directed edge from x to Rx gets voltage $\alpha(x)$, from x to Lx gets the identity voltage and x to Tx also gets identity voltage. The condition $\alpha(Rx) = (\alpha(x))^{-1}$ corresponds to the voltage condition on directed edges. In Figure 4(a) we have a voltage assignment from $\mathbb{Z}_3$ on the gem of K_4; the resulting map is shown in Figure 4(b) — it is the gem of the wheel W_9 with 9 rim vertices.

There is no loss of generality in having only the edges of the gem of the form $xR(x)$ receiving voltages. For it is necessary that the 4-cycles representing edges of the underlying graph lift to 4-cycles, so that in the lift they will again represent edges. Therefore, in any voltage assignment, these 4-cycles must get net voltage the identity. Any voltage assignment to the gem with this property is equivalent to one of the type we have defined. For (see GROSS AND TUCKER [**6, Theorem 2.5.3**]) there is an equivalent voltage assignment which assigns the identity voltage to all the edges of any fixed spanning tree. If we choose the spanning tree to include 3 of the 4 edges from each of the (vertex-disjoint) 4-cycles representing the edges, then

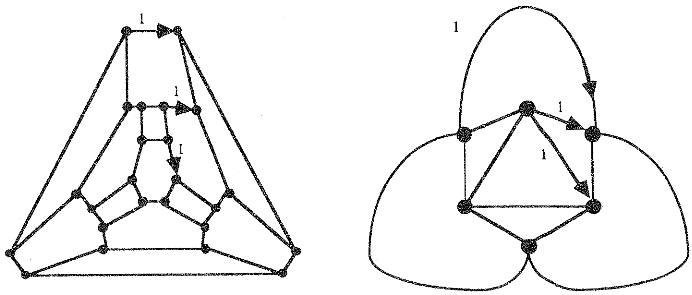


(a) (b)

FIGURE 4

the last edge also gets identity voltage. This shows there is an equivalent voltage assignment which assigns identity voltage to each edge of each of the 4-cycles. This leaves only the edges $xR(x)$ receiving non-identity voltages.

It is very natural to go between voltage assignments on $med(G)$ and prelifts. Each directed edge of $med(G)$ corresponds to a directed edge $xR(x)$ of the gem (in that contracting all the 4-cycles of the gem representing the edges yields the medial graph). The voltage assigned to the flag x (or the directed edge $xR(x)$ in the gem) is equal to the voltage of this directed edge of the medial. The converse operation takes a prelift to a voltage assignment on the medial. In Figure 5, the voltage assignment on the gem of K_4 is transferred to the medial.



(a) (b)

FIGURE 5

If a map G has a voltage assignment α, it is easy to transfer this to a voltage assignment of $gem(G)$. The directed edge e is assigned voltage $\alpha(e)$. The edge e is represented by a square $xL(x)LT(x)T(x)$ in $gem(G)$; we suppose x and $T(x)$ are the flags representing e at the origin of the directed edge e. Assign voltages $\alpha(e)$

to the directed edges $xL(x)$ and $T(x)LT(x)$ of $gem(G)$. Doing this for all edges e yields a voltage assignment of $gem(G)$ in which all edges of the form $xT(x)$ and $xR(x)$ get identity voltage. The square representing e gets identity net voltage and the lift of $gem(G)$ is the gem of the lift of G. This is illustrated in Figure 6.

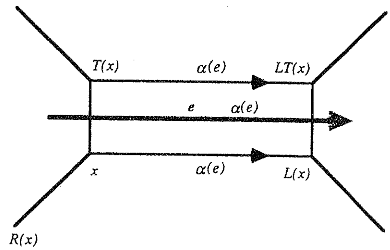


FIGURE 6

Similarly, currents on G lift to voltages on the edges $xT(x)$ and $L(x)LT(x)$ of $gem(G)$. One must be slightly careful here, because the interpretation of the lift in a current construction is not the same as in a voltage construction. The transfer of currents from G to $gem(G)$ described above is the same as from a voltage assignment on the dual G^* of G to $gem(G^*) = gem(G)$. The interpretation as to what are the vertices and faces needs to be properly understood: the current construction on G is exactly the voltage construction on G^*.

A simultaneous assignment of voltages to both a primal graph and its dual can be simultaneously, individually as above, transferred to the edges of the gem. In order that the squares representing edges have identity net voltage, however, it must be the case that the voltage ν assigned to an edge e and the voltage ν^* assigned to the edge e^* dual to e must commute, i.e. $\nu\nu^* = \nu^*\nu$. In the event this condition is satisfied for every dual pair of edges, then we get a simultaneous assignment with the property that the voltages around the faces determine how the faces lift and the (dual) voltages around the vertices (dual faces) determine how the vertices lift. We note that not every lift of an embedded graph can be realized by simultaneously assigning voltages to the primal and dual graph.

We remark that any of these assignments that we transfer to the gem is equivalent to some lift — that is, an assignment in which only the $xR(x)$ edges get nontrivial voltages.

In LINS [**11**] are described 6 maps obtained from the original map $M = (\mathcal{F}, R, L, T)$, namely M, its *dual* $(\mathcal{F}, R, T, L)$, its *antimap* $(\mathcal{F}, R, T, LT)$, the dual of the antimap (the *antiphial*) $(\mathcal{F}, R, LT, T)$, the *phial* $(\mathcal{F}, R, L, LT)$ and its dual (the *antidual*) $(\mathcal{F}, R, LT, L)$. Since these 6 maps all have the same involution R, which is the only one on which voltages need be assigned, and the various L and T are compositions of the original L and T, the 6 maps related to the derived map are the lifts of the 6 maps related to the base map. In particular, the dual of the base map lifts to the dual of the derived map and the *left-right walks* (or, equivalently, *Petrie polygons*) (see [**12**]) of the base map lift to the left-right walks of the derived map.

4. Voltages on Corners

We now describe a convenient way to represent a simultaneous current and voltage assignment on a graph without passing to the gem or to flags. For ease of description we begin with embeddings on orientable surfaces. Here an embedding is represented by the clockwise rotations of arcs (= longitudinally directed edges) around each vertex corresponding to the orbits of RT.

This point is probably best described by an example. In Figure 7 we have assigned voltages to the corners of K_4 in the plane. These are the same voltages that were assigned to the $xR(x)$ edges of $gem(K_4)$ in Figure 4. The new rotations of arcs around vertices are easily computed from the picture. The group is $\mathbb{Z}_3$ and so, for example, we have around the vertex A the arcs a, b, c. In the lift, then, we will have the single orbit $(a,0),(b,0),(c,0),(a,1),(b,1),(c,1),(a,2),(b,2),(c,2)$. Around the vertex B we have the arcs c, d, e which lift to the three orbits $(c,\varepsilon),(d,\varepsilon),(e,\varepsilon+1)$, $\varepsilon \in \mathbb{Z}_3$.

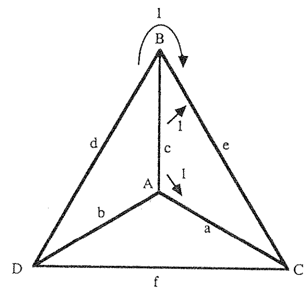


Figure 7

Notice that the vertices, arcs and faces can be read off directly from the voltages on the corners. We caution the reader, however, that there is no obvious way to label the vertices in the fibre above a given base vertex. The result for the particular example is shown in Figure 8.

This can be formalized. Let $P = RT$ denote the permutation of the arcs whose orbits are the vertices of the map. We can describe the corresponding permutation P^* in the lift by the following rule: $P^*(x,\varepsilon) = (P(x), \varepsilon\alpha(x))$, where $\alpha(x)$ is the voltage assigned to the corner from arc x to arc $P(x)$. The permutation P^* completely determines the embedding of the lift in the case the base surface is orientable.

The construction for nonorientable surfaces follows analagously. Here an embedding is represented by a permutation P which cyclically permutes the arcs emanating from each vertex and by a signature (± 1) assigned to each edge (for details

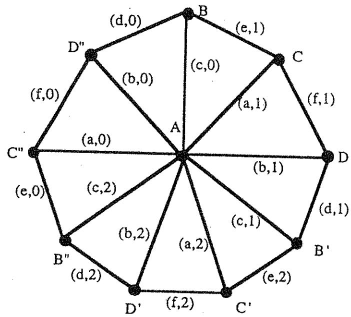


FIGURE 8

see STAHL [15] or GROSS AND TUCKER [6]). We assign voltages to corners as in the orientable case. The lift is described using the permutation P^* as before; the signature of each edge is the lift of the corresponding signature in the original map.

5. LIFTING MAP HOMOMORPHISMS

In this section, we describe a necessary and sufficient condition for a map homomorphism $f : M \to N$ to have a lift to $\tilde{f} : \tilde{M} \to \tilde{N}$.

Definition 5. Let $M = (\mathcal{F}_M, R_M, L_M, T_M)$ and $N = (\mathcal{F}_N, R_N, L_N, T_N)$ be maps. A *map homomorphism* is a function $f : \mathcal{F}_M \to \mathcal{F}_N$ such that, for each $S \in \{R, L, T\}$, $fS_M = S_N f$.

Examples of map homomorphisms are automorphisms and projections from covers. We write $f : M \to N$ to denote that f is a map homomorphism.

Definition 6. Two flags x and y are *adjacent* if $y \in \{Lx, Tx, Rx\}$.

Because R, L, T are involutions, adjacency is symmetric. This corresponds precisely to adjacency in the gem.

Proposition 7. (1) *A map homomorphism f preserves adjacencies, i.e. if x and $y = S_M x$ are adjacent in M, then $f(y) = S_N f(x)$, so $f(x)$ and $f(y)$ are adjacent in N.*

(2) *If $f : M \to N$ is a map homomorphism, then M is a branched covering of N.* $\square$

Definition 8. A *flag-walk* is a sequence of flags with consecutive terms adjacent. It is *closed* if the first and last flags are identical.

A simple consequence is that if W is a flag-walk and f is a map homomorphism, then $f(W)$ is also a flag-walk.

Let M be a map with lift α. Let $W = (x_1, x_2, \ldots, x_n)$ be a flag-walk, with $S_i \in \{R, L, T\}$ such that $x_i = S_i x_{i-1}$, for $i = 2, 3, \ldots, n$. For each $i = 1, 2, \ldots, n-1$, let $\gamma(x_i) = \alpha(x_i)$ if $S_i = R$; otherwise, $\gamma(x_i)$ is the identity. (Thus, $\gamma(x_i)$ is simply the voltage on the arc in the gem joining x_i to x_{i+1} in the walk corresponding to W.) Define $\gamma(W) = \gamma(x_1)\gamma(x_2)\cdots\gamma(x_{n-1})$.

Given two flag-walks, $W = (x_1, x_2, \ldots, x_s)$ and $W' = (y_1, y_2, \ldots, y_t)$, such that the origin y_1 of W' is the terminus x_s of W, their product WW' is the flag-walk $(x_1, x_2, \ldots, x_s, y_2, y_3, \ldots, y_t)$. The flag-walk W^{-1} is $(x_s, x_{s-1}, \ldots, x_2, x_1)$. Obviously, $\gamma(WW') = \gamma(W)\gamma(W')$ and $\gamma(W^{-1}) = (\gamma(W))^{-1}$.

Theorem 9. *Let M and N be maps, with lifts $\alpha_M : \mathcal{F}_M \to \Gamma_M$ and $\alpha_N : \mathcal{F}_N \to \Gamma_N$, respectively, and let $\tilde{M}$ and $\tilde{N}$ be the derived maps. Let ρ_M and ρ_N be the respective projections. Let $f : M \to N$ be a map homomorphism. There is a map homomorphism $\tilde{f} : \tilde{M} \to \tilde{N}$ such that $f\rho_M = \rho_N\tilde{f}$ if and only if, for each closed flag-walk W with origin a fixed flag x, if $\gamma_M(W)$ is the identity in Γ_M, then $\gamma_N(f(W))$ is the identity in Γ_N.*

Proof. **Necessity.** Suppose there is such a map homomorphism $\tilde{f}$. Let $W = (x_1, x_2, \ldots, x_n)$ be a closed flag-walk in M such that $x_n = x_1 = x$ and $\gamma_M(W)$ is the identity in Γ_M. Let $\varepsilon \in \Gamma_M$ and set $\mu_i = \gamma_M(x_i)$. Then

$$W' = \big((x_1, \varepsilon), (x_2, \varepsilon\mu_1), \cdots, (x_{n-1}, \varepsilon\mu_1\cdots\mu_{n-2}), (x_n, \varepsilon)\big)$$

is a closed flag-walk in $\tilde{M}$ such that $\rho_M(W') = W$.

Now, $f(W) = (f(x_1), f(x_2), \ldots, f(x_n))$. Set $y_i = f(x_i)$ and $\nu_i = \gamma_N(y_i)$. Then

$$\tilde{f}(W') = \big((y_1, \varepsilon'), (y_2, \varepsilon'\nu_1), \ldots, (y_n, \varepsilon'\nu_1\nu_2\cdots\nu_{n-1})\big),$$

since $f\rho_M = \rho_N\tilde{f}$ and $\tilde{f}$ is a map homomorphism. Since $\tilde{f}(W')$ must be closed, we see that $\gamma_N(f(W)) = \nu_1\cdots\nu_{n-1}$ is the identity in Γ_N, as required.

Sufficiency. Because $\tilde{M}$ is a map, for each flag y of M and each $\varepsilon \in \Gamma_M$, there is a flag-walk W with origin x and terminus y such that $\gamma_M(W) = \varepsilon$.

To define $\tilde{f}$, let (y, ε) be any flag of $\tilde{M}$. Let W be a flag-walk with origin x and terminus y such that $\gamma_M(W) = \varepsilon$. Now we let $\tilde{f}(y, \varepsilon) = (f(y), \nu\gamma_N(f(W)))$, where ν is an arbitrary, but fixed, element of Γ_N.

To see that $\tilde{f}$ is well-defined, let V be another flag-walk with origin x and terminus y such that $\gamma_M(V) = \varepsilon$. Then $\gamma_M(WV^{-1})$ is the identity in Γ_M, so $\gamma_N(f(WV^{-1}))$ is the identity in Γ_N and the result follows.

It remains to show that $\tilde{f}$ is a map homomorphism such that $\rho_N\tilde{f} = f\rho_M$. For the former, let $S \in \{R, L, T\}$. Then

$$\begin{aligned}
S_{\tilde{N}}\tilde{f}(y, \varepsilon) &= S_{\tilde{N}}\tilde{f}(y, \gamma_M(W)) = S_{\tilde{N}}\Big(f(y), \nu\gamma_N\big(f(W)\big)\Big) \\
&= \Big(S_N f(y), \nu\gamma_N\big(f(W)\big)\gamma_N\big(f(y)\big)\Big) = \Big(fS_M(y), \nu\gamma_N\big(f(W, (y))\big)\Big) \\
&= \tilde{f}\Big(S_M(y), \nu\gamma_M\big(W, (y)\big)\Big) = \tilde{f}S_{\tilde{M}}(y, \gamma_M(W)) = \tilde{f}S_{\tilde{M}}(y, \varepsilon).
\end{aligned}$$

To see that the diagram commutes, note that $\rho_N\tilde{f}(y, \varepsilon) = \rho_N(f(y), *) = f(y)$ and that

$f\rho_M(y, \varepsilon) = f(y)$. $\square$

It is easy to show that if f is a bijection, then $\tilde{f}$ is also a bijection. The point of the ν in the definition of $\tilde{f}$ is so that $\tilde{f}(x, id)$ is any particular $(f(x), \nu)$. It is also straightforward to show that there is a unique lift of f that takes (x, id) to $(f(x), \nu)$.

On a slightly different note, observe that if there is a group homomorphism $f^* : \Gamma_M \to \Gamma_N$ such that $f^*(\gamma_M(y)) = \gamma_N(f(y))$ for every flag y of M, then the lift $\tilde{f}$ exists and could be defined by $\tilde{f}(y, \varepsilon) = (f(y), \nu f^*(\varepsilon))$. In general, however, $\tilde{f}$ can exist without having $f^*(\gamma_M(y)) = \gamma_N(f(y))$. For example, if M is a directed 2-cycle with one arc getting voltage 0 and the other voltage 1 in $\mathbb{Z}_2$, then the automorphism that interchanges the arcs has a lift but does not satisfy $f^*(\gamma_M(y)) = \gamma_N(f(y))$ for either arc.

$$\begin{tikzcd}
\tilde{M} \arrow[r, "1"] \arrow[d, "1"'] & \tilde{M} \arrow[d, "p"] \\
\tilde{M} \arrow[r, "p"'] & M
\end{tikzcd}$$

FIGURE 9

Let M be the directed 2-cycle with voltages as in the preceding paragraph. Let $\tilde{M}$ be the lift of M. Put voltages 0 on the arcs of $\tilde{M}$ and consider the commutative diagram in Figure 9. In this case, the group homomorphism $f^*(\gamma_M(W)) = \gamma_N(f(W))$, where W ranges over all the closed walks in $\tilde{M}$ with origin a fixed vertex, is not an epimorphism from $\Gamma_M = \{0\}$ to $\Gamma_N = \mathbb{Z}_2$. Therefore the necessary and sufficient condition for the existence of a lift $\tilde{f}$ is this f^* is a group homomorphism.

6. Permutation Voltages

The "ordinary" lifts described in the previous sections generalize naturally "ordinary" voltage constructions. In the same way, one can generalize permutation voltage assignments.

Definition 10. Let $M = (\mathcal{F}, R, L, T)$ be a map and let Γ be a subgroup of the symmetric group S_n. A function $\alpha : \mathcal{F} \to \Gamma$ is a *permutation prelift* of M if, for every flag x, $\alpha(Rx) = (\alpha(x))^{-1}$.

In this work, we use the notation $j\varepsilon$ to denote the image of $j \in \{1, 2, \ldots, n\}$ under the permutation $\varepsilon \in S_n$.

If α is a permutation prelift of the map M, then it is easy to show that

$$\tilde{M} = (\mathcal{F} \times \{1, 2, \ldots, n\}, R^*, L^*, T^*)$$

is a premap, where $R^*(x, i) = (Rx, i\alpha(x))$, $L^*(x, i) = (Lx, i)$ and $T^*(x, i) = (Tx, i)$.

Definition 11. The permutation prelift α is a *permutation lift* of M if $\tilde{M}$ is a map.

We remark that this construction is a common generalization of the classical permutation voltages and the classical (though nonexistent) permutation currents. Also, note that once again, all six of the associated maps lift; in particular, the dual of the lift is the lift of the dual.

7. Permutation Lifts of Map Homomorphisms

In analogy with Theorem 9, we have a necessary and sufficient condition for the existence of a permutation lift of a map homomorphism $f : M \to N$, when M and N have simultaneous permutation lifts.

Theorem 12. *Let M and N be maps, with permutation lifts α_M and α_N, respectively and let $\tilde{M}$ and $\tilde{N}$ be the derived maps. Let ρ_M and ρ_N be the respective projections. Let $f : M \to N$ be a map homomorphism. There is a map homomorphism $\tilde{f} : \tilde{M} \to \tilde{N}$ such that $f\rho_M = \rho_N \tilde{f}$ if and only if, there is a $j \in \{1, 2, \ldots, n\}$ such that if W is a closed flag-walk with origin a fixed flag x, with $1\gamma_M(W) = 1$, then $j\gamma_N(f(W)) = j$.*

We remark that the permutation lift for M may be relative to $\{1, 2, \ldots, m\}$ and that for N to $\{1, 2, \ldots, n\}$ with $m \neq n$.

Proof. **Necessity.** Let $\tilde{f}(x, 1) = (f(x), j)$. Let W be a closed flag-walk in M with origin x such that $1\gamma_M(W) = 1$. Let $\tilde{W}$ be the lift of W with origin $(x, 1)$. Then $\tilde{W}$ is a closed flag-walk in $\tilde{M}$. Thus, $\tilde{f}(\tilde{W})$ is a closed flag-walk in $\tilde{N}$, which projects down to $f(W)$. Hence, $j\gamma_N(f(W)) = j$, as claimed.

Sufficiency. Define $\tilde{f}(y, i)$ by $(f(y), j\gamma_N(f(V)))$, where j is such that if W is a closed flag-walk with origin x such that $1\gamma_M(W) = 1$, then $j\gamma_N(f(W)) = j$ and V is a flag-walk with origin x such that $1\gamma_M(V) = i$. We prove $\tilde{f}$ is well-defined. The proof that it is a flag-walk is similar to the proof in Theorem 9.

Suppose $1\gamma_M(V) = i$ and $1\gamma_M(V') = i$. Then $1\gamma_M(VV'^{-1}) = 1$, and so, by assumption, $j\gamma_N(f(VV'^{-1})) = j$. Thus, $j\gamma_N(f(V)) = j\gamma_N(f(V'))$, so $\tilde{f}(y, 1\gamma_M(W)) = \tilde{f}(y, 1\gamma_M(V))$, as required. □

A lift as in Section 5 corresponds to a permutation lift where the permutation is given by the action of right multiplication on the group elements. Using this correspondence Theorem 9 follows from Theorem 12.

The advantage of our constructions is increased generality. The generalization of the ordinary voltage construction we have given in Section 3 gives all "regular branched coverings", where branch points are now allowed in both faces and vertices. The regularity condition means that the deck transformations act transitively on the fibres. The generalization of the permutation voltage construction yields all branched coverings, where again branch points are to be in both faces and vertices.

8. Centrally Symmetric Self-Dual Polyhedra

To conclude, we consider the existence of centrally symmetric self-dual spherical polyhedra. In [7], Grünbaum and Shephard ask if there is such a polyhedron.

Examples were independently discovered by ROTE [**14**] and ARCHDEACON [**2**]. The latter paper showed that some self-dual projective planar graphs lift to centrally symmetric self-dual spherical polyhedra. We use Theorem 9 to show that all such projective graphs lift.

Theorem 13. *Let $(\mathcal{F}, R, L, T)$ be a map on the projective plane which is isomorphic to its dual $(\mathcal{F}, R, T, L)$. Then the spherical double cover of this map is a centrally symmetric self-dual spherical map.*

Proof. The self-duality map ϕ must map contractible cycles to contractible cycles. Since these are the only cycles that receive the identity voltage, ϕ satisfies the hypothesis of Theorem 9 and, therefore, the self-duality map lifts to a self-duality map $\tilde{\phi}$ of the spherical double cover. □

ARCHDEACON AND NEGAMI [**3**] show that there are infinitely many self-dual projective planar maps. This, together with Theorem 13 shows that there are infinitely many centrally symmetric self-dual spherical maps. Moreover, if the projective planar map is 3-connected and 3-representative, then it is easy to see that the spherical double cover is 3-connected and, therefore, a polyhedron. ARCHDEACON AND NEGAMI also prove there are infinitely many 3-connected and 3-representative self-dual projective planar maps, so we can conclude that there are infinitely many centrally symmetric self-dual spherical polyhedra.

REFERENCES

1. D. Archdeacon, *The medial graph and voltage-current duality*, Discrete Math. **104** (1992), 111-141.
2. D. Archdeacon, *The construction of centrally symmetric self-dual polyhedra*, Univ. of Vermont Dept. of Math. and Stat. Research Report **91-10**.
3. D. Archdeacon and S. Negami, *The construction of self-dual projective polyhedra*, J. Combin. Theory Ser. B **59** (1993), 122-131.
4. D. Archdeacon and R.B. Richter, *The construction and classification of self-dual spherical polyhedra*, J. Combin. Theory Ser. B **54** (1992), 37-63.
5. H.S.M. Coxeter and W.O.J. Moser, *Generators and relations for discrete groups. 2d ed.*, Springer-Verlag, Berlin, 1965.
6. J. Gross and T. Tucker, *Topological Graph Theory*, John Wiley & Sons, New York, 1987.
7. B. Grünbaum and G.C. Shephard, *Some problems on polyhedra*, J. of Geometry **29** (1987), 182-190.
8. P. Gvozdjak and J. Širáň, *Regular maps from voltage assignments*, to appear.
9. G.A. James and J.S. Thornton, *Operations on maps and outer automorphisms*, J. Combin. Theory Ser. B **35** (1984), 93-103.
10. G.A. Jones and D. Singerman, *Theory of maps on orientable surfaces*, Proc. London Math. Soc. (3) **37** (1978), 273-307.
11. S. Lins, *Graph-encoded maps*, J. Combin. Theory Ser. B **32** (1982), 171-181.

12. S. Lins, R.B. Richter and H. Shank, *The Gauss code problem off the plane*, Aeq. Math. **33** (1987), 81-95.
13. G. Ringel, *Map Color Theorem*, Springer Verlag, Berlin, 1974.
14. G. Rote, *A centrally symmetric self-dual polyhedron*, preprint.
15. S. Stahl, *Generalized embedding schemes*, J. of Graph Theory **2** (1978), 41-52.
16. W.T. Tutte, *Graph Theory*, Cambridge University Press, Cambridge, 1984.

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