ON MINIMALLY k-EXTENDABLE GRAPHS

N. Anunchuen and L. Caccetta

School of Mathematics and Statistics
Curtin University of Technology
GPO Box U1987
Perth 6001
Western Australia

ABSTRACT:

Let G be a simple connected graph on 2n vertices with a perfect matching. G is **k-extendable** if for any set M of k independent edges, there exists a perfect matching in G containing all the edges of M. G is **minimally k-extendable** if G is k-extendable but G - uv is not k-extendable for every pair of adjacent vertices u and v of G. The problem that arises is that of characterizing k-extendable and minimally k-extendable graphs. k-extendable graphs have been studied by a number of authors whilst minimally k-extendable graphs have not been studied. In this paper, we focus on the problem of characterizing minimally k-extendable graphs. We establish necessary and sufficient conditions for a graph to be minimally k-extendable. In addition, we obtain a complete characterization of (n-1)-extendable graphs.

1. INTRODUCTION

All graphs considered in this paper are finite, connected, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [3]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices, $\varepsilon(G)$ edges and minimum degree $\delta(G)$. For $V' \subseteq V(G)$, $G[V']$ denotes the subgraph induced by V'. Similarly $G[E']$ denotes the subgraph induced by the edge set E' of G. $N_G(u)$ denotes the neighbour set of u in G and $\bar{N}_G(u)$ the non-neighbours of u. Note that $\bar{N}_G(u) = V(G) - N_G(u) - u$. The **join** $G \vee H$ of disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H.

A **matching M** in G is a subset of E(G) in which no two edges have a vertex in common. M is a **maximum matching** if $|M| \geq |M'|$ for any other matching M' of G. A vertex v is **saturated** by M if some edge of M is incident to v; otherwise v is said to be **unsaturated**. A matching M is **perfect** if it saturates every vertex of the graph. For simplicity we let V(M) denote the vertex set of the subgraph G[M] induced by M.

Let G be a simple connected graph on 2n vertices with a perfect matching. For $1 \leq k \leq n - 1$, G is **k-extendable** if for any matching M in G of size k there exists a perfect matching in G containing all the edges of M. We say that G is **minimally (critically) k-extendable** or simply **k-minimal (k-critical)** if it is k-extendable but $G - uv$ $(G + uv)$ is not k-extendable for any edge uv of G $(uv \notin E(G))$.

Observe that a cycle C_{2n} of order $2n \geq 4$ is 1-minimal but not 1-critical. The complete graph K_{2n} of order 2n and the complete bipartite graph $K_{n,n}$ with bipartitioning sets of order n are each k-extendable for $1 \leq k \leq n - 1$. Further, these graphs are k-critical. However, K_{2n} and $K_{n,n}$ are k-minimal if and only if $k = n - 1$; we will prove this in due course.

For convenience, we say that G is 0-extendable if G has a perfect matching. Plummer [7,8] proved the following result.

Theorem 1.1: Let G be a k-extendable graph on 2n vertices, $1 \le k \le n - 1$. Then

(a) G is (k-1)-extendable;

(b) G is (k+1)-connected;

(c) For any edge e of G, G - e is (k-1)-extendable. □

Theorem 1.1 allows us to make the following observations.

Remark 1: A k-minimal graph G need not be (k-1)-minimal. For example, the graph in Figure 1.1 is 2-minimal but not 1-minimal.

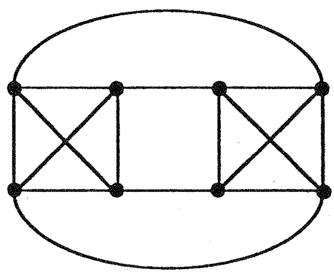

Figure 1.1

Remark 2: Consider any k-extendable graph G on 2n vertices, $1 \le k \le n - 1$. If $d_G(u) = k + 1$ or $d_G(v) = k + 1$ for any edge e = uv in G, then G is minimal. This implies that a (k+1)-regular k-extendable graph on 2n vertices, $1 \le k \le n - 1$, is minimal. Thus the k-cube Q_k which is a k-regular (k-1)-extendable graph (see Györi and Plummer

[4]), is (k-1)-minimal.

A number of authors have studied k-extendable graphs; an excellent survey is the paper of Plummer [9]. Anunchuen and Caccetta [1] characterized k-critical graphs of order 2n for k = 1,2,n-2 and n-1. k-minimal graphs have not been previously investigated; the characterization problem was posed to us (private communication) by M.D.Plummer. In this paper, we focus on this problem.

We establish necessary and sufficient conditions for a graph to be k-minimal. In addition, we prove that a graph G of order 2n is (n-1)-minimal if and only if it is (n-1)-extendable. The only (n-1)-extendable graphs on 2n vertices are shown to be $K_{n,n}$ and K_{2n}. We present a number of properties of k-minimal graphs, including an upper bound on the minimum degree.

Section 2 contains some preliminary results that we make use of in establishing our main results. In Section 3, we prove some properties of k-minimal graphs and establish necessary and sufficient conditions for k-minimal graphs. The complete characterization of (n-1)-extendable graphs and (n-1)-minimal graphs are given in Section 4.

2. PRELIMINARIES

In this section, we state a number of results on k-extendable graphs which we make use of in our work. We state only results which we use; for a more detailed account we refer to the paper of Plummer [9].

We begin with an important result of Berge (see [6] p. 90). Let M be a maximum matching in a graph G. The **deficiency** def(G) of G is defined as the number of M-unsaturated vertices of G. Denoting the number of odd components in a graph H by o(H) we can now state Berge's Formula :

Theorem 2.1: For any graph G

$$\mathrm{def}(G) = \max\{o(G - X) - |X| : X \subseteq V(G)\}. \qquad \square$$

We let M(S) denote a maximum matching in G[S]. One characterization of k-extendable graphs was proved by Lou [5]. The result is

Theorem 2.2: G is a k-extendable graph on 2n vertices, $1 \le k \le n - 1$ if and only if for any $S \subseteq V(G)$, $o(G - S) \le |S| - 2d$ where $d = \min\{|M(S)|, k\}$. □

Anunchuen and Caccetta [1] proved the following result.

Theorem 2.3: Let G be a k-extendable graph on 2n vertices with $\delta(G) = k + t$, $1 \le t \le k \le n - 1$. If $d_G(u) = \delta(G)$, then $|M(N_G(u))| \le t - 1$. □

Plummer [7] gave the following sufficient condition for graphs on 2n vertices to be k-extendable, $1 \le k \le n - 1$:

Theorem 2.4 : Let G be a graph on 2n vertices and $1 \le k \le n - 1$. If $\delta(G) \ge n + k$, then G is k-extendable. □

We conclude this section by stating Dirac's Theorem (see [3] p. 54).

Theorem 2.5: If G is a simple graph with $\nu(G) \geq 3$ and $\delta(G) \geq \frac{1}{2}\nu(G)$, then G is hamiltonian. □

3. PROPERTIES OF MINIMALLY k-EXTENDABLE GRAPHS

Consider a k-minimal graph G. Since for any edge e of G, G-e is not k-extendable, there exists a matching M in G-e of size k that does not extend to a perfect matching in G-e. Our first result concerns the size of a maximum matching in G-e-V(M).

Lemma 3.1: Let G be a k-minimal graph on 2n vertices, $1 \leq k \leq n - 1$ and e any edge of G. If M is a matching of size k in G-e that does not extend to a perfect matching in G-e, then G-e-V(M) has a maximum matching of size n-k-1.

Proof: Let M′ be a maximum matching of G′ = G-e-V(M). Since G is k-minimal, $|M'| \leq n-k-1$. Suppose that $|M'| \leq n-k-2$. Then

$$\begin{aligned} def(G') &= |V(G')| - 2|M'| \\ &= 2(n-k) - 2|M'| \\ &\geq 4. \end{aligned}$$

By Theorem 2.1, there exists a subset S′ of V(G′) such that

$$o(G' - S') - |S'| = def(G') \geq 4.$$

Let xy be an edge of M. Put $S'' = S' \cup \{x,y\}$ and $G'' = G' \cup \{x,y\}$. Then $o(G'' - S'') = o(G' - S')$ and hence

$$o(G'' - S'') - |S''| = o(G' - S') - |S'| - 2 \geq 2.$$

Then $def(G'') \geq 2$, implying that G-e is not (k-1)-extendable, contradicting Theorem 1.1(c). This completes the proof of the lemma.□

Our next two lemmas yield a characterization of k-minimal graphs.

Lemma 3.2: Let G be a k-extendable graph on 2n vertices, $1 \leq k \leq n - 1$. Then G is minimal if and only if for any edge e = uv of G there exists a matching M of size k in G-e such that $V(M) \cap \{u,v\} = \phi$ and for every perfect matching F, in G, containing M, $e \in F$.

Proof: The sufficiency is obvious, so we need only prove the necessity. Let e = uv be an edge of G and M a matching of size k in G-e that does not extend to a perfect matching in G-e. We need to show that $V(M) \cap \{u,v\} = \phi$.

Suppose to the contrary that $V(M) \cap \{u,v\} \neq \phi$. First we assume that $\{u,v\} \subseteq V(M)$. Then $G-V(M) = G-e-V(M)$. Hence, M is extendable in G only if it is extendable in G-e, a contradiction. Hence, $\{u,v\} \nsubseteq V(M)$. So we need only consider the case when exactly one of u or v belongs to V(M).

Without any loss of generality, assume that

$$\{u,v\} \cap V(M) = \{u\}.$$

Since M is a matching in G-e, there exists a vertex $u' \in V(G) - v$ such that $uu' \in M$. If F is a perfect matching in G containing M, then $uv \notin F$ since $uu' \in F$. Consequently, F is a perfect matching in G-e containing M which contradicts the choice of M. Hence, $u \notin V(M)$. This proves that $V(M) \cap \{u,v\} = \phi$.

Since M is a matching in G-e, M is also a matching in G. If there exists a perfect matching F′ in G containing M such that $uv \notin F'$, then F′ is a perfect matching in G-e containing M, a contradiction. Hence, every perfect matching in G containing M must contain edge uv. This proves our result. □

Recall that M(S) denotes a maximum matching in G[S]. We now establish another characterization of k-minimal graphs.

Lemma 3.3: Let G be a k-extendable graph on 2n vertices, $1 \le k \le n - 1$. Then G is minimal if and only if for any edge e = uv of G there exists a vertex set S of G-u-v such that :

(i) $|M(S)| \ge k$;

(ii) $o(G-e-S) = |S| - 2k + 2$;

and (iii) u and v belong to different odd components of G-e-S.

Proof: The sufficiency follows directly from Theorem 2.2. We need only consider the necessity. Let e = uv be an edge of G. Since G is minimal, G-e is not k-extendable. However, by Theorem 1.1 (c), G-e is (k-1)-extendable. Thus by Theorem 2.2, there exists a set $S_0 \subseteq V(G-e)$ such that $o(G-e-S_0) > |S_0| - 2d_0$, where $d_0 = \min\{|M(S_0)|,k\}$. Further, since G-e is (k-1)-extendable we have, for any $S_1 \subseteq V(G-e)$, $o(G-e-S_1) \le |S_1| - 2d_1$, where $d_1 = \min\{|M(S_1)|,k-1\}$.

Now if $|M(S_0)| \le k - 1$, then

$$o(G-e-S_0) > |S_0| - 2d_0 = |S_0| - 2|M(S_0)|$$

and

$$o(G-e-S_o) \le |S_o| - 2d_1 = |S_o| - 2|M(S_o)|,$$

a contradiction. Hence, $|M(S_o)| \ge k$, proving (i). Thus we have $d_o = k$ and $d_1 = k - 1$. Consequently,

$$o(G-e-S_o) > |S_o| - 2d_o = |S_o| - 2k$$

and

$$o(G-e-S_o) \le |S_o| - 2d_1 = |S_o| - 2(k-1).$$

Since $\nu(G)$ is even, S_0 and $o(G-e-S_0)$ have the same parity. Hence,

$$o(G-e-S_o) = |S_o| - 2k + 2,$$

proving (ii).

Now we establish (iii). Since G is k-extendable, by Theorem 2.2 and the fact that $|M(S_0)| \ge k$, we have

$$o(G-S_o) \le |S_o| - 2k.$$

Now making use of the fact that

$$o(G-e-S_o) \le o(G-S_o) + 2,$$

we conclude that

$$|S_o| - 2k + 2 = o(G-e-S_o) \le o(G-S_o) + 2 \le |S_o| - 2k + 2.$$

Hence,

$$o(G-e-S_o) = o(G-S_o) + 2.$$

This implies that e must be an edge joining two different odd components of $G-e-S_o$. Consequently, u and v belong to different odd components of $G-e-S_0$ and clearly $S_0 \cap \{u,v\} = \phi$. This proves (iii) and thus completes the proof of our lemma. □

Lemmas 3.2 and 3.3 together yield the following Theorem :

Theorem 3.1: Let G be a k-extendable graph on 2n vertices, $1 \le k \le n - 1$. Then the following are equivalent:

(a) G is minimal.

(b) For any edge e = uv of G there exists a matching M of size k in G-e such that $V(M) \cap \{u,v\} = \phi$ and for every perfect matching F, in G, containing M, $e \in F$.

(c) For any edge e = uv of G there exists a vertex set S of G-u-v such that : $|M(S)| \ge k$; $o(G-e-S) = |S| - 2k + 2$; and u and v belong to different odd components of G-e-S. □

Clearly the graphs $K_{n,n}$ and K_{2n} are k-extendable for each k, $1 \le k \le n - 1$. However, it is not so obvious that $K_{n,n}$ and K_{2n} are k-minimal if and only if $k = n - 1$. We prove this in our next result.

Theorem 3.2: (a) K_{2n} is k-minimal, $1 \le k \le n - 1$ if and only if $k = n-1$.

(b) $K_{n,n}$ is k-minimal, $1 \le k \le n - 1$ if and only if $k = n - 1$.

Proof: (a) First we will prove the sufficiency. Let $uv \in K_{2n}$. By Theorem 3.1 (b), there exists a matching M of size k in K_{2n} - uv such that $V(M) \cap \{u,v\} = \phi$ and for every perfect matching F, in K_{2n}, containing M, $e \in F$.

If $\nu(K_{2n} - (V(M) \cup \{u,v\})) \geq 2$, then there exists a perfect matching F_1 in K_{2n}, containing M such that $e \notin F_1$, since $K_{2n} - V(M)$ is a 1-factorable graph on 2n - 2k vertices, a contradiction. Consequently,

$$\nu(K_{2n} - (V(M) \cup \{u,v\})) = 0.$$

Hence, n = k + 1 as required.

Now we show that K_{2n} is (n-1)-minimal. Clearly, K_{2n} is (n-1)-extendable. Let e = xy be any edge of K_{2n}. Then $K_{2n}-\{x,y\} \cong K_{2n-2}$. Clearly K_{2n-2} contains a matching M_1 of size n-1 and M_1 does not extend to a perfect matching in K_{2n} - xy since M_1 saturates the neighbour set of x and y in K_{2n} - xy. Therefore, K_{2n} is minimal. This completes the proof of (a).

The proof of (b) is similar. □

Theorem 1.1 (b) implies that a k-extendable graph G has minimum degree at least k + 1. A useful result in our work on k-critical graphs was an upper bound on the minimum degree. Our next theorem establishes a similar upper bound on the minimum degree of a k-minimal graph.

Theorem 3.3: If $G \neq K_{2n}$ is a k-minimal graph on 2n vertices, $1 \leq k \leq n - 1$, then $\delta(G) \leq n + k - 1$.

Proof: If k = n - 1, then since $G \neq K_{2n}$, we have $\delta(G) \leq 2n - 2 = n + k - 1$ and we are done. Now we may assume $1 \leq k \leq n - 2$. Suppose to the contrary that $\delta(G) \geq n + k$. Let e be an edge of G. Since G - e is not k-extendable, $G - e$ has a matching M of size k such that M is not extendable to a perfect matching of $G - e$. On the other hand, we have $\delta(G - V(M)) \geq n + k - 2k = n - k = \frac{1}{2}\nu(G - V(M))$ and $\nu(G - V(M)) = 2(n - k) \geq 4$, since $k \leq n - 2$. Hence, by Theorem 2.5, $G - V(M)$ is hamiltonian and hence $G - V(M)$ has a hamiltonian cycle of even order $2(n - k)$. Since every even cycle has two disjoint perfect matchings, $G - V(M)$ has at least two disjoint perfect matchings M_1 and M_2. Clearly, either $e \notin M_1$ or $e \notin M_2$, since $M_1 \cap M_2 = \phi$. Without any loss of generality, assume that $e \notin M_1$. But then $F = M_1 \cup M$ is a perfect matching of $G - e$ with $M \subseteq F$, contradicting the assumption on M. Thus $\delta(G) \leq n + k - 1$. □

Theorems 2.4 and 3.3 together yield the following corollary:

Corollary: Let $G \neq K_{2n}$ be a graph on $2n$ vertices, $1 \leq k \leq n - 1$. If $\delta(G) \geq n + k$, then G is k-extendable but not k-minimal. □

The upper bound of $n + k - 1$ given in Theorem 3.3 is not always achievable. The characterization of (n-1)-minimal graphs given in the next section shows that the bound is not achievable for the case $k = n - 1$. On the other hand, our characterization of (n-2)-minimal graphs given in [2] shows that the bound is achievable for the case $k = n - 2$; an example is the graph $\bar{K}_2 \vee K_{2n-2}\backslash\{\text{a hamiltonian cycle}\}$. It would be interesting to determine when the bound is achievable.

4. CHARACTERIZATION OF (n-1)-EXTENDABLE AND (n-1)-MINIMAL GRAPHS

We begin with the following lemma which establishes the possible values of the minimum degree of (n-1)-extendable graphs.

Lemma 4.1: If $G \neq K_{2n}$ is an (n-1)-extendable graph on $2n \geq 4$ vertices, then $\delta(G) = n$.

Proof: We shall first establish that $\delta(G) \leq n$. Suppose to the contrary that $n + 1 \leq \delta(G) \leq 2n - 2$. Let u be a vertex of G with $d_G(u) = \delta(G) = r$ and M a maximum matching in $G[N_G(u)]$. By Theorem 2.3 and the fact that $r = (n - 1) + (r - n + 1) \leq 2n - 2$, we have

$$|M| \leq (r - n + 1) - 1 = r - n.$$

Let x and y be vertices of $N_G(u) - V(M)$; x and y exist since $r - 2|M| \geq 2n - r \geq 2$. Since $\delta(G) = r$ and M is a maximum matching in $G[N_G(u)]$, we have

$$\begin{aligned} 2r \leq d_G(x) + d_G(y) &\leq 2|M| + 2(2n - r) \\ &\leq 2(r - n) + 2(2n - r) \\ &= 2n. \end{aligned}$$

But then $r \leq n$, contradicting the fact that $r \geq n + 1$. This proves that $\delta(G) \leq n$.

On the other hand, we have, by Theorem 1.1(b), G is n - connected and hence $\delta(G) \geq n$. Thus $\delta(G) = n$ as required. □

We now characterize the (n-1)-extendable graphs on 2n vertices.

Theorem 4.1: G is an (n-1)-extendable graph on $2n \geq 4$ vertices if and only if $G \cong K_{n,n}$ or K_{2n}.

Proof: We need only prove the necessity condition as $K_{n,n}$ and K_{2n} are clearly (n-1)-extendable. So suppose that G is (n-1)-extendable and $G \not\cong K_{n,n}$ and K_{2n}. Then by Lemma 4.1, $\delta(G) = n$.

Let $d_G(u) = n$. By Theorem 2.3, $N_G(u)$ is independent. Consequently, every vertex in $N_G(u)$ is adjacent to every vertex in $\bar{N}_G(u)$. Consider any vertex v of $N_G(u)$, $d_G(v) = n$ and so $N_G(v)$ is independent. Hence, $\bar{N}_G(u)$ is an independent set and therefore $G \cong K_{n,n}$, a contradiction. This completes the proof of the theorem. □

By Theorems 3.2 and 4.1 we have the following corollary.

Corollary: G is an (n-1)-minimal graph on $2n \geq 4$ vertices if and only if $G \cong K_{n,n}$ or K_{2n}. □

Remark: By Theorem 4.1 and its corollary, every (n-1)-extendable graph on $2n \geq 4$ vertices is minimal. The result is best possible in the sense that there is an (n-2)-extendable graph on $2n \geq 6$ vertices which is not minimal. Such a graph is $H = \bar{K}_2 \vee K_{2n-2}$. Clearly H is (n-2)-extendable, but it is not minimal since H - uv is also (n-2)-extendable where $u \in \bar{K}_2$, $v \in K_{2n-2}$.

It is interesting to observe that, by Theorem 3.2, $K_{n,n}$ and K_{2n} are not k-minimal for $1 \leq k \leq n - 2$. It turns out that characterizing k-minimal graphs, $1 \leq k \leq n - 2$, on 2n vertices is a much more challenging task than that of characterizing the (n-1)-minimal graphs. We have completely characterized the (n-2)-minimal graphs in a lengthy paper [2]. Our main result is :

Theorem 4.2: Let G be an (n-2)-extendable graph on $2n \geq 10$ vertices. G is minimal if and only if G :

(1) is an (n-1)-regular bipartite graph, or

(2) is a (2n-3)-regular graph, or

(3) contains one vertex of degree 2n-1 and 2n-1 vertices of degree 2n-3, or

(4) contains 2n-2 vertices of degree 2n-3 and two vertices ,u and v say,of degree 2n-2 such that $N_G(u) - v = N_G(v) - u$. □

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