A CHARACTERIZATION OF WELL COVERED BLOCK-CACTUS GRAPHS

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ABSTRACT

A graph G is well covered, if any two maximal independent sets of G have the same number of vertices. A graph is called a block-cactus graph if each block is complete or a cycle. In this paper we characterize the well covered block-cactus graphs.

1. TERMINOLOGY AND INTRODUCTION

In this paper we consider finite, undirected, and simple graphs G with the vertex set $V(G)$. The degree $d(x, G)$ of a vertex x of G is the number of edges incident with x. We denote by K_n the complete graph of order n. For $A \subseteq V(G)$ let $G[A]$ be the subgraph induced by A. Moreover, $N(x, G)$ denotes the set of vertices adjacent to the vertex x and, more generally, $N(X, G) = \bigcup_{x \in X} N(x, G)$ for a subset X of $V(G)$. We write $N[x, G]$ and $N[X, G]$ instead of $N(x, G) \cup x$ and $N(X, G) \cup X$. A cycle of length n is denoted by $C_n = x_1 x_2 ... x_n x_1$. A vertex c of a graph G is called a cut vertex of G if $G - c$ has more components than G. A connected graph with no cut vertex is called a block. A block of a graph G is a subgraph of G which is itself a block and which is maximal with respect to that property. A graph G is a block graph if every block of G is a complete graph. A graph G is called a block-cactus graph if every block is complete or a cycle. A set $I \subseteq V(G)$ is an independent set of G, if $N(x, G) \cap I = \emptyset$ for every $x \in I$. Let $i(G)$ and $\alpha(G)$ denote the minimum and maximum cardinality of a maximal independent set in G. A graph G is said to be well covered if every maximal independent set in G is a maximum independent set in G. Equivalently, G is well covered if $i(G) = \alpha(G)$.

The concept of well covered graphs was introduced by Plummer [6] and studied in a few papers. In particular, the well covered bipartite graphs were characterized by Favaron [2], Ravindra [9], and Staples [10]. The cubic, planar, and 3-connected well covered graphs have been characterized in [1] by Campbell and Plummer. Recently, Finbow, Hartnell, and Nowakowski [3] and Prisner, Topp, and Vestergaard [8] have described the well covered graphs of girth at least five, and the well covered simplicial and chordal graphs, respectively. Additional examples and properties of well covered graphs may be found in the survey paper of Plummer [7]. In this paper we characterize the well covered block-cactus graphs.

2. PRELIMINARY RESULTS

The following simple property of well covered graphs was first observed by Campbell and Plummer [1].

Proposition 2.1 ([1]). If G is a well covered graph, then for each vertex $v \in V(G)$, the graph $G - N[v, G]$ is well covered.

Proposition 2.2. If X and Y are two independent sets of a graph G with $|X| > |Y|$ and $N[X, G] \subseteq N[Y, G]$, then G is not well covered.

Proof. Assume to the contrary that G is well covered. Then, for every maximal independent set I with $Y \subseteq I$, we have $|I| = \alpha(G)$. Furthermore, it follows from $N[X, G] \subseteq N[Y, G]$ that $J = (I - Y) \cup X$ is an independent set such that

$$\alpha(G) \geq |J| = |(I - Y) \cup X| = |I| - |Y| + |X| > |I| = \alpha(G).$$

This contradiction yields the desired result. $\square$

A vertex v of a graph G is simplicial if every two vertices of $N(v, G)$ are adjacent in G. Equivalently, a simplicial vertex is a vertex that appears in exactly one clique. A clique of a graph G containing at least one simplicial vertex of G is called a simplex of G. A graph G is said to be simplicial if every vertex of G is simplicial or is adjacent to a simplicial vertex of G. Certainly, if G is simplicial and $S_1, S_2, \ldots, S_n$ are the simplexes of G, then $V(G) = \bigcup_{i=1}^{n} V(S_i)$. The next result is a special case of Proposition 2.2.

Proposition 2.3 ([8]). If G is a well covered graph, then all its simplexes are pairwise vertex-disjoint.

Proof. Suppose that S_1 and S_2 are two distinct simplexes of G containing a common vertex v and the simplicial vertices v_1 and v_2, respectively. Then the independent sets $X = \{v_1, v_2\}$ and $Y = \{v\}$ of G fulfill the condition $N[X, G] \subseteq N[Y, G]$, a contradiction to Proposition 2.2. □

The following characterization of well covered block graphs was first found by Topp and Volkmann [11].

Theorem 2.1 ([11]). A block graph G is well covered if and only if every vertex of G belongs to exactly one simplex of G.

For extensions and generalizations of Theorem 2.1 we refer the reader to the papers of Topp and Volkmann [12], Hattingh and Henning [5], and Prisner, Topp and Vestergaard [8].

A 5-cycle C_5 of a graph is called basic, if C_5 does not contain two adjacent vertices of degree three or more in G (see [3]). We call a 4-cycle C_4 basic if it contains two adjacent vertices of degree two, and if the remaining two vertices belong to a simplex or a basic 5-cycle of G. A graph G is in the family $\mathcal{SQC}$, if $V(G)$ can be partitioned into three disjoint subsets S, Q, and C: The subset S contains all vertices of the simplexes of G, and the simplexes of G are vertex disjoint; the subset C consists of the vertices of the basic 5-cycles and the basic 5-cycles form a partition of C; the remaining set Q contains all vertices of degree two of the basic 4-cycles. Let us recall that a unicyclic graph is a connected graph with exactly one cycle.

Theorem 2.2 ([13]). A unicyclic graph G is well covered if and only if G is a member of $\{C_4, C_7\} \cup \mathcal{SQC}$.

For a connected graph G with blocks $\{B_i\}$ and cut vertices (cutpoints) $\{c_j\}$, the block-cutpoint graph of G, denoted by $bc(G)$, is defined as the bipartite graph with the partition sets $\{B_i\}$ and $\{c_j\}$ such that c_j is adjacent with B_i if and only if c_j is in B_i.

Proposition 2.4 ([4]). The block-cutpoint graph is a tree.

3. MAIN RESULTS

Theorem 3.1. If G is a graph of the family $\mathcal{SQC}$, then G is well covered.

Proof. Let I be a maximal independent set of G.
a) It is a simple matter to show that $|I \cap S| = s_G$, where s_G denotes the number of the simplexes of G.
b) For a basic 5-cycle C' of G it is obvious that $1 \leq |I \cap V(C')| \leq 2$. If $I \cap V(C')$ consists of a single vertex v, then there exists a vertex w of C' which is not adjacent to v such that $d(w, G) = 2$. But then $I \cup \{w\}$ is also an independent set of G, a contradiction. So we have $|I \cap V(C')| = 2$ and therefore $|I \cap C| = \frac{2|C|}{5}$.
c) If x and y are the two vertices of degree two of a basic 4-cycle, then it easy to see that $|I \cap \{x, y\}| = 1$, and hence we obtain $|I \cap Q| = \frac{|Q|}{2}$.
Since G is in $\mathcal{SQC}$, it follows together with a), b), and c) for an arbitrary maximal independent set I that

$$|I| = |I \cap V(G)| = |I \cap S| + |I \cap C| + |I \cap Q| = const.$$

Consequently, the graph G is well covered. □

Theorem 3.2. A connected block-cactus graph G is well covered if and only if G is an element of $\{C_4, C_7\} \cup \mathcal{SQC}$.

Proof. If $G \in \{C_4, C_7\} \cup \mathcal{SQC}$, then we are done by Theorem 3.1.

For the converse, assume that G is a well covered connected block-cactus graph. Suppose on the contrary that $G \notin \{C_4, C_7\} \cup \mathcal{SQC}$. In addition, let G be of minimum order with these properties. From Theorem 2.1 we can immediately deduce that G is not a block graph. Thus, G contains a cycle C_p with $p \geq 4$, which is also a block of G. Combining this with Theorem 2.2, we see that there exists a further block of G with at least three vertices. So we obtain our first claim.

Claim 1. The graph G contains at least two blocks with at least three vertices, and one block of G is a cycle of length greater or equal to four.

Therefore, G has at least two end blocks. Now we shall prove the next claim.

Claim 2. Every end block of G is a K_2, and thus a simplex of order two.
Suppose that there exists an end block B of G with $V(B) = \{x_1, x_2, ..., x_p\}$ and $p \geq 3$. Let without loss of generality x_1 be the unique cut vertex of B.
Case 1. Let $B = K_p$.
First, we shall show that the graph $G - x_3$ is also well covered. If not, then there exist two maximal independent sets I^* and J^* in $G - x_3$ with $|I^*| < |J^*|$. Since G is well covered, it follows that $I = I^* \cup \{x_3\}$ is an independent set in G, and therefore $I^* \cap V(B) = \emptyset$. Consequently, $I^* \cup \{x_2\}$ is an independent set in $G - x_3$, a contradiction.
Since $G - x_3$ is a connected well covered block-cactus graph with fewer vertices than G, it is an element of $\mathcal{SQC}$. But now it is a simple matter to verify that G is also in $\mathcal{SQC}$ and hence, Case 1 is not possible.
Case 2. Let $B = C_4 = x_1x_2x_3x_4x_1$.
If the vertex $x_0 \notin V(B)$ is adjacent to x_1, then the independent sets $\{x_0, x_2, x_4\}$ and $\{x_0, x_3\}$ of G have the property $N[\{x_0, x_2, x_4\}, G] = N[\{x_0, x_3\}, G]$. According to Proposition 2.2, the graph G is not well covered, a contradiction.
Case 3. Let $B = C_5 = x_1x_2x_3x_4x_5x_1$.
After Proposition 2.1, the graph $G - N[x_4, G]$ is well covered, and hence it is a member of $\mathcal{SQC}$. Now it is straightforward to show that G is also in the family $\mathcal{SQC}$, but this contradicts our assumption.
Case 4. Let $B = C_6 = x_1x_2x_3x_4x_5x_6x_1$.
According to Proposition 2.1, the graph $G - N[x_4, G]$ is well covered. But x_1 is a common vertex of two simplexes of $G - N[x_4, G]$, a contradiction to Proposition 2.3.
Case 5. Let $B = C_p = x_1x_2...x_px_1$ with $p \geq 7$.
It follows again from Proposition 2.1 that the graph $H = G - N[x_{p-1}, G]$ is well covered. The two independent sets $X = \{x_{p-3}, x_{p-5}, x_{p-7}\}$ and $Y = \{x_{p-4}, x_{p-7}\}$ (for $p = 7$, the vertex x_0 is an adjacent vertex of x_1 with $x_0 \notin V(B)$) of H have the property $N[X, H] = N[Y, H]$. This is a contradiction to Proposition 2.2, and therefore Claim 2 is proved.

Now we choose a longest path P in the block-cutpoint tree $bc(G)$. If A_1 and A_2 are the end vertices of P, then A_1 and A_2 are end blocks of G. Claim 2 implies that A_1 and A_2 are simplexes of order two. Let s_i be the simplicial and x_i the non simplicial vertex of A_i for $i = 1, 2$. It is clear that $x_1 \neq x_2$. In addition, let B_i be the block on the path P such that $x_i \in V(B_i)$ for $i = 1, 2$.

Claim 3. The cut vertex x_i belongs only to the blocks A_i and B_i for $i = 1, 2$.
Suppose that there exists a further block D of G with $x_i \in V(D)$. In view of Claim 2 and Proposition 2.3, D is not an end block of G. But then we obtain a contradiction to the fact that P is a longest path in the block-cutpoint tree $bc(G)$.

Claim 4. The blocks B_1 and B_2 are different.
If $B_1 = B_2$, then the path P has length 4. From Claim 1, Claim 2, and Claim 3 we deduce that there exists a block $B \neq A_1, A_2, B_1$ which is not an end block such that $x_1, x_2 \notin V(B)$. Let P_1 be a shortest path in the block-cutpoint tree $bc(G)$ between the vertices A_1 and B. It is easy to see that the length of P_1 is at least 4. Since B is not an end block, there is a longer path than P in the graph $bc(G)$, but this is impossible.

Claim 5. B_i has exactly one cut vertex y_i of G that belongs to no end block of G for $i = 1, 2$.
Since B_1 and B_2 are different, we find the desired cut vertex y_i on our longest path P. If we suppose that there exists a further cut vertex in B_i which belongs to no end block, then we obtain a contradiction to the fact that P is a longest path in $bc(G)$.

In the following we need the induced subgraph $G_i = G - N[s_i, G]$ which is according to Proposition 2.1 also well covered for $i = 1, 2$. From Claim 1 and Claim 2 we see that G_i is connected and not a cycle. Hence, G_i is an element of $\mathcal{SQC}$. By S_i, Q_i, and C_i we denote the decomposition of $V(G_i)$ in the sense of the definition above.

Claim 6. The block B_i is neither a simplex nor a basic 4-cycle nor a basic 5-cycle of G for $i = 1, 2$.
We prove Claim 6 for $i = 1$. Since A_1 is simplex, it follows from Proposition 2.3 that B_1 is not a simplex.
Suppose that B_1 is a basic 4-cycle of G with the two vertices v_1 and v_2 of degree two. By the definition of a basic 4-cycle, the vertex y_1 of B_1 (see Claim 5) belongs to a simplex or to a basic 5-cycle. Since v_1 and v_2 induce a simplex of order 2 in G_1, the decomposition

$$S' = (S_1 - \{v_1, v_2\}) \cup \{s_1, x_1\}, \quad Q' = Q_1 \cup \{v_1, v_2\}, \quad C' = C_1$$

of $V(G)$ shows that G is in $\mathcal{SQC}$, a contradiction. Consequently, the block B_1 is not a basic 4-cycle.
Suppose that B_1 is a basic 5-cycle of G. From $d(x_1, G) \geq 3$ and $d(y_1, G) \geq 3$ we deduce that x_1 and y_1 are not adjacent. If $w \neq x_1$ is the second vertex of B_1 which is not adjacent to y_1, then we conclude $N[\{s_1, y_1, w\}, G] = N[\{y_1, x_1\}, G]$ for the independent sets $\{s_1, y_1, w\}$ and $\{y_1, x_1\}$ of G, a contradiction to Proposition 2.2. This completes the proof of Claim 6.

Claim 7. The block B_i is not a clique of G for $i = 1, 2$.
We prove Claim 7 for $i = 1$. Suppose that $B_1 = K_p$. Using Claim 6, we see that every vertex of B_1 is a cut vertex of G.

If $p \geq 3$, then it is straightforward to verify that $S' = S_1 \cup \{x_1, s_1\}$, $Q' = Q_1$, and $C' = C_1$ is a decomposition of $V(G)$, a contradiction to the assumption that G is not in $\mathcal{SQC}$.

Now we investigate the more difficult case $p = 2$. If $y_1 \in V(B_1)$ is either a non-simplicial vertex of a simplex of G or a vertex of a basic 5-cycle of G, then the decomposition $S' = S_1 \cup \{x_1, s_1\}$, $Q' = Q_1$, and $C' = C_1$ of $V(G)$ yields a contradiction to the assumption that G is not a member of $\mathcal{SQC}$. Therefore, it remains to discuss the following three possible cases:

Case 1. The vertex y_1 is the only simplicial vertex of a simplex B of G_1.

Case 2. The vertex y_1 is a vertex of a basic 4-cycle B of G_1 with $d(y_1, G_1) = 2$.

Case 3. The vertex y_1 is a vertex of a basic 5-cycle B of G_1 with $d(y_1, G_1) = 2$ such that at least one neighbour, say v, of y_1 fulfills the condition $d(v, G_1) \geq 3$.

In Case 1, we choose for every vertex of $B - y_1$ exactly one neighbour in G_1 which is not contained in B. If we denote this vertex set by J, then it is obvious that J is an independent set. Furthermore, we observe that $N[J \cup \{y_1, s_1\}, G] = N[J \cup \{x_1\}, G]$. Since the two sets $J \cup \{y_1, s_1\}$ and $J \cup \{x_1\}$ are independent in G, we obtain by Proposition 2.2 a contradiction to the well coveredness of G.

In Case 2, let $w \in V(B)$ be the non adjacent vertex of y_1. Then the two independent sets $\{w, y_1, s_1\}$ and $\{w, x_1\}$ yield analogously to Case 1 a contradiction.

In Case 3, let $a \in V(B)$ be the vertex which is adjacent to neither v nor y_1. In addition, let $b \notin V(B)$ be adjacent to v. Now, analogously to the cases above, we consider the independent sets $\{b, a, y_1, s_1\}$ and $\{b, a, x_1\}$ of G, and we see that Case 3 is also not possible.

Since G is a block-cactus graph, it follows from Claim 7 that the blocks B_1 and B_2 are cycles of length at least four. Finally, we consider two cases.

Case 1. All vertices of the cycle B_1 or B_2 are cut vertices of G.

Using Claim 2, Claim 5, and Proposition 2.3, we see that every vertex $x \in (V(B_i) - y_i)$ is contained in exactly one end block of order 2, and in no further block. Now it is straightforward to verify that G is in the family $\mathcal{SQC}$, a contradiction.

Case 2. Both of the blocks B_1 and B_2 contain at least one vertex, say u_1 and u_2, that are not cut vertices of G.

From now on let $i = 1, 2$. The cycle B_i belongs also to G_{3-i} and $d(u_i, G_{3-i}) = 2$. Since G_{3-i} is an element of $\mathcal{SQC}$, we deduce that $u_i \in S_{3-i}$, $u_i \in C_{3-i}$, or $u_i \in Q_{3-i}$. It is obvious that $u_i \in S_{3-i}$ is impossible. But if B_i is a basic 5-cycle of G_{3-i}, then $V(A_i) \cap V(B_i) \neq \emptyset$, a contradiction, because A_i is simplex of G_{3-i}. Hence, B_i is a basic 4-cycle of G_{3-i}. If $V(B_1) \cap V(B_2) = \emptyset$, then it is immediate that B_1 and B_2 are also basic 4-cycles of G. Otherwise, we have $y_1 = y_2$, and y_1 belongs to a further block D that is a simplex or a basic 5-cycle of G_1 and G_2. Thus, the block D is also contained in G, and therefore B_1 and B_2 are again basic 4-cycles in G. This is a contradiction to Claim 6, and the proof of Theorem 3.2 is complete. □

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