Series and infinite products related to binary expansion of integers

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Abstract

Various problems related to the binary expansion of integers are investigated. The Thue-Morse sequence, which gives the parity of the number of 1's in the binary expansion of an integer, appears in several identities and constants. We illustrate techniques, based on Dirichlet series or elementary methods, used to obtain such formulae.

The starting point is the sum $s_2(n)$ of the bits of the binary representation of n and the Thue-Morse sequence, $\epsilon(n) = (-1)^{s_2(n)}$. The next three relations will be our leading thread;

$$\sum_{n=1}^{+\infty} \frac{s_2(n)}{n(n+1)} = 2 \ln 2, \tag{1}$$

$$\prod_{n=0}^{+\infty} \left(\frac{2n+1}{2n+2}\right)^{\epsilon(n)} = \frac{\sqrt{2}}{2}, \tag{2}$$

$$\sum_{k=0}^{n} \epsilon(3k) > 0. \tag{3}$$

All these examples relate to the domain of automatic sequences whose prototype is the Thue-Morse sequence. Such sequences are at the frontier between the theory of formal languages and the theory of numbers. Their definitions are often simple but the techniques often involve Dirichlet series and asymptotic analysis.

1. Series

Formula 1 is excerpted from the Putnam competition (1981) and was generalized by Shallit [12] into

$$\sum_{n=1}^{+\infty} \frac{s_B(n)}{n(n+1)} = \frac{B}{B-1} \ln B, \tag{4}$$

where $s_B(n)$ is the sum of the digits of n for radix B. We show an example which extends this result. We take the sequence $u(n)$ which counts the number of blocks 11 in the binary expansion of n. For instance, since $14 = \overline{1110}$, then $u(14) = 2$. Introducing an appropriate function $f(x)$, which will be defined later, we consider the two series

$$\sum_{n=0}^{+\infty} u(n)\, f(2n+1), \qquad \sum_{n=0}^{+\infty} u(2n+1)\, f(2n+1).$$

The only reasonable idea we can make use of is to split these series according to the parity of the integer n:

$$\sum_{n=0}^{+\infty} u(n)\, f(2n+1) = \sum_{n=0}^{+\infty} u(2n)\, f(4n+1) + \sum_{n=0}^{+\infty} u(2n+1)\, f(4n+2),$$

$$\sum_{n=0}^{+\infty} u(2n+1)\, f(2n+1) = \sum_{n=0}^{+\infty} u(4n+1)\, f(4n+1) + \sum_{n=0}^{+\infty} u(4n+3)\, f(4n+3).$$

But the sequence $u(n)$ satisfies the relations

$$u(n) = u(2n) = u(4n+1),$$
$$u(4n+3) = u(2n+1) + 1.$$

By subtraction, we obtain

$$\sum_{n=0}^{+\infty} u(n)\, f(2n+1) - \sum_{n=0}^{+\infty} u(2n+1)\, f(2n+1) = \sum_{n=0}^{+\infty} f(4n+3)$$

and eventually

$$\sum_{n=0}^{+\infty} u(n)\, [f(n) - f(2n) - f(2n+1)] = \sum_{n=0}^{+\infty} f(4n+3).$$

We apply this formula with

$$f(n) = \frac{1}{n^s} - \frac{1}{(n+1)^s}, \qquad f(0) = 0,$$

and we get

$$(5) \qquad \left(1 - \frac{1}{2^s}\right) \sum_{n=1}^{+\infty} u(n) \left[\frac{1}{n^s} - \frac{1}{(n+1)^s}\right] = \sum_{n=0}^{+\infty} \left[\frac{1}{(4n+3)^s} - \frac{1}{(4n+4)^s}\right].$$

The behaviour of the Riemann and Hurwitz zeta functions near 1 are known [13, §13.21]:

$$\zeta(s) \underset{s\to 1}{=} \frac{1}{s-1} + \gamma + o(1),$$

$$\zeta(s,a) \underset{s\to 1}{=} \frac{1}{s-1} - \frac{\Gamma'}{\Gamma}(a) + o(1),$$

and the local analysis of the terms in Formula 5 in the neighbourhood of 1 gives

$$\sum_{n=1}^{+\infty} \frac{u(n)}{n(n+1)} = \frac{1}{2}\left(-\frac{\Gamma'}{\Gamma}(3/4) - \gamma\right).$$

According to Gauss [13, §12.16] [8, p. 94], we have

$$\frac{\Gamma'}{\Gamma}(p/q) = -\gamma - \ln(2q) - \frac{\pi}{2}\,\mathrm{cotg}\,\frac{\pi p}{q} + 2 \sum_{0<n<q/2} \cos\frac{2\pi pn}{q}\,\ln\sin\frac{\pi n}{q},$$

hence

$$\sum_{n=1}^{+\infty} \frac{u(n)}{n(n+1)} = \frac{3}{2}\ln 2 - \frac{\pi}{4}.$$

In the same vein, one can prove the formulae [4]

$$\sum_{n=2}^{+\infty} s_2(n)\, \frac{(2n+1)}{n^2(n+1)^2} = \frac{\pi^2}{9},$$

$$\sum_{n=1}^{+\infty} s_2(n)^2 \frac{8n^3+4n^2+n-1}{4n(n^2-1)(4n^2-1)} = \frac{17}{24} + \ln 2,$$

$$\sum_{n=5}^{+\infty} u(n) \frac{8n^3-4n^2-n-9}{(n-3)(n+1)(2n-3)(4n^2-1)} = \frac{95}{168} + \frac{5}{4}\ln 2 - \frac{\pi}{8}.$$

The use of the function

$$f(n) = \frac{1}{n^s} - \frac{1}{(n+1)^s}$$

is more than a trick. The crux of the matter is that $f(x)$ is an eigenvector of operator T,

$$Tg(x) = g(x) - \sum_{0\le j<B} g(B\,x+j).$$

More precisely an eigenvector $g(x)$ such that the series $\sum_n g(n)$ converges and $g(x) \sim x^r$ for x in a neighbourhood of 0 is essentially $f(x)$.

2. Infinite products

Shallit [12] proved Formula 2 by real analysis and Allouche and Cohen [3, 1] gave another proof by means of Dirichlet series. The logarithm of the infinite product is

$$\sum_{n=0}^{+\infty} \epsilon(n) \ln\left(1 - \frac{1}{2\,(n+1)}\right) = -\sum_{k=0}^{+\infty} \frac{1}{k\,2^k} \sum_{n=0}^{+\infty} \frac{\epsilon(n)}{(n+1)^k}.$$

Therefore it is natural to introduce the Dirichlet series

$$f(s) = \sum_{n=0}^{+\infty} \frac{\epsilon(n)}{(n+1)^s}.$$

Splitting the sum according to the parity of n yields

$$\left(1+\frac{1}{2^s}\right) f(s) = \sum_{n=0}^{+\infty} \frac{\epsilon(n)}{2^s(n+1)^s} \left(1 - \frac{1}{2(n+1)}\right)^{-s}$$

and by binomial expansion

$$\left(1+\frac{1}{2^s}\right) f(s) = \sum_{k=0}^{+\infty} \frac{1}{2^{k+s}} \binom{s+k-1}{k} f(s+k).$$

The infinite functional equation

$$f(s) = \sum_{k=1}^{+\infty} \frac{1}{2^{k+s}} \binom{s+k-1}{k} f(s+k),$$

permits us to extend function $f(s)$ throughout the entire complex plane. Moreover

$$f'(0) = \sum_{k=1}^{+\infty} \frac{1}{k\,2^k} f(k)$$

and that is the quantity we are looking for. The function

$$g(s) = \sum_{n=1}^{+\infty} \frac{\epsilon(n)}{n^s}$$

satisfies

$$\left(1-\frac{1}{2^s}\right) g(s) = \left(1+\frac{1}{2^s}\right) f(s)$$

and can be extended in the same manner as $f(s)$. The only difference lies in the fact that $g(0) = -1$ and this gives

$$f'(0) = -1,$$

hence Formula 2.

Another method to obtain Formula 2 is to apply a greedy algorithm which computes sequentially the exponents $\alpha(n) = \pm 1$ such that

$$\left(\frac{1}{2}\right)^{+1} \left(\frac{3}{4}\right)^{-1} \left(\frac{5}{6}\right)^{+1} \left(\frac{7}{8}\right)^{+1} \cdots \left(\frac{2\,n+1}{2\,n+2}\right)^{\alpha(n)} \cdots = \frac{\sqrt{2}}{2}.$$

The partial products approach the limit alternating from below or above, as do the partial sums of the series

$$\sum_{n=0}^{+\infty} \epsilon(n) \ln \frac{2\,n+1}{2\,n+2}.$$

As a result the two sequences α and ϵ are equal.

More generally, for each pattern w there exists a rational function $g(n)$ such that

$$\prod_{n \geq 0} g(n)^{\beta(n)} = \frac{\sqrt{2}}{2},$$

where $\beta(n)$ equals $+1$ or -1 according to the parity of the number of w in the binary expansion of n. For example,

$$\prod_{n \geq 0} \left(\frac{(2\,n+1)^2}{(n+1)(4\,n+1)}\right)^{\beta(n)} = \frac{\sqrt{2}}{2},$$

if $\beta(n) = (-1)^{u(n)}$ is the Rudin-Shapiro sequence associated to pattern $w = 11$.

In the same vein, the Flajolet product [6]

$$\prod_{n \geq 0} \left(\frac{2\,n}{2\,n+1}\right)^{\epsilon(n)}$$

is not known. We point out that the rational fraction $2\,n/(2\,n+1)$ is related to pattern $w = 01$.

3. Newman-Coquet sequence

This is our last example. The first few terms of the Thue-Morse sequence $\epsilon(n)$ are (with $+ \equiv +1$ and $- \equiv -1$)

$$+ - - + - + + - - + + - + - - + - + + - + - - + + -$$

and the first few terms of the subsequence $\epsilon(3\,n)$, the Newman-Coquet sequence, are

$$+ + + + + + + - + + + + + + - + + + + + + + + + - + + .$$

The abundance of $+$ is startling [10] and it is natural to wonder about the sign of

$$s_N = \sum_{n<N} \epsilon(3\,n).$$

In fact there are asymptotically as many minuses as there are pluses and s_N/N is $o(N)$. Newman and Slater [11] have showed that s_N is of the same order of magnitude as N^α, with $\alpha = \log 3/\log 4$, though not admitting an asymptotic equivalent $C\,N^\alpha$. Next Coquet [5, 7] proved that

$$\sum_{n<x} \epsilon(3\,n) \underset{x\to\infty}{=} x^\alpha\, F\left(\frac{\log x}{\log 4}\right) + O(1),$$

where F is 1-periodic, continuous but nowhere differentiable.

By a greedy algorithm, every sequence of ± 1 can be written as an infinite product

$$\prod_w b_w(n),$$

where $b_w(n)$ is $+1$ or -1 according to the parity of the number of w in the binary expansion of n [9]. For the Coquet sequence, one obtains [2]

$$\epsilon(3\,n) = b_{111}(n) b_{11011}(n) \cdots b_{11(01)^i1}(n) \cdots .$$

The rarity of the blocks $11(01)^i1$ explains the excess of $+$ in the sequence.

Bibliography

[1] Allouche (J.-P.), Cohen (H.), France (M. Mendès), and Shallit (J. O.). – De nouveaux curieux produits infinis. *Acta Arithmetica*, vol. XLIX, 1987, pp. 141–153.

[2] Allouche (J.-P.), Morton (P.), and Shallit (J.). – Pattern spectra, substring enumeration, and automatic sequences. *Theoretical Computer Science*, vol. 94, 1992, pp. 161–174.

[3] Allouche (Jean-Paul) and Cohen (Henri). – Dirichlet series and curious infinite products. *Bulletin of the London Mathematical Society*, vol. 17, 1985, pp. 531–538.

[4] Allouche (Jean-Paul) and Shallit (Jeffrey). – The ring of k-regular sequences. *Theoretical Computer Science*, vol. 98, 1992, pp. 163–197.

[5] Coquet (J.). – A summation formula related to binary digits. *Inventiones Mathematicae*, vol. 73, 1983, pp. 107–115.

[6] Flajolet (P.) and Martin (G. N.). – Probabilistic counting algorithms for data base applications. *Journal of Computer and System Sciences*, vol. 31, n° 2, October 1985, pp. 182–209.

[7] Flajolet (Philippe), Grabner (Peter), Kirschenhofer (Peter), Prodinger (Helmut), and Tichy (Robert). – Mellin transforms and asymptotics: Digital sums. *Theoretical Computer Science*, 1993. – To appear.

[8] Knuth (Donald E.). – *The Art of Computer Programming*. – Addison-Wesley, 1968, volume 1: Fundamental Algorithms. Second edition, 1973.

[9] Morton (P.) and Mourant (W.). – Paper folding, digit patterns, and groups of arithmetic fractals. *Proceedings of the London Mathematical Society*, vol. 59, 1989, pp. 253–293.

[10] Newman (Donald J.). – On the number of binary digits in a multiple of three. *Proceedings of the American Mathematical Society*, vol. 21, 1969, pp. 719–721.

[11] Newmann (D. J.) and Slater (M.). – Binary digit distribution over naturally defined sequences. *Transactions of the American Mathematical Society*, vol. 213, 1975, pp. 71–78.

[12] Shallit (J. O.). – On infinite products associated with sums of digits. *Journal of Number Theory*, vol. 21, 1985, pp. 128–134.

[13] Whittaker (E. T.) and Watson (G. N.). – *A Course of Modern Analysis*. – Cambridge University Press, 1927, fourth edition. Reprinted 1973.