A COMBINATORIAL GENERALISATION OF THE JORDAN CURVE THEOREM

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ABSTRACT. We generalise to the setting of 3-graphs a combinatorial analogue of the Jordan curve theorem due to Stahl [9, 10]. More specifically, using combinatorial techniques only, we give a graph theoretic version of the theorem that the first Betti number of a surface is the largest number of closed curves that can be drawn on the surface without dividing it into two or more regions.

1. INTRODUCTION

In recent years, several authors have investigated topological graph theory from a combinatorial viewpoint. In particular, graph theoretic versions of the Jordan curve theorem have been proved in [5, 9, 10, 13]. In some of these papers ([5, 13]) the development is in terms of 3-graphs, which are defined as cubic graphs endowed with a proper edge colouring in three colours. In the present paper, this work is extended to a graph theoretic version of the theorem that the first Betti number of a surface is the largest number of closed curves that can be drawn on the surface without dividing it into two or more regions. Again the treatment is in terms of 3-graphs.

The paper is divided into seven sections. Section 2 covers the standard graph theoretic definitions and notation which permeate the paper. The next three sections provide the motivation for our main theorems. Section 3 gives a brief account of the use of 3-graphs in topological graph theory. In Section 4 further concepts related to 3-graphs are introduced, and it is shown how these concepts relate to topological ideas. That section also introduces our main theorems, and contains the proof of a corollary which is the graph theoretic version of the Jordan curve theorem that appeared in [5]. Section 5 uses this version of the Jordan curve theorem to derive another one due

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to Stahl [9, 10]. (This section may be omitted without loss of continuity.) Our main results appear in the last two sections.

Many of the ideas in the early sections of this paper have appeared elsewhere, but are discussed again here in order to make this paper as self-contained as possible.

2. Basic Definitions and Notation

Throughout this paper, the sum of sets is defined as their symmetric difference.

The graphs we consider lack loops, unless we indicate otherwise, but may have multiple edges. This paper is concerned only with *finite* graphs, those graphs G in which the vertex set VG and edge set EG are both finite. Two distinct edges of a graph are said to be *adjacent* if they are incident on a common vertex. We write $c(G)$ for the number of components in a graph G. If $X \subseteq VG$, then the *coboundary* $\partial_G X$ of X is the set of edges in G that join a vertex in X to a vertex in $VG - X$. If $T \subseteq EG$, then we write $G[T]$ for the subgraph of G whose edge set is T and whose vertex set is the set of all vertices of G incident with at least one edge of T. We sometimes write $VT = VG[T]$ when no ambiguity results.

A *path* P joining two vertices, a and b, in the same component of G is the edge set of a minimal connected subgraph of G containing a and b. The path is *trivial* if $a = b$. We call a and b the *terminal vertices* of P. The edges of P incident on a or b are the *terminal edges* of P. The other vertices and edges of $G[P]$ are *internal* vertices and edges, respectively, of P. If x and y are vertices or edges of a path P, then we denote by $P[x, y]$ the edge set of the unique minimal connected subgraph of $G[P]$ containing x and y. A *circuit* in G is the edge set of a non-empty connected subgraph in which each vertex has degree 2. If C is a circuit, the elements of VC are sometimes referred to as *vertices* of C. The *length* of a path or circuit is its cardinality.

The *cycle space* of G is the vector space (over the field of residue classes modulo 2) spanned by the set of circuits of G. We denote it by $\mathcal{C}(G)$, and its elements are *cycles*.

3. Gems

Let K be a cubic graph. A *proper edge colouring* of K is a colouring of the edges so that adjacent edges receive distinct colours. A 3-*graph* is defined as an ordered triple $(K, \mathcal{P}, \mathcal{O})$ where K is a cubic graph endowed with a proper edge colouring $\mathcal{P}$ in three colours and $\mathcal{O}$ is a ordering of the three colours. We shall assume throughout that the three colours are red, yellow and blue. We write $K = (K, \mathcal{P}, \mathcal{O})$ when no ambiguity results. The set obtained from EK by deletion of the edges of a specified colour is the union of a set of disjoint circuits, called *bigons*. Thus bigons are of three types: red-yellow, red-blue and blue-yellow. Following Lins [4], we define a *gem* to be a 3-graph in which the red-blue bigons are quadrilaterals (circuits of length 4).

A 2-cell embedding of a graph G, which may have loops, in a closed surface $\mathcal{S}$ can be modelled by means of a gem in the following way (see [1, 2, 4, 6]). First construct the barycentric subdivision Δ of the embedding of G, and colour each vertex of Δ with blue, yellow or red according to whether it represents a vertex, edge or face of the embedding. Each edge of Δ then joins vertices of distinct colours, and may be coloured with the third colour. Let K be the dual graph of Δ, each edge of K being coloured with the colour of the corresponding edge of Δ. Then each red-blue bigon of

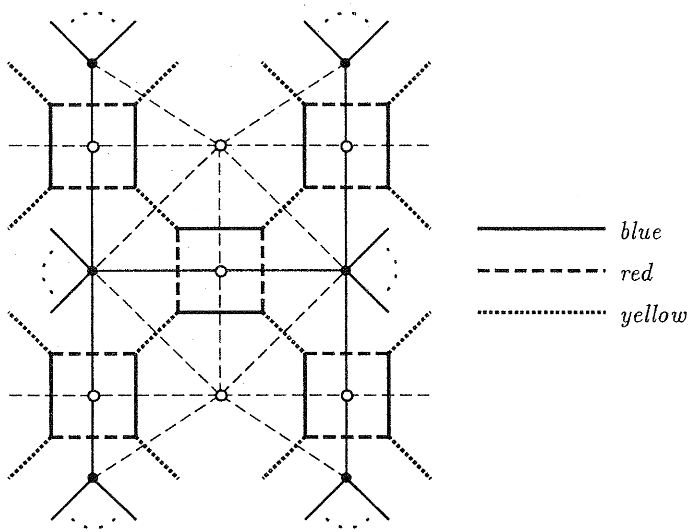

FIGURE 1.

the 3-graph K is a quadrilateral, so that K is a gem. (See Figure 1. In this figure, the vertices of G are the solid circles and the edges are the thin solid lines joining such circles. All the circles are vertices of Δ. The edges of Δ are thin solid line segments and the thin dashed lines. The edges of K are thicker and coloured as indicated in the figure. The vertices of K should be self-evident.)

This construction can be reversed. Given K, we first contract each red-yellow bigon to a single vertex. Each red-blue bigon then becomes a digon (a circuit of length 2) whose edges are both blue. The identification of the two edges in each of these digons yields G. Thus there is a 1:1 correspondence between gems and 2-cell embeddings of graphs in closed surfaces. Also, if $S \subseteq EK$ then the blue edges in S appear in G. The set T of such edges of G is also said to *correspond* to S, and vice versa, but this correspondence is not 1:1. In general, T corresponds to several subsets of EK. We also say that each of these subsets *represents* T.

If G is obtained from a gem K in this way, we will sometimes write $G(K)$ instead of G. We also say that K *represents* the embedding of $G(K)$. Recall that $G(K)$ may have loops. An edge that is not a loop will be called a *link.*

Lins also shows that the surface $\mathcal{S}$ is orientable if and only if K is bipartite. A generalisation of this result appears in [11].

The vertices of G are in 1:1 correspondence with the red-yellow bigons of K, the edges of G with the red-blue bigons of K, and the faces of the embedding of G with the blue-yellow bigons of K. Thus if we denote by $\kappa(K)$ the number of bigons of K, we have $\kappa(K) = |VG| + |EG| + |FG|$, where FG is the set of faces of the embedding of G. The Euler characteristic $\chi(\mathcal{S})$ of $\mathcal{S}$ is therefore

$$\begin{aligned} |VG| - |EG| + |FG| &= \kappa(K) - 2|EG| \\ &= \kappa(K) - \frac{|VK|}{2} \end{aligned}$$

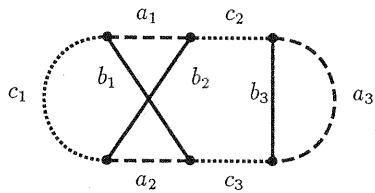

FIGURE 2.

since $|VK| = 4|EG|$.

Gems appeared first in the doctoral dissertation of Robertson [7] and subsequently in work of Ferri and Gagliardi [3]. The correspondence between gems and embeddings was developed by Lins in [4], though his account was not expressed in terms of the barycentric subdivision of the embedding. In [5], gems were used to prove a graph theoretic version of the Jordan curve theorem. This version is equivalent to a theorem of Stahl [9, 10] on pairs of permutations. Further graph theoretic versions of the Jordan curve theorem appeared in [13]. Although it is gems that correspond to embeddings, in this paper we work in the more general setting of 3-graphs. The topological implications of this paper are discovered by specialising the main theorems to the case of gems.

4. THE MAIN RESULTS AND THEIR TOPOLOGICAL IMPLICATIONS

We now develop the concepts needed for our main theorems. Some of these ideas appeared originally in the work of Stahl [8, 9, 10] on permutation pairs.

Let K be a 3-graph. A non-empty set C of edges of K is called a *b-cycle* if C is the union of disjoint circuits with at least one blue edge in each. A set $\mathcal{S}$ of b-cycles *induces* a b-cycle C if each blue edge of C is an element of $\bigcup \mathcal{S}$. In the special case where $\mathcal{S} = \{D\}$ for a b-cycle D, we also say that D *induces* C. For example, any b-cycle induces itself. The *boundary space* of K is the subspace of $\mathcal{C}(K)$ spanned by the set of bigons of K. A b-cycle is said to *separate* if it induces a b-cycle which is a member of the boundary space. A set $\mathcal{S}$ of b-cycles is said to *separate* if it induces a b-cycle which separates. A b-cycle is *connected* if it is a circuit.

EXAMPLE 1. *Consider the 3-graph of Figure 2 and let $C = \{b_1, c_3, b_3, c_2, a_1\}$. The b-cycle C separates since it induces the b-cycle $\{b_3, a_3\}$. However C is not a member of the boundary space since it is not a sum of bigons.*

If C is a b-cycle, its *necklace* $N(C)$ is the set of red-yellow bigons it meets. The elements of its necklace are the *beads* of C. The *poles* of a bead B (with respect to C) are the vertices of B incident with a blue edge of C. If C is connected and each bead has just two poles, then C is a *semicycle*.

EXAMPLE 2. *Consider the 3-graph of Figure 3 and let C be the connected b-cycle $\{b_2, c_2, a_2, b_4, a_4, c_3\}$. Let $B_1 = \{a_1, c_2, a_2, c_1\}$ and $B_2 = \{a_3, c_3, a_4, c_4\}$. Then $N(C) = \{B_1, B_2\}$. The poles of B_1 with respect to C are v_2 and v_4. Likewise the poles of B_2 with respect to C are w_2 and w_4. Hence C is a semicycle. The connected b-cycle $\{a_1, b_2, c_3, b_3, a_2, b_4, c_4, b_1\}$ is not a semicycle since the poles of B_1 are v_1, v_2, v_3 and v_4.*

Note that if a semicycle C induces a b-cycle C', then C' must be a semicycle with the same blue edges as C.

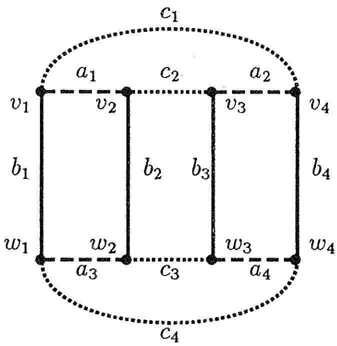

FIGURE 3.

The concept of a semicycle was introduced by Stahl [9] in the setting of permutation pairs, but the motivation for it is best explained by considering the case where K is a gem. As we indicated earlier, K then corresponds to a 2-cell embedding, in a closed surface $\mathcal{S}$, of a graph G. Under this interpretation, the beads of a semicycle C of K correspond to vertices of G. The requirements that C should be connected and have a blue edge, and that each bead should have just two poles, reveal that C corresponds to a circuit D of G or a path of length 1. In the former case, observe that if D divides $\mathcal{S}$ into two regions, then D is the sum of the boundaries of the faces inside one of those regions. (We consider an edge to belong to the boundary of a face F if and only if it separates two distinct faces, one of which is F.) If R is either of those regions, then C is the sum of the blue-yellow bigons of K corresponding to the faces inside R, the red-yellow bigons corresponding to vertices in the interior of R, the red-blue bigons corresponding to edges in the interior of R, and possibly some of the red-yellow and red-blue bigons corresponding to the vertices and edges, respectively, of D. We infer that C is a member of the boundary space, and hence separates. Conversely, if the semicycle C separates, then C is a sum of bigons, and it follows that D divides $\mathcal{S}$ into two regions. The vertices, edges and faces in the interior of one of these regions correspond to those bigons in the sum which are not beads of C or red-blue bigons which meet C.

On the other hand, suppose that C corresponds to a path Q in G of length 1. Then C has just two beads. Either C is the red-blue bigon corresponding to the unique edge of Q, or C is the sum of this bigon and one or both of the beads. In any case, C separates.

Note also that each circuit of G has a non-empty family of semicycles of K which correspond to it.

The members of a set $\mathcal{S}$ of b-cycles are *b-independent* if each b-cycle in $\mathcal{S}$ contains a blue edge not in any of the others. The b-cycles in $\mathcal{S}$ are *b-dependent* if they are not b-independent. Hence the members of $\mathcal{S}$ are b-dependent if there exists $C \in \mathcal{S}$ induced by $\mathcal{S} - \{C\}$.

EXAMPLE 3. *Consider the gem of Figure 4. Let* $C_1 = \{b_5, c_2, a_4, c_3, b_2, c_4\}$ *and* $C_2 = \{b_4, c_6, b_1, c_1, a_3, c_2\}$. *Hence* $\mathcal{S} = \{C_1, C_2\}$ *is a set of two b-independent semicycles since the blue edge* $b_4 \in C_2$ *is not in* C_1 *and similarly the blue edge* $b_5 \in C_1$ *is not in* C_2. *Now let* $C_3 = \{a_1, b_2, a_2, b_1\}$. *Then* $\{C_1, C_2, C_3\}$ *is a set of b-dependent semicycles*

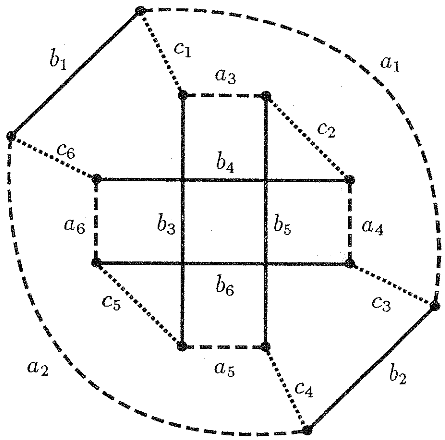

FIGURE 4.

since C_3 is induced by $\{C_1, C_2\}$.

We define the *first homology space* $\mathcal{H}(K)$ of K to be the orthogonal complement in $\mathcal{C}(K)$ of the boundary space of K. We now state the main theorem of the paper, which is proved in Section 7.

THEOREM 1. *If K is a 3-graph then the cardinality of a maximum set of b-independent semicycles which does not separate is* $\dim \mathcal{H}(K)$.

In the remainder of this section, we explain the topological significance of this result and show how it can be used to deduce the version of the Jordan curve theorem which appeared in [9].

The *conjugate* C^* of a b-cycle C is defined as $C + (\bigcup N(C))$, and is also a b-cycle. We define $I(C) = C \cap (\bigcup N(C))$ and $O(C) = C^* \cap (\bigcup N(C))$. We note that $\{I(C), O(C)\}$ is a partition of $\bigcup N(C)$. We call $I(C)$ and $O(C)$ the *sides* of C. This terminology is suggested by the interpretation of a gem as a model for an embedding of a graph.

EXAMPLE 4. *Consider C_1 and C_2 of Example 3. Then $I(C_1) = \{c_2, a_4, c_3, c_4\}$ and $O(C_1) = \{a_3, c_1, a_1, a_2, c_6, a_6, c_5, a_5\}$. Similarly, $I(C_2) = \{c_6, c_1, a_3, c_2\}$ and $O(C_2) = \{a_6, c_5, a_5, c_4, a_2, a_1, c_3, a_4\}$.*

Note that C^* separates if and only if C does.

Now let P be a non-trivial path in K whose terminal edges are blue. For such paths we define necklaces, beads, poles and conjugates as for b-cycles, and we use analogous notation for these concepts. If each bead of the necklace $N(P)$ has just two poles, we call P a *semipath* of K. In addition, a red-yellow bigon is called a *terminal bead* of P if it contains a terminal vertex of P. P therefore has just one or two terminal beads. The elements of $N(P)$ are sometimes called *internal beads* of P. Each internal bead of a semipath has just two poles, and so none of them is terminal.

EXAMPLE 5. *Consider the gem of Figure* 4 *and let* $P = \{b_1, c_6, b_4\}$. *Then* P *is a semipath with one terminal bead* $\{c_1, a_3, c_2, a_4, c_3, a_1\}$ *and one internal bead* $\{c_6, a_6, c_5, a_5, c_4, a_2\}$.

If K is a gem representing an embedding of a graph G, then a semipath P represents a path or circuit in G according to whether P has two terminal beads or just one. If P represents a path, then the internal and terminal beads of P correspond to the internal and terminal vertices, respectively, of the path. If P represents a circuit, then the internal beads and the unique terminal bead correspond to the vertices of the circuit.

A b-cycle C and a semipath P are said to *miss* if they are disjoint and no internal bead of P is a bead of C. (If K is a gem, C a semicycle in K corresponding to a circuit D in $G(K)$, and P a semipath in K corresponding to a path Q in $G(K)$, then C and P miss if and only if D and Q are disjoint and no internal vertex of Q is a vertex of D. If Q is a circuit rather than a path, then C and P miss if and only if D and Q are disjoint and no vertex of Q corresponding to an internal bead of P is a vertex of D.) If C and P miss, then P is said to *link the sides* of C if one terminal vertex is in $VI(C)$ and the other is in $VO(C)$. (We note that the terminal vertices of P cannot be poles of a bead of C, since $P \cap C = \emptyset$.)

EXAMPLE 6. *The semipath* P *of Example* 5 *links the sides of the semicycle* C_1 *of Example* 3.

A b-cycle C is *normal* if no blue edge of C is adjacent to edges of C with distinct colours. Clearly the conjugate of a normal b-cycle is also normal. Furthermore, no b-cycle of odd length can be normal.

The following theorem is proved in Section 6.

THEOREM 2. *A necessary and sufficient condition for a b-cycle* C *to separate a* 3*-graph* K *is for* C *to induce a normal b-cycle* C' *such that no semipath links the sides of* C'.

Theorem 2 asserts that a b-cycle separates if and only if it induces a normal b-cycle whose sides are not linked by a semipath. This theorem also has a topological interpretation, which may be discerned from the following considerations. Let K be a gem corresponding to a 2-cell embedding of a graph G in a closed surface $\mathcal{S}$, and let D be a circuit in G corresponding to a normal semicycle C in K. Then the two sides of C determine a pair of complementary subsets E_1 and E_2 of the set $\partial_G VD$. Each of these subsets consists of all the edges in $\partial_G VD$ whose corresponding red-blue bigons in K meet a given side of C. Intuitively, E_1 and E_2 may be thought of as representing the sides of D in the embedding of G, since C is normal. Now let P be a semipath in K of length greater than 1. If P corresponds to a path Q in G, then P links the sides of C in K if and only if Q joins two vertices of D, meets both E_1 and E_2 and none of its internal vertices is a vertex of D. On the other hand, if Q is a circuit rather than a path then P links the sides of C if and only if Q meets both E_1 and E_2 and has just one vertex in common with D.

Intuitively, Theorem 2 therefore asserts that C separates if and only if D divides $\mathcal{S}$ into two regions in such a way that if Q is a path in G with no internal vertex in

VD, or a circuit in G having at most one vertex in VD, then the vertices and edges of Q that are not in D are collectively confined to one region.

EXAMPLE 7. *Consider the b-cycle C of Example 1. C induces the normal b-cycle $C' = \{b_3, a_3\}$. Moreover $I(C') = \{a_3\}$ and $O(C') = \{c_3, a_2, c_1, a_1, c_2\}$ and therefore it is evident that no semipath links the sides of C'. This result of course agrees with Theorem 2.*

If T is either a set of edges of a 3-graph K or a subgraph of K, we denote by $\beta(T)$ the set of all blue edges of T. If C is a semicycle of K which induces another b-cycle C', then C' is a semicycle such that $\beta(C') = \beta(C)$. Now suppose that K is a gem corresponding to a 2-cell embedding of a graph G in a closed surface $\mathcal{S}$. It follows from the above observation that C and C' correspond to the same path or circuit of G. Note also that a set of semicycles in a gem is b-independent if and only if the corresponding paths and circuits in G have the property that each contains an edge not in the union of the others. Thus a necessary and sufficient condition for a set of semicycles to be a set S of b-independent semicycles which does not separate is for S to correspond to a set of circuits in G which are collectively drawn so as not to divide $\mathcal{S}$ into two or more regions and have the property that each contains an edge not in any of the others. According to Theorem 1, the cardinality of a maximum set of such semicycles is the dimension of the first homology space of K. It is shown in [5] that the dimension of the boundary space is $\kappa(K) - c(K)$. Therefore the dimension of the first homology space $\mathcal{H}(K)$ is

$$\begin{aligned} |EK| - |VK| + c(K) - (\kappa(K) - c(K)) &= 2c(K) - \kappa(K) + \frac{|VK|}{2} \\ &= 2c(G) - \chi(\mathcal{S}) \end{aligned}$$

since K is cubic and $c(K) = c(G)$. This number is the first Betti number of $\mathcal{S}$. These observations show that Theorem 1 is a graph theoretic version of the theorem that the first Betti number of a surface is the largest number of closed curves that can be drawn on the surface without dividing it into two or more regions.

The considerations above also show that $\mathcal{S}$ is the sphere if and only if the dimension of the first homology space of K is 0. Thus G is planar if and only if the set of bigons of K spans $\mathcal{C}(K)$.

In order to obtain a graph theoretic version of the Jordan curve theorem, let us suppose first that $\mathcal{S}$ is the sphere. Since the sphere is orientable, K is bipartite. Let C be a normal semicycle in K. By Theorem 1, $\{C\}$ separates, and therefore induces a b-cycle which separates. This b-cycle induces a normal b-cycle D whose sides are not linked by a semipath. D therefore separates. But D, being normal and induced by $\{C\}$, must be C or C^*. In either case we conclude that C separates.

On the other hand, suppose that K is bipartite but that $\mathcal{S}$ is not the sphere. Then K has a semicycle C which does not separate. C therefore does not induce a b-cycle which is a sum of bigons. But it is easy to show that C induces a normal semicycle. We begin by observing that if b is a blue edge of C adjacent to edges of C with distinct colours, then by adding to C a bead containing one of these edges we obtain a semicycle induced by C in which the edges adjacent to b are of the same colour. By applying this construction to all but one of the blue edges of C, we can find a

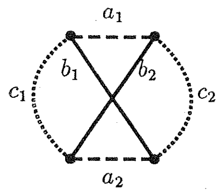

FIGURE 5.

semicycle C' which is induced by C and has the property that at most one blue edge of C' is adjacent to edges of C' with distinct colours. The requirement that $|C'|$ be even, because K is bipartite, then forces C' to be normal. Since C' is induced by C, it cannot be a sum of bigons. Any b-cycle induced by C' is obtained from C' by the addition of beads of C', and therefore is not a sum of bigons either. We conclude that C' does not separate.

In summary, we have shown that if K is bipartite then G is planar if and only if every normal semicycle of K separates. This is the graph theoretic version of the Jordan curve theorem which appeared in [5]. It implies another version, due to Stahl [9], as we explain in the next section.

In the case where K is non-bipartite, it may not be possible to construct a normal semicycle, separating or non-separating. Consider the gem in Figure 5. The four possible semicycles in this gem are $\{b_1, c_1, a_2\}$ and $\{b_2, c_2, a_2\}$ and their conjugates. In all cases the semicycle is not normal.

5. PERMUTATION PAIRS

The concept of a *permutation pair*, an ordered pair of permutations of a finite set A, has been studied in several papers [8, 9, 10]. The image of an element $a \in A$ under a permutation P will be written as aP, and the composition of permutations therefore read from left to right. It is shown in [5] that each permutation pair (P, Q) corresponds to a bipartite 3-graph $K(P, Q)$. The vertex set of this graph is the Cartesian product $A \times \{-1, 1\}$. For each $a \in A$, $K(P, Q)$ contains a red edge joining $(a, 1)$ and $(a, -1)$, a blue edge joining $(a, 1)$ and $(aQ, -1)$ and a yellow edge joining $(a, -1)$ and $(aP, 1)$. (These choices of colours, though different from the choices in [5], are more convenient for our present purpose.) Conversely each bipartite 3-graph represents a permutation pair.

We give an example of a permutation pair and the corresponding 3-graph. It is Example 1 of [9]. The corresponding 3-graph is given in Figure 6.

EXAMPLE 8. $P = (123)(456)(789)$, $Q = (1)(23)(567)(48)(9)$.

Any permutation can be written as a product of disjoint cycles, which we call *orbits*. The orbits of P are in 1:1 correspondence with the red-yellow bigons of $K(P, Q)$ and the orbits of Q with the red-blue bigons of $K(P, Q)$. Inspection of Figure 7 also reveals the existence of a similar 1:1 correspondence between the orbits of QP and the blue-yellow bigons of $K(P, Q)$. If Q is an involution then the red-blue bigons are quadrilaterals, so that $K(P, Q)$ is a gem. Thus the notion of a permutation pair generalises the concept of a 2-cell embedding of a graph in an orientable surface. Note also that each $a \in A$ corresponds to a red edge $a' \in EK(P, Q)$.

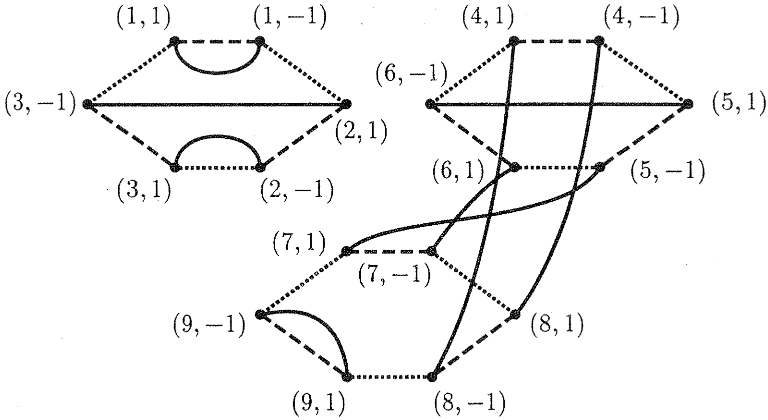

FIGURE 6.

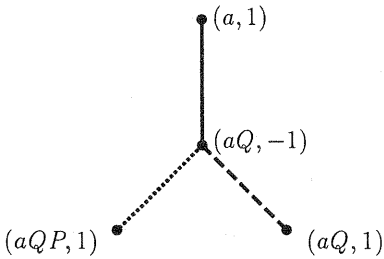

FIGURE 7.

Stahl [9] defines an *xy-semipath* D in (P,Q) as a sequence

$$(x, a_1, P_1, b_1, a_2, P_2, b_2, \dots, a_t, P_t, b_t, y)$$

for which

(1) $P_1, \dots, P_t$ are distinct orbits of P,
(2) $a_i, b_i \in P_i$ for each i, and
(3) there exist $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_t \in \{1, -1\}$ satisfying the condition that $b_i Q^{\varepsilon_i} = a_{i+1}$ for each integer i such that $0 \le i \le t$, where $b_0 = x$ and $a_{t+1} = y$.

Likewise a *semicycle* C in (P,Q) is a sequence

$$(a_1, P_1, b_1, a_2, P_2, b_2, \dots, a_t, P_t, b_t),$$

where $t > 0$, for which (1) and (2) hold and there exist $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t \in \{1, -1\}$ such that $b_i Q^{\varepsilon_i} = a_{i+1}$ for each $i \le t$, where $a_{t+1} = a_1$.

The orbits $P_1, P_2, \dots, P_t$ are the *vertices* of D and C in (P,Q). The *arcs* of D are the ordered pairs $(x, a_1), (b_1, a_2), (b_2, a_3), \dots, (b_{t-1}, a_t), (b_t, y)$, and the *arcs* of C are $(b_1, a_2), (b_2, a_3), \dots, (b_{t-1}, a_t), (b_t, a_1)$.

If $t > 1$ and neither x nor y appears in any of the orbits $P_1, \dots, P_t$, then the xy-semipath D in (P,Q) given above correponds in the following way to a family $\mathcal{D}$ of semipaths in $K(P,Q)$ such that the internal beads of each member of $\mathcal{D}$ are the red-yellow bigons $P_1', P_2', \dots, P_t'$ of $K(P,Q)$ corresponding to $P_1, P_2, \dots, P_t$ respectively. First, condition (3) shows that for each arc there is a corresponding blue edge of $K(P,Q)$ adjacent to the red edges corresponding to the members of the arc. (This blue edge is not necessarily unique.) For each member of $\mathcal{D}$ the set of blue edges is obtained by selecting one such blue edge for each arc. Let us refer to a vertex as a pole if it is incident on one of the selected blue edges. For each P_i there exist paths in P_i' joining poles. Condition (1) implies that for each i there are exactly two choices for such a path, the choices being complementary subsets of P_i'. The red and yellow edges of a member of $\mathcal{D}$ are supplied by choosing such a path for each i. It is immediate that each member of $\mathcal{D}$ is a semipath in $K(P,Q)$.

In a similar way we find that if $t > 2$ then each semicycle

$$(a_1, P_1, b_1, \dots, a_t, P_t, b_t)$$

in (P,Q) corresponds to a family of semicycles in $K(P,Q)$. The beads of each member of this family are the red-yellow bigons $P_1', P_2', \dots, P_t'$ of $K(P,Q)$ corresponding to $P_1, P_2, \dots, P_t$ respectively. For each arc, every member of the family contains a corresponding blue edge adjacent in $K(P,Q)$ to the red edges corresponding to the members of the arc. Since $K(P,Q)$ is bipartite, we find that for each semicycle in (P,Q) the corresponding family of semicycles in $K(P,Q)$ contains a normal semicycle.

For instance, in Example 8, Stahl defines the semicycle

$$C_6 = (4, (456), 6, 7, (789), 8)$$

in (P,Q). An example of a normal semicycle in $K(P,Q)$ corresponding to C_6 is the one with vertex set

$$\{(4,1), (6,-1), (6,1), (7,-1), (7,1), (9,-1), (9,1), (8,-1)\}.$$

Note that the semipath (x,y) in (P,Q) corresponds in $K(P,Q)$ to a semipath consisting of a single blue edge.

Next, suppose that D is the semipath (x, a_1, P_1, b_1, y) in (P, Q), where neither x nor y appears in the orbit P_1. If either $a_1 \neq b_1$ or $x \neq y$, then D corresponds to a family of semipaths in $K(P, Q)$ by the construction given above. If $a_1 = b_1$ and $x = y$, then D still corresponds to a family of semipaths in $K(P, Q)$ provided that (xa_1) is an orbit of Q, because in that case there are two blue edges in $K(P, Q)$ adjacent to both x' and a_1'. However if $a_1 = b_1$, $x = y$ and (xa_1) is not an orbit of Q, then $(x, a_1) = (y, b_1)$ but only one blue edge of $K(P, Q)$ is adjacent to x' and a_1'. Only under these circumstances does D not correspond to a family of semipaths in $K(P, Q)$. In this case we describe D as *singular*.

Any semicycle in (P, Q) of the form (a_1, P_1, b_1) corresponds to a family of semicycles in $K(P, Q)$ by the construction given above. Stahl defines a semicycle of the form (a_1, P_1, a_1) to be *degenerate*. The family of corresponding semicycles in $K(P, Q)$ consists of a red-blue digon and its conjugate.

Now consider a semicycle in (P, Q) of the form $(a_1, P_1, b_1, a_2, P_2, b_2)$. By the argument used above for the semipath (x, a_1, P_1, b_1, y), this semicycle corresponds to a family of semicycles in $K(P, Q)$ except in the case where $a_1 = b_1, a_2 = b_2$ and $(a_1 a_2)$ is not an orbit of Q. Stahl also defines a semicycle of the form $(a_1, P_1, a_1, a_2, P_2, a_2)$ to be *degenerate*, and *singular* if in addition $(a_1 a_2)$ is not an orbit of Q. Thus a singular semicycle in (P, Q) has no corresponding family of semicycles in $K(P, Q)$.

Stahl illustrates these ideas in Example 8 with the following semicycles:

$$\begin{aligned}
C_1 &= (1, (123), 1), \\
C_2 &= (2, (123), 3), \\
C_3 &= (5, (456), 6), \\
C_4 &= (4, (456), 4, 8, (789), 8), \\
C_5 &= (6, (456), 6, 7, (789), 7), \\
C_6 &= (4, (456), 6, 7, (789), 8).
\end{aligned}$$

Of these, only C_5 is singular. In Figure 6 the semicycles with vertex sets

$$\{(1,1), (1,-1)\}$$

$$\{(3,1), (2,-1)\}$$

$$\{(5,1), (6,-1), (4,1), (4,-1)\}$$

$$\{(4,1), (8,-1), (8,1), (4,-1)\}$$

$$\{(4,-1), (4,1), (6,-1), (6,1), (7,-1), (8,1)\}$$

are members of the families corresponding to C_1, C_2, C_3, C_4, C_6 respectively.

Note also that every semipath in $K(P, Q)$ corresponds to a semipath in (P, Q), and every semicycle in $K(P, Q)$ to a semicycle in (P, Q). Moreover the vertices of a semipath or semicycle in (P, Q) correspond to the (internal) beads of each member of the corresponding family of semipaths or semicycles in $K(P, Q)$.

An arc (b, a) of a semipath D or semicycle C in (P, Q) is said to be *coherent* if $a = bQ \neq bQ^{-1}$, *retrograde* if $a = bQ^{-1} \neq bQ$ and *neutral* if $a = bQ = bQ^{-1}$. If C or D corresponds to a family of semicycles or semipaths in $K(P, Q)$, then in any member of this family the blue edges corresponding to neutral arcs of this kind are those that belong to red-blue quadrilaterals of $K(P, Q)$. However arcs of the form

(a, a) are also said to be neutral. Blue edges corresponding to this kind of neutral arc belong to red-blue digons of $K(P,Q)$. Note also that if C or D corresponds to a family of semicycles or semipaths in $K(P,Q)$, then an arc (b,a) in C or D corresponds to a blue edge joining vertices $(b,1)$ and $(a,-1)$ if the arc is coherent, and to a blue edge joining vertices $(b,-1)$ and $(a,1)$ if the arc is retrograde.

A semipath and a semicycle in (P,Q) are said to *intersect* if they share a vertex. Corresponding semipaths and semicycles in $K(P,Q)$ therefore do not miss.

We come now to Stahl's conception of the sides of a non-degenerate semicycle. First, he lets Fix(Q) denote the set of all elements of A that are fixed by Q. In $K(P,Q)$ they correspond to the red edges that appear in red-blue digons.

Now let C be the non-degenerate semicycle

$$(a_1, P_1, b_1, a_2, P_2, b_2, \dots, a_t, P_t, b_t).$$

Choose $i \leq t$. If $a_i \neq b_i$, define $L(P_i) = \{a_iP, a_iP^2, \dots, a_iP^{k-1}\}$, where k is the smallest positive integer for which $a_iP^k = b_i$. Suppose now that $a_i = b_i$. Since C is non-degenerate, it follows that $t > 1$ and that the arc (b_{i-1}, a_i) cannot be neutral. (Here, and in the sequel, subscripts are to be read modulo t.) Suppose that this arc is coherent. Then $a_i = b_{i-1}Q$. If $a_{i+1} = b_iQ^{-1}$, then $a_{i+1} = a_iQ^{-1} = b_{i-1}$ in contradiction to the assumption that C is non-degenerate. We infer that if the arc (b_{i-1}, a_i) is coherent, then so is (b_i, a_{i+1}). Similarly if the former arc is retrograde then so is the latter. We define $L(P_i) = P_i - \{a_i\}$ if (b_{i-1}, a_i) is coherent, and $L(P_i) = \emptyset$ if (b_{i-1}, a_i) is retrograde. In both cases there is a unique blue edge of $K(P,Q)$ corresponding to the arc (b_{i-1}, a_i). In the former case, this blue edge joins the vertices $(b_{i-1}, 1)$ and $(a_i, -1)$; in the latter case it is the vertices $(b_{i-1}, -1)$ and $(a_i, 1)$ that are so joined.

In $K(P,Q)$, let $\rho(P_i)$ be the set of red edges corresponding to elements of $L(P_i)$. Let P_i' be the red-yellow bigon in $K(P,Q)$ corresponding to P_i. Let u_i and v_i be the vertices of P_i' incident on the blue edges corresponding to the arcs (b_{i-1}, a_i) and (b_i, a_{i+1}) respectively. (If the arc (b_{i-1}, a_i) is neutral, then the corresponding blue edge is not unique. In this case we define $u_i = (a_i, -1)$. Similarly if (b_i, a_{i+1}) is neutral, then we define $v_i = (b_i, 1)$.) Let $I(P_i')$ be the path included in P_i' which joins u_i and v_i and contains, as a terminal edge, the red edge incident on u_i if (b_{i-1}, a_i) is retrograde, but not if that arc is coherent or neutral. Then $I(P_i')$ contains the red edge incident on v_i if and only if the arc (b_i, a_{i+1}) is retrograde. Moreover the set of internal red edges of $I(P_i')$ is $\rho(P_i)$. Observe also that $L(P_i) - \text{Fix}(Q)$ corresponds to the set of internal red edges of $I(P_i')$ which do not appear in a red-blue digon.

Next, let $\beta(C)$ denote a set of blue edges of $K(P,Q)$ corresponding to the arcs of C, and chosen in such a way that $\bigcup_{i=1}^t I(P_i') \cup \beta(C)$ is a semicycle C' in $K(P,Q)$ corresponding to C. Then the beads of C' are $P_1', P_2', \dots, P_t'$, and $I(C') = \bigcup_{i=1}^t I(P_i')$. Moreover, by the considerations of the previous paragraph we see that C' is normal, because a blue edge of C' corresponding to a retrograde pair is adjacent in C' only to red edges, and a blue edge corresponding to a coherent or neutral pair only to yellow edges.

Stahl defines $L(C)$ as the complement of Fix(Q) in the union of $\bigcup_{i=1}^t L(P_i)$ with all the sets of the form $\{b_{i-1}, a_i\}$ where (b_{i-1}, a_i) is a retrograde arc. $L(C)$ therefore corresponds in $K(P,Q)$ to the set of red edges of $I(C')$ which do not appear in a red-blue digon.

Similarly, for each i Stahl defines $R(P_i) = P_i - (L(P_i) \cup \{a_i, b_i\})$. Then $R(C)$ is defined as the complement of $\text{Fix}(Q)$ in the union of $\bigcup_{i=1}^{t} R(P_i)$ with all the sets of the form $\{b_{i-1}, a_i\}$ where (b_{i-1}, a_i) is a coherent arc. The set of corresponding red edges in $K(P, Q)$ is the set of red edges of $O(C')$ which do not belong to a red-blue digon or to a red-blue quadrilateral which contains a blue edge of C'. The *sides* of C are $L(C)$ and $R(C)$.

An xy-semipath D is said by Stahl to *link the sides* of a non-degenerate semicycle C if $x \in L(C)$, $y \in R(C)$ and C and D do not intersect. Thus C and D have no vertex in common. Therefore neither x nor y appears in a vertex of D. Moreover $x \neq y$. Hence D corresponds to a family of semipaths in $K(P, Q)$. Let D' be a member of this family. Since C is non-degenerate, C corresponds to a family of semicycles in $K(P, Q)$. Let C' be a normal member of this family. Without loss of generality, we may assume that $L(C)$ corresponds in $K(P, Q)$ to a subset of $I(C')$, and $R(C)$ to a subset of $O(C')$. Moreover the terminal edges of D' are adjacent in $K(P, Q)$ to the red edge $x' \in I(C')$ and to the red edge $y' \in O(C')$ respectively. Note that if b is a blue edge of C' then b cannot be adjacent to a red edge of $I(C')$ and to a red edge of $O(C')$, since C' is normal. Hence $D' \neq \{b\}$, and since no internal bead of D' is a bead of C' it follows that C' and D' are disjoint, and therefore miss. Hence D' links the sides of C' in $K(P, Q)$.

Stahl defines a semicycle C of (P, Q) to *separate* if no semipath links its sides. Thus every degenerate semicycle separates. A non-degenerate semicycle separates (P, Q) if and only if a corresponding normal semicycle separates $K(P, Q)$.

The number of orbits of a permutation P is denoted by $\|P\|$. The *genus* $\gamma(P, Q)$ of the permutation pair (P, Q) is defined as

$$c(P, Q) - \frac{\|P\| + \|Q\| + \|PQ\| - |A|}{2},$$

where $c(P, Q)$ is the number of orbits of the group generated by P and Q. In other words, $c(P, Q)$ is the number of components of $K(P, Q)$. It is known that $\gamma(P, Q)$ is a non-negative integer. Since $|VK(P, Q)| = 2|A|$, we have

$$\gamma(P, Q) = c(P, Q) - \frac{\kappa(K(P, Q))}{2} + \frac{|VK(P, Q)|}{4},$$

which is half the dimension of the first homology space of $K(P, Q)$.

Our version of the Jordan curve theorem therefore asserts that every normal semicycle of $K(P, Q)$ separates if and only if $\gamma(P, Q) = 0$. Stahl's version asserts that every semicycle of (P, Q) separates if and only if $\gamma(P, Q) = 0$. We now prove that this version follows from ours.

We must show that every semicycle of (P, Q) separates if and only if every normal semicycle of $K(P, Q)$ separates. Suppose that every normal semicycle of $K(P, Q)$ separates. It is then immediate that every semicycle of (P, Q) separates. Conversely, suppose that every semicycle of (P, Q) separates. Let C' be a normal semicycle of $K(P, Q)$, and let D' be a semipath linking its sides. It is immediate that neither terminal vertex of D' can be a vertex of a red-blue digon which meets C'. Since C' is normal, the red edges adjacent to a blue edge of C' are either both in $I(C')$ or both in $O(C')$. If these red edges belong to a red-blue quadrilateral, it follows that no vertex of this quadrilateral can be a terminal vertex of D'. Therefore in (P, Q) the semipath corresponding to D' links the sides of the semicycle corresponding to C'. This contradiction establishes that every normal semicycle of $K(P, Q)$ separates.

6. A Condition for a b-cycle to Separate

This section is devoted to a proof of Theorem 2.

THEOREM 2. *A necessary and sufficient condition for a b-cycle D to separate a 3-graph K is for D to induce a normal b-cycle D' such that no semipath links the sides of D'.*

The proof of this theorem consists of several lemmas. We may assume that K is connected, since the general case follows by applying the theorem to each component separately.

LEMMA 1. *Let $C = \sum \mathcal{U}$ where $\mathcal{U}$ is a set of red-blue and blue-yellow bigons in a 3-graph K. Let b be a blue edge, not in $\beta(C)$, joining vertices v and w. Then either $\{v, w\} \subseteq VC$ or $\{v, w\} \cap VC = \emptyset$.*

Proof. Suppose $v \in VC$. Let a_1 and c_1 be the red and yellow edges respectively incident on v. Let a_2 and c_2 be the red and yellow edges respectively incident on w. Since $a_1 \in C$ we have $b \in \bigcup \mathcal{U}$. Hence $a_2, c_2 \in C$ and therefore $w \in VC$. Similarly one can show that $w \notin VC$ when $v \notin VC$. □

LEMMA 2. *If a b-cycle D separates, then D induces a normal b-cycle D' such that no semipath links the sides of D'.*

Proof. Since D separates, D induces a b-cycle $\sum \mathcal{U}$ where $\mathcal{U}$ is a set of bigons. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ denote the set of red-blue and blue-yellow bigons, respectively, included in $\mathcal{U}$, and let $C = \sum(\mathcal{U}_1 \cup \mathcal{U}_2)$. Now consider the cycle $D' = C + \bigcup \mathcal{B}$ where $\mathcal{B}$ is the set of red-yellow bigons included in C. (We include $\bigcup \mathcal{B}$ in the above sum of bigons to ensure that all circuits in D' contain a blue edge.) Since $\emptyset \neq \beta(\sum \mathcal{U}) = \beta(D')$, then D' is a b-cycle induced by D. Also, the two edges of D' adjacent to a given blue edge of D' must belong to the same bigon, and hence D' is a normal b-cycle. We claim that no semipath links the sides of D'. Assume by way of contradiction that P is a semipath that links the sides of D'. Let v denote the terminal vertex of P that is in $VI(D')$, and let b be the blue edge incident on v (b is therefore in P). Let b join v to w and let B denote the red-yellow bigon that contains w. By Lemma 1, $w \in VC$ and therefore $B \in N(C)$. If $B \in N(D')$, then $P = \{b\}$ and $v, w \in VI(D')$, a contradiction. Hence we conclude that $B \in \mathcal{B}$ and $B \subseteq C$. Therefore the blue edge of $P - \{b\}$ incident on a vertex of VB must have both end vertices in VC. Proceeding inductively along P, we find that the terminal vertex x of P other than v must lie in VC. Since the red-yellow bigon that contains x must be in $N(D')$ it follows that $x \in VI(D')$, a contradiction. The lemma follows. □

Lemma 2 proves half of Theorem 2. Accordingly we henceforth assume that D is a b-cycle such that no b-cycle induced by D is a sum of bigons. Let D' be an arbitrary normal b-cycle induced by D. We shall show that the sides of D' are linked by a semipath.

Let $K^\dagger$ be a graph whose vertices are the red-blue and blue-yellow bigons of K and whose edges are the blue edges of K. Any edge $b \in EK^\dagger$ is to join the two bigons containing b in K. Clearly $K^\dagger$ is connected since K is connected.

LEMMA 3. *The graph* $K^\dagger - \beta(D')$ *is connected.*

Proof. Suppose $K^\dagger - \beta(D')$ is unconnected, and let $L^\dagger$ be a component of $K^\dagger - \beta(D')$. Let us consider the b-cycle $D'' = \sum VL^\dagger + \bigcup \mathcal{B}$ where $\mathcal{B}$ is the set of red-yellow bigons included in $\sum VL^\dagger$. (Recall that $\sum VL^\dagger$ is the symmetric difference of the bigons that constitute the vertex set of $L^\dagger$.) By the construction $\beta(D'') \subseteq \beta(D') \subseteq \beta(D)$ and therefore D'' is a b-cycle induced by D that is a sum of bigons. This contradicts our assumption. The lemma follows. □

If there is an edge b of $\beta(K) - \beta(D')$ with an end vertex in $VI(D')$ and one in $VO(D')$ then $\{b\}$ links the sides of D'. Henceforth we suppose there is no blue edge with this property, and partition the set $\beta(K) - \beta(D')$ into three classes: the set I of edges with an end vertex in $VI(D')$, the set O of edges with an end vertex in $VO(D')$, and the set M of edges incident with no vertex in $VI(D') \cup VO(D')$.

LEMMA 4. *The edge sets* I *and* O *are non-empty.*

Proof. Assume that I is empty. Then there is no vertex of VD' incident on a red edge and a yellow edge of D'. Since D' is normal it must therefore be the sum of a set of red-blue and blue-yellow bigons, a contradiction. A similar argument applied to the conjugate of D' shows that $O \neq \emptyset$. □

LEMMA 5. *Let* P *be a path with terminal vertices* v *and* w *and blue terminal edges. Suppose that the red-yellow bigons containing* v *and* w *are not in* $N(P)$. *Then there exists a semipath* P', *joining* v *and* w, *such that* $\beta(P') \subseteq \beta(P)$ *and* $N(P') \subseteq N(P)$.

Proof. We use induction on $|\beta(P)|$. If $|\beta(P)| = 1$ then P is the required semipath. Now suppose the lemma holds for all paths with fewer than $|\beta(P)|$ blue edges, where $|\beta(P)| > 1$. Let b denote the blue terminal edge of P incident on v. Let B denote the red-yellow bigon in $N(P)$ containing a vertex x incident with b. (B exists since $|\beta(P)| > 1$.) Let y be the vertex of $VP \cap VB$ that minimises $|P[w, y]|$. Thus $Q = P[w, y]$ is a path with fewer blue edges than P. Furthermore $B \notin N(Q)$ by the choice of y. By the inductive hypothesis, there exists a semipath P' with terminal vertices y and w such that $\beta(P') \subseteq \beta(Q) \subseteq \beta(P)$ and $N(P') \subseteq N(Q) \subseteq N(P)$. Let Q' be a path included in B which joins x and y. Then $P' \cup Q' \cup \{b\}$ is the required semipath. □

LEMMA 6. *There exists a semipath that links the sides of* D'.

Proof. Case i) Suppose there exists a vertex Y in $VK^\dagger$ incident on an edge $b \in O$ and an edge $b' \in I$. We may choose b and b' so that they are terminal edges of a path P included in the bigon Y such that $\beta(P) - \{b, b'\} \subseteq M \cup D'$. Let $\beta(P) = \{b_1, b_2, \dots, b_n\}$, where $b_1 = b$ and $b_j \in P[b_i, b_k]$ whenever $i < j < k$. Thus $b_n = b'$. For each positive integer $i < n$ let a_i be the edge of P joining an end-vertex w_i of b_i to an end-vertex v_{i+1} of b_{i+1}, and let d_i and c_{i+1} be the edges of $EK - Y$ incident on w_i and v_{i+1} respectively. (See Figure 8.)

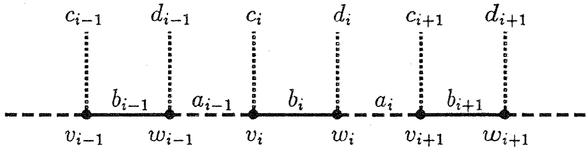

FIGURE 8.

Suppose $b_j \in D'$ for some $j < n$. Choose j to be as small as possible subject to this requirement. We have $j > 1$ since $b_1 \notin D'$. Hence $b_{j-1} \in M \cup O$. It follows that $w_{j-1} \notin VD'$, and so $c_j \in D'$. Therefore $d_j \in D'$ since D' is normal. If $j+1 < n$ then $b_{j+1} \notin M$; hence $\{c_{j+1}, b_{j+1}, d_{j+1}\} \subseteq D'$. By induction $\{c_{n-1}, b_{n-1}, d_{n-1}\} \subseteq D'$. Therefore $a_{n-1} \notin D'$. Since $b' \notin D'$, we obtain the contradiction that $b' \in O$. Hence $\{b_2, b_3, \ldots, b_{n-1}\} \subseteq M$, and so $N(P) \cap N(D') = \emptyset$. By Lemma 5, there exists a semipath P' with terminal edges b_1 and b_n such that P' misses D'. Therefore P' links the sides of D', as required.

Case ii) Suppose there is no bigon in $VK^\dagger$ incident on an edge in I and on an edge in O. By Lemma 3 and Lemma 4 there exists a path $P^\dagger \subseteq M$ in $K^\dagger - \beta(D')$ with terminal vertices Y_1 and Y_2 such that $Y_1 \cap O \neq \emptyset$ and $Y_2 \cap I \neq \emptyset$. We may assume $P^\dagger$ chosen so that $\beta(L) \cap (I \cup O) = \emptyset$, where $L = \bigcup(VP^\dagger - \{Y_1, Y_2\})$. (Recall that $VP^\dagger$ is the set of bigons in K that make up the vertices in $P^\dagger$.) This choice guarantees that $N(Z) \cap N(D') = \emptyset$ for each internal vertex Z of $P^\dagger$. Let P_1 be a path in Y_1 with blue terminal edges $b \in O$ and $b_1 \in (L \cup Y_2) \cap Y_1$ such that the blue internal edges of P_1 are edges in M. Clearly $N(P_1) \cap N(D') = \emptyset$, for otherwise an internal blue edge of P_1 would not be in M. Similarly let P_2 be a path in Y_2 with blue terminal edges $b' \in I$ and $b_2 \in (L \cup Y_1) \cap Y_2$ such that the blue internal edges of P_2 are edges in M. Again, $N(P_2) \cap N(D') = \emptyset$. It follows that there is a path P in K, with terminal edges b and b', such that $N(P) \cap N(D') = \emptyset$. By Lemma 5 we can construct a semipath P' with terminal edges b and b' that misses D'. P' links the sides of D', as required. □

7. FUNDAMENTAL SETS OF SEMICYCLES

A set of m b-independent semicycles that does not separate is said to be an m-*fundamental* set. With this definition, Theorem 1 can be restated in the following way.

THEOREM 1. *If K is a 3-graph then the maximum size of an m-fundamental set is* $\dim \mathcal{H}(K)$.

This section is devoted to a proof of Theorem 1. We shall show that the size of a maximum m-fundamental set is

$$\dim \mathcal{H}(K) = 2c(K) - \kappa(K) + \frac{|VK|}{2}.$$

If $K_1, K_2, \ldots, K_{c(K)}$ denote the components of K then

$$\begin{aligned} \sum_{i=1}^{c(K)} \dim \mathcal{H}(K_i) &= \sum_{i=1}^{c(K)} \left(2 - \kappa(K_i) + \frac{|VK_i|}{2}\right) \\ &= \dim \mathcal{H}(K). \end{aligned}$$

Therefore if Theorem 1 holds for each K_i then it holds for K. We henceforth assume K to be connected.

LEMMA 7. *If $\mathcal{S}$ is an m-fundamental set then $m \leq \dim \mathcal{H}(K)$.*

Proof. Let C be a semicycle in $\mathcal{S}$ and suppose a is a red or yellow edge contained in C. Let B denote the red-yellow bigon that contains a. Clearly $C' = C + B$ is a semicycle such that $a \notin C'$ and $\beta(C') = \beta(C)$. Let $\mathcal{S}' = (\mathcal{S} - \{C\}) \cup \{C'\}$. Since C is not induced by $\mathcal{S} - \{C\}$ then C' is not induced by $\mathcal{S}' - \{C'\}$, and therefore $\mathcal{S}'$ is a b-independent set. Furthermore $\beta(\bigcup \mathcal{S}) = \beta(\bigcup \mathcal{S}')$ and therefore $\mathcal{S}'$ is an m-fundamental set such that the number of semicycles in $\mathcal{S}'$ containing a is one less than in $\mathcal{S}$. Proceeding inductively we obtain an m-fundamental set $\mathcal{S}''$ such that there exists an edge in each red-yellow bigon of K that is not in $\bigcup \mathcal{S}''$.

Since $\mathcal{S}''$ does not separate, each semicycle in $\mathcal{S}''$ is a cycle that does not lie in the boundary space of K. Also, $\mathcal{S}''$ is a linearly independent set of cycles since it is b-independent. Let $\mathcal{T}$ be a set of bigons in K comprising all the bigons except for exactly one arbitrarily chosen bigon. It is shown in [5] that $\mathcal{T}$ is a basis for the boundary space of K. If $m > \dim \mathcal{H}(K)$ then $\mathcal{T} \cup \mathcal{S}''$ is linearly dependent since $|\mathcal{T} \cup \mathcal{S}''| > \dim \mathcal{C}(K)$ and hence there exists a cycle D belonging to the boundary space which is a sum of semicycles in $\mathcal{S}''$. By the construction of $\mathcal{S}''$, it is impossible for D to include a red-yellow bigon. We conclude that D must be a b-cycle. Moreover, D is induced by $\mathcal{S}''$, which is a contradiction. Hence we conclude that $m \leq \dim \mathcal{H}(K)$. □

The rest of this section is concerned with the construction of a $(\dim \mathcal{H}(K))$-fundamental set $\mathcal{S}$. This construction together with Lemma 7 gives us our theorem. It also implies that $\mathcal{S}$ is a basis for the first homology space and that $\mathcal{T} \cup \mathcal{S}$ is a basis for the cycle space of K, where $\mathcal{T}$ is a set of bigons in K comprising all the bigons except for exactly one arbitrarily chosen bigon.

A major tool in the construction of a $(\dim \mathcal{H}(K))$-fundamental set is the concept of a blue 1-dipole in K. Let v and w be a pair of adjacent vertices in a 3-graph K. Suppose that v and w are linked by a single edge b, which is blue. Following Ferri and Gagliardi [3], we say that b is a *blue 1-dipole* if the red-yellow bigons A and B passing through v and w respectively are distinct. Let c_1 and c_2 be the yellow edges incident on v and w respectively. Let a_1 and a_2 be the red edges incident on v and w respectively. Let v_1, v_2, w_1, w_2 be the vertices other than v and w incident on c_1, a_1, c_2, a_2 respectively. The *cancellation* of this blue 1-dipole b is the operation of deletion of the vertices v and w followed by the insertion of edges c and a linking v_1 to w_1 and v_2 to w_2 respectively. (See Figure 9.) We denote the resulting 3-graph by $K - [b]$. We observe that A and B have coalesced into one red-yellow bigon A'. Since the number of vertices has dropped by two and the number of bigons by one, the dimension of the first homology space is invariant under this operation. The inverse operation is referred to as *creation* of a blue 1-dipole.

An intriguing result (see [2, 3, 12]) is that two 3-graphs, H and K, can be obtained from one another by a finite sequence of 1-dipole cancellations and creations if and only if H and K are both bipartite or both non-bipartite, and $\dim \mathcal{H}(H) = \dim \mathcal{H}(K)$. This result is a combinatorial analogue to the famous "classification of surfaces" theorem of Dehn and Heegaard.

Let K be a 3-graph with a blue 1-dipole b. Let C' be a semicycle in the 3-graph $K' = K - [b]$. The following uses the notation in Figure 9. If C' does not meet A' then all the edges of C' are in K and we define $C = C'$. If C' meets A' then let x and y be

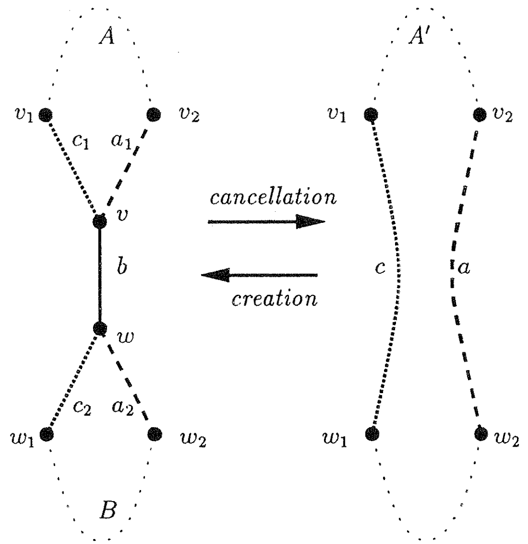

FIGURE 9.

the two poles of A' with respect to C'. Let $P' = A - \{a_1, c_1\}$ and $Q' = B - \{a_2, c_2\}$. Assume without loss of generality that $x \in VP'$ and consider the following two cases.

Case a) $y \in VP'$. Let P be a path in A that links x to y. Then we define $C = (C' - A') \cup P$.

Case b) $y \in VQ'$. Let P be a path in A that links x to v and Q be a path in B that links w to y. We define $C = (C' - A') \cup P \cup Q \cup \{b\}$.

In the cases presented above, C is clearly a semicycle in K such that $\beta(C) - \{b\} = \beta(C')$. We say that C is a semicycle (in K) *implied* by the semicycle C' (in K').

More generally, let $\mathcal{S}' = \{C'_1, C'_2, \dots, C'_n\}$ be a set of semicycles in K'. For each i let C_i be a semicycle in K implied by C'_i, and let $\mathcal{S} = \{C_1, C_2, \dots, C_n\}$. We say that $\mathcal{S}$ is a set of semicycles (in K) *implied* by the set $\mathcal{S}'$ of semicycles (in K'). Clearly $\beta(\bigcup \mathcal{S}) - \{b\} = \beta(\bigcup \mathcal{S}')$.

LEMMA 8. *Let K be a 3-graph with blue 1-dipole b, and let $K' = K - [b]$. If $\mathcal{S}'$ is an m-fundamental set in K' then a set $\mathcal{S}$ of semicycles in K implied by $\mathcal{S}'$ is an m-fundamental set in K.*

Proof. First we show that $\mathcal{S}$ is b-independent. Suppose not, and let C be a semicycle in $\mathcal{S}$ that is induced by $\mathcal{S} - \{C\}$. Therefore $\beta(C) \subseteq \beta(\bigcup(\mathcal{S} - \{C\}))$. Let C' denote the semicycle in $\mathcal{S}'$ that implies C. The fact that $\beta(C') = \beta(C) - \{b\} \subseteq \beta(\bigcup(\mathcal{S} - \{C\})) - \{b\} = \beta(\bigcup(\mathcal{S}' - \{C'\}))$ implies that $\mathcal{S}'$ is not b-independent, a contradiction. Hence we conclude that $\mathcal{S}$ is b-independent.

The following uses the notation of Figure 9. Let Y denote the red-blue bigon in K that includes $\{a_1, b, a_2\}$ and let Y' be the red-blue bigon $(Y - \{a_1, b, a_2\}) \cup \{a\}$ in K'. Similarly, let R denote the blue-yellow bigon in K that includes $\{c_1, b, c_2\}$ and let R' be the blue-yellow bigon $(R - \{c_1, b, c_2\}) \cup \{c\}$ in K'.

Suppose that $\mathcal{S}$ induces a b-cycle $D = \sum \mathcal{U}$ where $\mathcal{U}$ is a set of bigons. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ denote the set of red-blue and blue-yellow bigons, respectively, included in $\mathcal{U}$. Let $D_1 = \sum(\mathcal{U}_1 \cup \mathcal{U}_2) + \bigcup \mathcal{B}$ where $\mathcal{B}$ is the set of red-yellow bigons included in $\sum(\mathcal{U}_1 \cup \mathcal{U}_2)$. Then D_1 is also a b-cycle induced by $\mathcal{S}$ that separates. If $Y \in \mathcal{U}_1$, then let $\mathcal{U}_1' = (\mathcal{U}_1 - \{Y\}) \cup \{Y'\}$; otherwise let $\mathcal{U}_1' = \mathcal{U}_1$. If $R \in \mathcal{U}_2$, then let $\mathcal{U}_2' = (\mathcal{U}_2 - \{R\}) \cup \{R'\}$; otherwise let $\mathcal{U}_2' = \mathcal{U}_2$. Let $D' = \sum(\mathcal{U}_1' \cup \mathcal{U}_2') + \bigcup \mathcal{B}'$ where $\mathcal{B}'$ is the set of red-yellow bigons included in $\sum(\mathcal{U}_1' \cup \mathcal{U}_2')$. If $D' = \emptyset$ then D would have exactly one blue edge, namely b. By the definition of a b-cycle, D would consist of one circuit, comprising b and some red and yellow edges that belong to a red-yellow bigon. This contradicts the fact that b is a blue 1-dipole. Hence we conclude that D' is a b-cycle which separates K'. Moreover D' is induced by $\mathcal{S}'$ since $\beta(D') = \beta(D_1) - \{b\}$. However, this is a contradiction since $\mathcal{S}'$ does not separate. Hence we conclude that $\mathcal{S}$ is an m-fundamental set in K. □

Now suppose that K' is obtained from K by a finite sequence of blue 1-dipole cancellations, and that C' is a semicycle in K'. Then we apply the definition of an implied semicycle inductively to obtain a semicycle C in K that is *implied* by C'. Similarly, we speak of a set of semicycles in K' *implying* a set of semicycles in K. By Lemma 8, if $\mathcal{S}'$ is an m-fundamental set in K' then the set $\mathcal{S}$ of semicycles in K implied by $\mathcal{S}'$ is an m-fundamental set in K.

Suppose K to be a 3-graph with a unique red-yellow bigon B. Let b be a blue edge in K joining vertices x and y. Let P be a path in B with terminal vertices x and y. The blue edge b can be used to define a semicycle in K, namely $C = \{b\} \cup P$. We say that C is a semicycle *formed* from b.

Recall that $K^\dagger$ is a graph whose vertices are the red-blue and blue-yellow bigons of K and whose edges are the blue edges of K. Any edge $b \in EK^\dagger$ is to join the two bigons containing b in K. Let T be a spanning tree of $K^\dagger$, and let $T' = EK^\dagger - ET$.

Lemma 9. $|T'| = \dim \mathcal{H}(K)$.

Proof. Since K has exactly one red-yellow bigon, we have $|VK^\dagger| = \kappa(K) - 1$. The number of edges in a spanning tree for the graph $K^\dagger$ is $|VK^\dagger| - 1 = \kappa(K) - 2$. Observe that $|EK^\dagger| = |\beta(K)| = |VK|/2$. Hence

$$|T'| = |EK^\dagger| - |ET| = \frac{|VK|}{2} - (\kappa(K) - 2). \quad \square$$

Lemma 10. *If K is a connected 3-graph with one red-yellow bigon then there exists a $(\dim \mathcal{H}(K))$-fundamental set in K.*

Proof. Let $j = \dim \mathcal{H}(K) = |T'|$, where T' is defined as above. For each $b_i \in T'$ where $i = 1, \dots, j$, let C_i be a semicycle formed from b_i. Let $\mathcal{S} = \{C_1, C_2, \dots, C_j\}$. Hence $\beta(\bigcup \mathcal{S}) = T'$. Since all the b_i's are distinct, $\mathcal{S}$ is a set of j b-independent semicycles in K. Assume that $\mathcal{S}$ induces a b-cycle which separates. Then $\mathcal{S}$ induces a b-cycle D of the form $\sum \mathcal{U}$, for some set $\mathcal{U}$ of bigons in K. Since we may add the red-yellow bigon to $\mathcal{U}$ and still have a b-cycle induced by $\mathcal{S}$ which separates, we may assume that $\mathcal{U}$ does not contain the red-yellow bigon. Therefore, $\beta(D) = \partial_{K^\dagger} \mathcal{U}$. Hence there must exist an edge in T that is in $\beta(D)$. However, this is impossible since $\beta(D) \subseteq \beta(\bigcup \mathcal{S}) = T'$. Therefore $\mathcal{S}$ is a $(\dim \mathcal{H}(K))$-fundamental set in K. □

We now combine the results of the preceeding lemmas to produce a proof of Theorem 1.

Proof. By Lemma 7 we have $m \leq \dim \mathcal{H}(K)$. Hence we are required to show that there exists a $(\dim \mathcal{H}(K))$-fundamental set in K. Cancel blue 1-dipoles from K one at a time until none is left, and let K' denote the resulting graph. Therefore K' has exactly one red-yellow bigon. By Lemma 10 there exists a $(\dim \mathcal{H}(K'))$-fundamental set $\mathcal{S}'$ in K'. Since K' is obtained from K by a finite sequence of blue 1-dipole cancellations, by Lemma 8 the set of semicycles in K implied by $\mathcal{S}'$ is a $(\dim \mathcal{H}(K))$-fundamental set in K. □

REFERENCES

1. C. P. Bonnington, *Combinatorial maps and the foundations of topological graph theory*, Ph.D. thesis, Massey University, 1991.
2. C. P. Bonnington and C. H. C. Little, *The classification of combinatorial surfaces using 3-graphs*, Australas. J. Combin **5** (1992), 87–102.
3. M. Ferri and C. Gagliardi, *Crystallization moves*, Pacific J. Math. **100** (1982), 85–103.
4. S. Lins, *Graph-encoded maps*, J. Combin. Theory Ser. B **32** (1982), 171–181.
5. C. H. C. Little, *Cubic combinatorial maps*, J. Combin. Theory Ser. B **44** (1988), 44–63.
6. C. H. C. Little and A. Vince, *Embedding schemes and the Jordan curve theorem*, Topics in Combinatorics and Graph Theory (Heidelberg) (R. Bodendiek and R. Henn, eds.), Physica- Verlag, Heidelberg, 1990, pp. 479–489.
7. N. Robertson, *Graphs minimal under girth, valency and connectivity*, Ph.D. thesis, University of Waterloo, 1971.
8. S. Stahl, *Permutation-partition pairs: A combinatorial generalization of graph embeddings*, Trans. Amer. Math. Soc. **1** (1980), 129–145.
9. ———, *A combinatorial analog of the Jordan curve theorem*, J. Combin. Theory Ser. B **35** (1983), 28–38.
10. ———, *A duality for permutations*, Discrete Math. **71** (1988), 257–271.
11. A. Vince, *Combinatorial maps*, J. Combin. Theory Ser. B **34** (1983), 1–21.
12. ———, *Recognizing the 3-sphere*, Graph Theory and its Applications: East and West (M. F. Capobianco, M. Guan, D.F. Hsu, and F. Tian, eds.), New York Academy of Sciences, December 1989, pp. 571–583.
13. A. Vince and C. H. C. Little, *Discrete Jordan curve theorems*, J. Combin. Theory Ser. B **47** (1989), 251–261.

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