Chapter 5

FROM THE GILBREATH PRINCIPLE TO THE MANDELBROT SET

One of the great new discoveries of modern card magic is called the Gilbreath Principle. It is a new invariant that lets the spectator shuffle a normal deck of cards and still concludes in a grand display of structure.

One of the great new discoveries of modern mathematics is called the Mandelbrot set. It's a new invariant that takes a "shuffle" of the plane and still concludes in a grand display of structure.

The above is wordplay; the connections between the invariants of a random riffle shuffle and the universal structure in the Mandelbrot set lie far below the surface. We'll only get there at the end. This chapter gives some very good card tricks and explains them using our new "ultimate" Gilbreath Principle. Later in this chapter, the Mandelbrot set is introduced. This involves pretty pictures and some even more dazzling universal properties that say that the pretty pictures are hidden in virtually every dynamical system. We'll bet you can't yet see any connection between the two parts of our story.

Right now, let's begin with our tricks.

THE GILBREATH PRINCIPLE

To try the Gilbreath Principle, go and get a normal deck of cards. Turn them face-up and arrange them so that the colors alternate red, black, red, black, and so on, from top to bottom. The suits and values of the cards don't matter, just the colors. With this preparation, you're ready

Figure 1. A riffle shuffle

to fool yourself. Give the deck a complete cut, any place you like. Hold the deck face-down as if your are about to deal cards in a card game. Deal about half the deck face-down into a pile on the table. The actual number dealt doesn't matter; it's a free choice. You now have two piles, one on the table, one in your hands. Riffle shuffle these two piles together. Most people know how to shuffle (see figure 1). Again, the shuffle doesn't have to be carefully done. Just shuffle the cards as you normally do, and push the packets together.

Here comes the finale: Pick the deck up into dealing position, and deal off the top two cards. They will definitely be one red/one black. Of course, this isn't so surprising since it happens half the time in a well-shuffled deck. Deal off the next two cards. Again, one red/one black. Keep going. You'll find each consecutive pair alternates in color. In a well-shuffled deck, one might naively expect that this would happen about $\overbrace{1/2 \times 1/2 \times \cdots \times 1/2}^{26\text{ terms}}$ of the time (which is less than two chances in a hundred million). In fact, the odds are actually somewhat better than that, namely, about one chance in seven million. We will explain how we arrive at this number at the end of this chapter.

Before proceeding, you might want to figure out how it works. It's pretty easy to see that no matter how the cards are cut, dealt, and shuffled, it's a sure thing that the top two cards are one of each color. When we try this out on our students, it's quite rare for anyone to be able to see why the next pair is red/black. We don't recall a single student providing a full, clear argument for the whole story.

What is described above is called Gilbreath's First Principle. It was discovered by Californian Norman Gilbreath, a mathematician and lifelong magician, in the early 1950s. We'll have more to say about

Gilbreath at the end of this chapter. The second and ultimate principles are coming.

The red/black trick can be performed as just described. Thus, set up a deck of cards alternating red/black and put them in the card case. Find a spectator and proceed as directed above. Take the cards out of the case. Ask the spectator to give the deck a few straight cuts, deal off any number of cards into a pile on the table, and then riffle shuffle the pile on the table with the pile still in hand. As this happens, you might appear to be carefully studying the spectator (you might even fake making a few notes on a pad). You can promise that “This is an ordinary deck, not prepared in any way.” Take the deck of cards and put them under the table. Say that you’re going to try to separate the cards by sense of touch: “I promise I won’t look at the cards in any way. You know, red ink and black ink are made up of quite different stuff. It used to be that red ink had nitroglycerine in it. Guys in prison used to scrape if off the cards. Anyway, I’ll try to feel the difference between red and black and pair them up.”

All you do is take the cards off the top in pairs. Pretend to feel carefully and perhaps occasionally say, “I’m not sure about these,” and so on. If, as you display the pairs, you order them so each has a red on top followed by a black (you’ll find they appear to come out in random order), assemble the pairs in order and you’re ready to repeat the procedure instantly.

To be honest, the trick as just described is only “okay.” It’s a bit too close to the surface for our tastes. Over the years, magicians have introduced many extensions and variations to build it up into something terrific. As an example, we’ll now describe a fairly elaborate presentation developed by the magician and insurance executive Paul Curry. The following unpublished creation has many lessons.

The performer gets a spectator to stand up and asks two quite personal questions: “Are you good at telling if someone else is lying? If you had to, do you think you could lie so that we couldn’t tell?” It’s a curious asymmetry of human nature that a large number of people answer yes to both questions (we owe this observation to Amos Tversky).

The props for this effect are a deck of cards and a personal computer (used as a score keeper). The performer asks the spectator to cut and shuffle the cards. Then two piles of ten or so cards are dealt off. The performer takes one of them and calls out the colors for each card, red or black. The spectator’s job is to guess when the performer lies. This is carried through, one card at a time. After each time, the performer shows the actual card and enters a score of correct or incorrect. This is continued for ten or so steps. At the end, the computer gives a tally of, say, "Seven correct out of ten, above average."

Now the tone changes. It is the spectator's turn to lie or tell the truth. What's more, it won't be the performer who guesses "lie or truth." The computer will act as a lie detector. The spectator looks at the top card of the pack and decides (mentally) whether to lie or not. Depending on which decision he or she makes, the spectator taps the "R" or "B" key on the computer to indicate red or black. The computer responds, accurately determining if a lie is told. The messages vary from time to time but the computer is always right. This has an eerie effect, quite out of proportion to the trick's humble means.

How does it work? The deck is set up initially with red/black alternating throughout. The spectator cuts the deck several times and deals some cards into a single pile on the table. The performer might patter about poker and bluffing or lie detector machines. The two piles are riffle shuffled together by the spectator, who then deals them into two piles, alternating left, right, left, right, and so on, until ten or so cards are in each pile. The spectator hands either pile to the performer.

Here is the key to the trick. Because of the Gilbreath Principle, each consecutive pair of cards contains one red and one black after the riffle shuffle. Dealing alternately into two piles ensures that the cards are of opposite colors in the two piles as we work from top to bottom. Thus, if the top twenty cards of the deck after shuffling are RBBRBRRBRB-BRRBBRRBBR, then after dealing into two alternate piles, we have

R	B
B	R
B	R
R	B
R	B
B	R
R	B
B	R
R	B
B	R

If the top of the left-hand pile is red, the top of the right-hand pile is black. The same holds for the second cards, and so on. The spectator hands either packet to the performer who looks at the cards, calls out colors, and lies or not each time. There is no preset pattern. Just do as you please, using funny tones of voice and making faces if that's your style. The spectator guesses "lie or truth," the performer shows the card, and enters "C" or "W" each time, depending on whether the guess is correct or wrong.

The second secret lies here. After each guess is entered on the computer, the performer taps the space bar if the actual card in question is red, and does not tap if the actual card is black. This variation goes unnoticed amidst all the banter, and it tells the computer the actual colors of the cards in the performer's pile. By taking opposites, the computer now knows the actual color of each card in the spectator's pile. When the spectator goes through his or her pile (and whatever complex thought processes are required), he or she finally presses the "R" or "B" key. The computer compares each of the spectator's entries with the known color and determines if a lie has been told.

It will help the presentation if a separate set of messages is preprogrammed for each card. Thus, the computer might announce, "You lie" or "Tsk-tsk—don't try that again" for lies, or "You're trying to trick me—you told the truth" when the spectator isn't lying. This takes a modest amount of preparation but is worth the effort.

When Paul Curry first performed this for us, personal computers and programmable calculators were far in the future. He hand-built a complicated gadget with displays, wires, and switches all over it to carry out this simple task. He later published a pencil and paper version of the trick in his wonderful book *Paul Curry Presents.* Because this loses the wonderful effect of the computer as lie detector, it is not as good as the version above.

We will not give programming details here. If you know a bit about programming, it's an hour's work (oh, all right, a few hours' work). If you don't, go find a teenager. The Curry trick is a great example of how thought and presentation can turn a humble mathematical trick into great theater. Curry also invented perhaps the greatest red/black trick of all time: Out of This World. We can't explain it here but it is definitely worth hunting down.

So far we have explained Gilbreath's First Principle. In 1966, Gilbreath stunned the magical world by introducing a sweeping generalization, known as Gilbreath's Second Principle. In the first principle, alternating red/black patterns are used. Gilbreath discovered that *any* repeated pattern can be used. For example (go get a deck of cards), arrange a normal deck so that the suits rotate: clubs, hearts, spades, diamonds, clubs, hearts, spades, diamonds, and so on. Give the deck a random cut, deal any number of cards onto the table face-down in a pile (reversing their order), and riffle shuffle the two piles together. The top four cards will consist of one of each suit, no repeats, the next four cards will have one of each suit, and so on through to the bottom four cards in the deck.

Here is a simple variation. Remove all four aces, twos, threes, fours, and fives from the deck (twenty cards in all). Arrange them in rotation:

1, 2, 3, 4, 5, 1, 2 , 3, 4, 5, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5.

Cut this packet randomly, deal any number of cards face-down into a pile on the table, and riffle shuffle this pile with the rest of the cards from the packet. The top five cards will be {1, 2, 3, 4, 5} in some order, the same for the next five, the next five, and the last five cards. Dozens of tricks have been invented using these ideas. Often this principle is combined with some sleight of hand, making the trick unsuitable for this book. We have cleaned up one of them to make quite a performable trick. Those knowing a bit of sleight of hand will be able to dress it more handsomely (see chapter 11 if you want to learn more).

The following has served us well. It uses Gilbreath's Second Principle together with ideas from Ronald Wohl and Herbert Zarrow.

The rough effect is this: The performer asks if someone would like a lesson in cheating at cards. "A key to making big money is that you must learn to deal someone a good hand but also deal a better hand to yourself (or your partner)." With these preliminaries, the spectator cuts, shuffles, and deals the cards (with a little help and kidding from the performer). The spectator deals a pat poker hand—a high straight (ace, king, queen, jack, ten), to one player and a better hand, a flush (all five cards of the same suit), to himself. At the end, the spectator is as mystified as everyone else. The performer cautions that the new skills are to be used for entertainment purposes only.

To perform this trick, the top twenty-five cards of the deck must be prearranged. Remove any ten spades and three each of aces, kings, queens, jacks, and tens (of any suits). These are arranged with five spades on the top, five spades on the bottom, and the middle fifteen cards in the rotation ace, king, queen, jack, ten, and so on, that is, as:

S S S S S A K Q J 10 A K Q J 10 A K Q J 10 S S S S S.

Put these twenty-five cards on top of the rest of the cards and put the cards in the card case.

Ask for a volunteer who wants to learn about cheating at cards. This may involve some funny interactions with the audience. Ask the volunteer (let's call her Susan) if she knows how to play poker—with all the poker on TV, many people do. But still, many people don't. Take out the deck, turn it face-up and display a few poker hands, explaining one pair, two pairs, three of a kind, straight, and flush. Do this without disturbing the original top twenty-five cards. Saying you'll start easy, break the deck (it's still face-up) at the run of five middle spades, so you have only the original top twenty-five in hand. Turn these face-down and hand them to Susan. Say you're going to get an idea of her dealing skills—ask her to deal any number of cards into a single pile on the table. The actual number doesn't matter but it must be five or more, and at most twenty. Now ask if she can shuffle, and have her riffle shuffle the dealt pile with the rest of the twenty-five cards. Tell her that poker is played by dealing around—have her deal five hands as in a normal poker game. Comment on her technique. Have her look at one of the hands (turn it face-up, without changing the order, and comment on its value). Now have her assemble the five hands in any order, keeping the packets of five together.

Say, "That was practice; here comes the real thing. There is a high roller in second position and your partner is playing the first hand. I'd like you to use your skills to deal a pat hand to the second player but make sure you give your partner a better hand." Susan may look at you as if you are out of your mind. Anyway, have her deal five hands in the normal fashion. Turn up the cards in second position one at a time. They will form an ace-high straight—shake her hand, and act as if the trick is over. "Susan, you're really talented." Then remember, "Wait. An ace-high straight is almost impossible to beat. The odds of getting an ace-high straight are about one in twenty-five hundred. What about your partner?" Slowly turn over the cards in the first player's hand one at a time. They will form a flush in spades, handily beating the ace-high straight. Offer her your hand again with the comment, "Susan, you're a poker genius."

That's a lot of dressing but it makes for a very entertaining few minutes. For you, our reader, understanding how it all comes together is a nice lesson in the beginnings of combinatorics. A fancier version of this trick involving some sleight of hand appears as U-shuffle Poker in *Zarrow, A Lifetime of Magic* by David Ben.[2]

THE ULTIMATE GILBREATH PRINCIPLE

Up to now, we have seen two grand applications of Gilbreath's two principles, utilizing reds and blacks as well as rotating sequences. It is natural to ask what other properties or arrangements are preserved by our riffle shuffle. This is actually a hard, abstract math question. What do we mean by "property or arrangement" and "preserved"? After all, if a deck of cards labelled {1, 2, 3, . . . , 52} is shuffled in any way, it still contains all these numbers only once. Clearly, this doesn't count. Is looking at every other card allowed?

Let us start by carefully defining what we mean by "shuffle." Consider a deck of N cards labeled 1, 2, 3, . . . , N. A normal deck has $N = 52$. The deck starts out in order, with card 1 on top, card 2 next, and card N on the bottom. By a *Gilbreath shuffle* we mean the following permutation. Fix a number between 1 and N, call it j. Deal the top j cards into a pile face-down on the table, reversing their order. Now, riffle shuffle the j cards with the remaining $N - j$ cards. For example, if $N = 10$ and $j = 4$, the shuffle might result in:

1					4
2					5
3		5			6
4		6	4		3
5	→	7	3	→	7
6		8	2		2
7		9	1		8
8		10			9
9					1
10					10

What we want to understand is just what arrangements are possible after one Gilbreath shuffle? Two answers will be given. First, we will count how many different arrangements are possible. Second, we will give a simple description of the possible arrangements, which we modestly call the Ultimate Gilbreath Principle.

Counting. The number of different permutations of N cards is $N \times (N-1) \times (N-2) \times \cdots \times 2 \times 1 = N!$ (read "N factorial"). These numbers grow very rapidly with N. For example, if $N = 10$ then $N! = 3{,}628{,}800$, more than three and a half million. When $N = 60$, $N!$ is larger than the number of atoms in the universe. Put another way, $60! \approx 8.32 \times 10^{81}$, while the estimated number of atoms in the universe (according to the current theories) is less than 10^{81}.

Of course, after one Gilbreath shuffle, not all arrangements are possible. At the end of this chapter we show that, with a deck of N cards, only 2^{N-1} arrangements can occur. When $N = 10$, $2^{N-1} = 512$. When $N = 52$, $2^{51} \approx 2.25 \times 10^{15}$. This is still a large number (which makes the tricks confusing and interesting). As an example, the reader may check that, with four cards, the eight possible Gilbreath arrangements are:

1	2	2	2	3	3	3	4
2	1	3	3	2	2	4	3
3	3	1	4	1	4	2	2
4	4	4	1	4	1	1	1.

As an aside, we first did the examples for decks of sizes $N = 1, 2, 3, 4$. By enumerating all possibilities by hand, we saw the answer 2^{N-1}. If this is the right answer, it is so neat that there must be an easy proof. Notice that having a neat count is different from having a neat description. We give our descriptions next, followed by an appeal for help in inventing a good trick. The proofs are given at the end of the chapter.

THE ULTIMATE INVARIANT(S)

To describe the results, we need some way of writing things down. For a deck of cards originally in order 1, 2, 3, . . . , N, record a new order (we call it π) by letting $\pi(1)$ be the card at position 1, $\pi(2)$ be the card at position 2, . . . , and $\pi(N)$ be the card at position N. Thus, if the new order of a five-card deck is 3, 5, 1, 2, and 4, then $\pi(1) = 3$, $\pi(2) = 5$, $\pi(3) = 1$, $\pi(4) = 2$ and $\pi(5) = 4$. This may seem like a complex way to talk about something simple, but we can't proceed without it. Thus, we can now say that "π is a Gilbreath permutation" is shorthand for "A deck of N cards starting in order 1, 2, 3, . . . , N is in final order $\pi(1)$, $\pi(2)$, . . . , $\pi(N)$ after one Gilbreath shuffle."

The final thing we need is the notion of the remainder modulo j. If we take a fixed number j (e.g., $j = 3$), then any number (e.g., 17) has some remainder when divided by j. For example, 17 has remainder 2 when divided by 3. In this case, we say 17 is 2 modulo 3. A set of numbers are *distinct modulo j* if their remainders are distinct. Thus, 12 and 17 have remainders 0 and 2 and so are distinct modulo 3, whereas

14 and 17 are not, since 14 and 17 both have the same remainder 2 modulo 3. With all these prerequisites, here is our main result. The abstract-looking statement is followed by some very concrete examples. The proof is given below.

THEOREM. The Ultimate Gilbreath Principle. For a permutation π of $\{1, 2, 3, \ldots, N\}$, the following four properties are equivalent:

1. π is a Gilbreath permutation.
2. For each j, the top j cards $\{\pi(1), \pi(2), \pi(3), \ldots, \pi(j)\}$ are distinct modulo j.
3. For each j and k with $kj \leq N$, the j cards $\{\pi((k-1)j+1), \pi((k-1)j+2), \ldots, \pi(kj)\}$ are distinct modulo j.
4. For each j, the top j cards are consecutive in $1, 2, 3, \ldots, N$.

Here is an example illustrating the theorem. For a ten-card deck, we can deal off four cards into a small pile on the table (one by one) and then riffle shuffle them to lead to the arrangement π below:

4
5
6
3
7
2
8
9
1
10.

Thus, π is a Gilbreath permutation, so it satisfies (1) by definition. The theorem now says that π has many special properties. For example, consider property (2). For each choice of j, the remainders modulo j of the top j cards are distinct. When $j = 2$, the top two cards, 4 and 5, have distinct remainders 0 and 1 modulo 2. When $j = 3$, the top three cards, 4, 5, and 6, are 1, 2, 0 modulo 3. This works for all j up to N, no matter what Gilbreath shuffle is performed.

Property (3) is our refinement of the original general Gilbreath Principle. For example, if $j = 2$, it says that, after any Gilbreath shuffle, each consecutive pair of cards contains one even value and one odd value. If the even cards are red and the odd cards are black in the original arrangement, we have Gilbreath's First Principle. The small refinement is that we do not need to assume that N is divisible by j; the last k cards still have distinct remainders when divided by j, provided $k \leq j$ and the number of cards preceding these cards is a multiple of j.

The final one, property (4), needs some explanation. Consider our Gilbreath permutation π (written sideways to conserve space):

$$4\ 5\ 6\ 3\ 7\ 2\ 8\ 9\ 1\ 10.$$

The top four cards (here 4 5 6 3) were consecutive in the original deck. (They are out of order, but the set of four started out as consecutive). Similarly, for any j the top j cards were consecutive in the original deck for any j.

The point of all of this is that *any one of these* parts gives a complete characterization. For example, if π is a Gilbreath permutation then π satisfies property (3) for all j. Conversely, if π is any permutation satisfying property (3) for all j, then π arises from a Gilbreath shuffle. In one sense, this is a negative result. It says that there are no new hidden invariants—Gilbreath discovered them all. On the other hand, now we know and can stop brooding about this.

Property (2) is our new Ultimate Gilbreath Principle. We haven't seen it elsewhere and it is the key to proving the theorem. What we don't see is any way of making a good trick. In the hope of angering some readers into making progress in this direction, here is an unsuccessful attempt.

You, the performer, show ten cards, each with a unique number, 1, 2, 3, . . . , 10. The patter goes as follows: "Did you ever have to help your kids with their math homework? It's getting pretty complicated. Our kids are doing binary, ternary, and octal arithmetic. They came home with something they call 'modulo j.'" Explain modulo j (as we did before) and then continue, "Their teacher says the following stunt always works." The cards are arranged in order, say, 1, 2, . . . , 10. They can be ordinary playing cards or index cards with bold numbers written on them. Have the spectator cut the packet, deal any number onto the table, and then shuffle the two packets together. Explain: "The top two cards are a full set modulo 2, so one should be even and the other odd. Let's take a look. Now, the top three cards form a complete set modulo 3. Turn over the next card." Explain how it's true: "Let's see—4, 5, 3, well 3 is 0 modulo 3, and 4 is 1 modulo 3 and 5 is 2 modulo 3, so it worked then. Let's see the next card. . . ." This continues for as long as you have the nerve to keep talking.

To be honest again, we haven't had the courage to try this trick out on our friends. It just doesn't seem very good. What's worse, the pattern described in property (4) of the theorem might be obvious. Indeed, this is the way (4) was discovered. We had proved the equivalence of (1), (2), and (3) without knowing (4). When we tried the trick out, we noticed (4). Its discovery makes the proof much easier. Take a look at the proof in the following section.

THE MANDELBROT SET

The Mandelbrot set is one of the most amazing objects of mathematics. Figure 2 shows a picture of the Mandelbrot set. A close look reveals a "leafy" quality on the edge of everything. Consider the bottom region of figure 2. We blow this area up in figure 3. Now, new "leafy" fixtures appear. The bottom region of figure 3 is expanded to reveal the dazzling structure in figure 4. Figures 5 and 6 take closer and closer looks. Each reveals a rich, detailed structure.

There are many computer programs on the Web that allow exploration of the Mandelbrot set.[3] The appearance of refined structure keeps going forever. It has engaged the best minds in mathematics, physics, and biology. Moreover, as explained below, the pattern is "universal." It appears in many other seemingly unrelated systems.

This is a chapter on shuffling cards and the Gilbreath Principle. We hope the reader is as surprised as we were to learn that there is an intimate connection between shuffling cards and the Mandelbrot set. The story is hard to tell, so here is a roadmap to what's coming. We begin with a simple procedure: squaring and adding. This is really all that is needed to define the Mandelbrot set. Next, we determine when repeated squaring and adding leads to a periodic sequence. Card shuffling and the Gilbreath Principle now enter to describe the way the points of this sequence are ordered. All of the activity up to now has taken place with one-dimensional, "ordinary" numbers. The Mandelbrot set lives in *two* dimensions. Only then can the Mandelbrot set be properly defined. At the end, we give a whirlwind tour of the Mandelbrot set, explain its universality, and enter a plea for help in finding two-dimensional shuffles that will explain the last remaining mysteries.

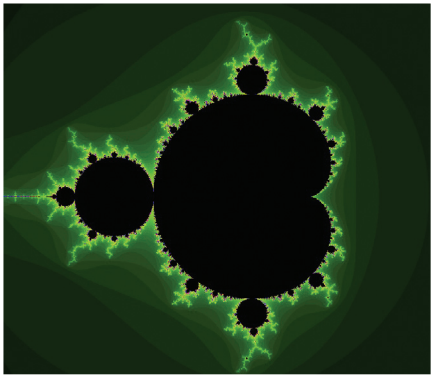

Figure 2. The Mandelbrot set (image created by Paul Neave, neave.com)

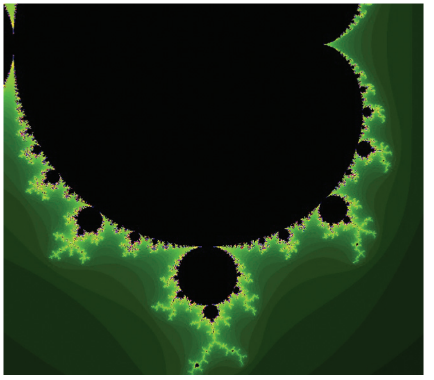

Figure 3. Enlarging part of the Mandelbrot set (image created by Paul Neave, neave.com)

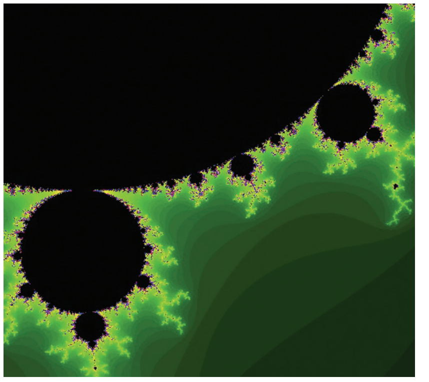

Figure 4. A further enlargement of part of the Mandelbrot set (image created by Paul Neave, neave.com)

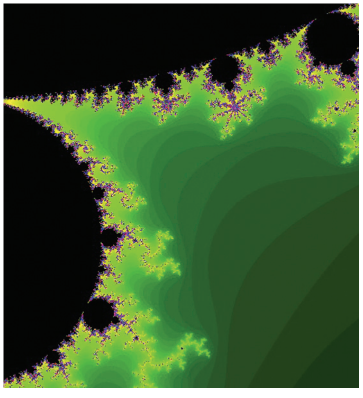

Figure 5. A further enlargement (image created by Paul Neave, neave.com)

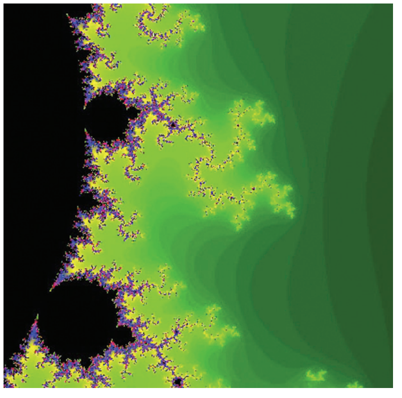

Figure 6. An even further enlargement (image created by Paul Neave, neave.com)

Squaring and Adding. Repeated squaring is a familiar procedure. Starting with 2, we get 2, 4, 16, 256, 65536, . . . , off to infinity. Starting with a number less than 1, say with 1/2, we get 1/2, 1/4, 1/16, 1/256, 1/65536, ... This sequence tends to zero. We will have to deal with negative numbers. Starting at –1 and repeatedly squaring gives –1, 1, 1, 1, 1, 1, ... Things become more interesting if a fixed number is added each time after. Suppose 1 is added each time. Starting with 0, squaring and adding 1 gives $0^2 + 1 = 1$, squaring and adding 1 repeatedly gives $1^2 + 1 = 2$, $2^2 + 1 = 5$, $5^2 + 1 = 26$, . . . off to infinity. If instead we add –1 each time, we get 0, $0^2 - 1 = -1$, $(-1)^2 - 1 = 0$, $0^2 - 1 = -1$, $(-1)^2 - 1 = 0$, ... This sequence bounces back and forth between 0 and −1 forever. It's the same if we add −2 each time. This time the sequence goes 0, −2, 2, 2, 2, 2, ... Adding any number smaller than −2 or larger than 0 leads to a sequence that tends to infinity. Starting points between −2 and 0 lead to bounded sequences. (They don't get arbitrarily far from 0 as time goes on). They are in the Mandelbrot set.

Periodic Points. Adding certain special numbers leads to sequences that cycle in a fixed pattern. Let c be the value added after each squaring. Thus, the sequences are:

$$0,\ 0^2 + c = c,\ c^2 + c,\ (c^2 + c)^2 + c = c^4 + 2c^3 + c^2 + c, \ldots$$

If such a sequence is to return to 0, then eventually one of the iterated terms must vanish. Consider the term $c^2 + c$. When is this 0? If $c^2 + c = 0$ then either $c = 0$ or $c+1 = 0$, i.e., $c = -1$. We saw above that adding −1 each time gives 0, −1, 0, −1, 0, −1, . . . , a pattern with "period 2." Consider the next term $c^4 + 2c^3 + c^2 + c$. Which values of c make this 0? The value $c = 0$ works but we have seen this before. If $c \neq 0$, we can divide through and consider $c^3 + 2c^2 + c + 1$. This is a cubic equation and there is a rather complicated formula for the roots of a cubic polynomial that shows that in this case, the value

$$c = -\frac{\sqrt[3]{100 + 12\sqrt{69}}}{6} - \frac{2}{\sqrt[3]{100 + 12\sqrt{69}}} - \frac{2}{3} = -1.75487\ldots$$

works. Using this value for c, we get

$$0,\ -1.75487\ldots,\ (-1.75487\ldots)^2 - 1.75487\ldots = 1.32471\ldots,\ 0, \ldots$$

This pattern continues, repeating every third step. We say that $c = -1.75487\ldots$ is a "period three" point.

The same scheme works to get points of a higher period. For example, squaring $c^4 + 2c^3 + c^2 + c$ and adding c gives $c^8 + 4c^7 + 6c^6 + 6c^5 + 5c^4 + 2c^3 + c^2 + c$. This gives two new values of c, both of which lead to points of period four. These are $c = -1.3107\ldots$ and $c = -1.9407\ldots$. These in turn lead to the repeated sequences

$$c = -1.3107\ldots: \qquad 0,\ -1.3107\ldots,\ 0.4072\ldots,\ -1.1448\ldots,\ 0, \ldots$$

and

$$c = -1.9407\ldots: \qquad 0,\ -1.9407\ldots,\ 1.8259\ldots,\ 1.3931\ldots,\ 0, \ldots.$$

New periodic sequences occur for each possible period. These can be found by finding the values of c where the n^{th} iterate of "squaring and adding c" vanishes. They turn out to be exactly described by Gilbreath permutations.

The Shuffling Connection. To make the connection with shuffling cards, write down a periodic sequence starting at zero. Write a one above the smallest point, a two above the next smallest point and so on. For example, if $c = -1.75487\ldots$ (a period three point), we have:

2	1	3
0	−1.75487 . . .	1.32471 . . . .

For the two period four sequences, we get for $c = -1.3107\ldots$:

3	1	4	2
0	−1.3107 . . .	0.4072	−1.1448 . . .

and for $c = -1.9407\ldots$:

2	1	4	3
0	−1.9407 . . .	1.8259	1.3931 . . . .

For a fixed value of c, the numbers written on top code up a permutation that is a Gilbreath shuffle. Here is the decoding operation. For example, when $c = -1.3107\ldots$, the numbers on top are 3 1 4 2. Start with the 1 and go to the left (going around the corner if you have to). This gives (1324). This is "cycle notation" for a permutation. It is read as "1 goes to 3, 3 goes to 2, 2 goes to 4, and 4 goes back to 1." Rewrite this by putting the numbers 1, 2, 3, 4 in a row, and under them put what they go to in the cycle, as:

1	2	3	4
3	4	2	1.

The reader may practice by taking the example $c = -1.9407\ldots$. As we have seen above, it is 2, 1, 4, 3. Starting with 1 and going to the left gives the cycle (1234), and finally the two-line arrangement

1	2	3	4
2	3	4	1.

The point of all this decoding is that the arrangement on the bottom line is always a Gilbreath permutation, and furthermore, *every* cyclic Gilbreath permutation of length n appears exactly once from a period n value of c.

We were told this result by Dennis Sullivan, who attributes it to the great mathematicians John Milnor and William Thurston. These are three of the greatest mathematicians of the twentieth century—the latter two are winners of the Fields Medal (often called "the Nobel Prize of mathematics"). Whomever this result belongs to, it sets up a fascinating connection that is just beginning to be understood. We state it as a formal theorem below along with further comments.

The Full Mandelbrot Set at Last. All of the activity above has been confined to the one-dimensional line. The Mandelbrot set lives in two dimensions. There is a notion of "square and add c" in two dimensions. The values of c where repeated squaring and adding stays bounded are exactly the points of the Mandelbrot set. Working in the plane, the values of c are two-dimensional: $c = (c_1, c_2)$.

Figure 1 shows the Mandelbrot set. The values *on* the x-axis between −2 and 0 are the points discussed above. The big central heart-shaped region is called the cardioid. It is surrounded by blobs, and each of these is in turn surrounded by smaller blobs (and so ad infinitum). One of the main open research problems concerning the Mandelbrot set has to do with the values of c (now (c_1, c_2)) that give periodic sequences from the squaring and adding process. It is conjectured that each blob (the big ones, the smaller ones, and so ad infinitum) contains one of those periodic points. Proving this conjecture would lead to the resolution of the outstanding *local connectivity* conjecture. Sullivan told us about the connection with shuffling because shuffles parameterize the periodic points on the x-axis. Is there some kind of two-dimensional shuffle that parameterizes the two-dimensional periodic points? We don't know but we're thinking hard about it.

SOME MATH WITH A BIT OF MAGIC

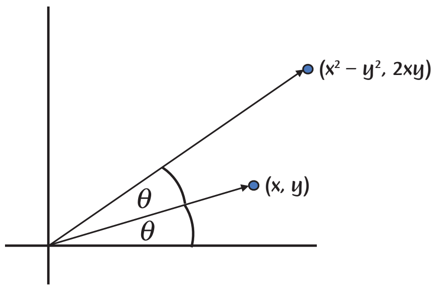

Figure 7. Squaring a complex point

Squaring and adding makes perfect sense in two dimensions, taking a point z to $z^2 + c$. There is a simple geometric description: A point z in two dimensions is described by its coordinates (x, y). Figure 7 shows (x, y) plotted as a dot with the line connecting the dot to 0. Also shown is the angle θ that the point (x, y) makes with the x-axis. To square the point (x, y), we square the distance along the line connecting (x, y) to 0, double the angle to 2θ, and plot the new point. The new point can be given in coordinates as $(x^2 - y^2, 2xy)$. Call this (x', y'). Adding $c = (c_1, c_2)$ gives $(x' + c_1, y' + c_2)$. This is repeated using the same value of c each time. If this procedure, starting at 0, leads to points that stay inside a large enough circle around 0, we put c in the Mandelbrot set. Figure 2 shows all such values of c.

A comprehensive picture book about the Mandelbrot set is *Chaos and Fractals: New Frontiers of Science*, by H. O. Peitgen, H. Jürgens, and D. Saupe.[4] A discussion of shuffling and the Mandelbrot set can be found in the paper "Bounds, Quadratic Differentials, and Renormalization Conjectures" by D. Sullivan.[5] To observe professionals talking among themselves about the Mandelbrot set, see T. Lei's book *The Mandelbrot Set: Theme and Variations*.[6]

Let us state the basic connection between shuffling cards and real points in the Mandelbrot set more carefully. Define a sequence of polynomials $P_1, P_2, P_3, \ldots$, iteratively as $P_1(x) = x$, $P_2(x) = x^2 + x$, $P_3(x) = P_2(x)^2 + x = (x^2 + x)^2 + x = x^4 + 2x^3 + x^2 + x, \ldots, P_n(x) = P_{n-1}(x)^2 + x$. Thus, the top degree of P_n is 2^n. Dennis Sullivan proved that the real zeroes of P_n are "simple." Each real zero can be used as the additive constant in the "squaring and adding" iteration.

> THEOREM. Define $P_1(x) = x$ and $P_{k+1} = P_k^2 + x$ for $k < n$. The real zeroes c of P_n that lead to periodic sequences of period n are in one-to-one correspondence with Gilbreath permutations that are n-cycles. The correspondence goes as follows: From c, form the iteration $0, c, c^2 + c, c^4 + 2c^3 + c^2 + c, \ldots$. Label the smallest of those values 1, the next smallest 2, . . . , and the largest n. Read these in right-to-left order as a cyclic permutation. Transform this to two-rowed notation. The resulting bottom row is a Gilbreath permutation (characterized at the beginning of this chapter). Each cyclic Gilbreath permutation occurs exactly once through this correspondence.

Note that not every Gilbreath permutation gives rise to an n-cycle. For example, removing the top card and inserting it into the middle of the deck is a Gilbreath permutation that is not an n-cycle. The number of n-cycles among Gilbreath permutations has been determined by Rogers and Weiss.[7] They show that this number is exactly

$$\frac{1}{2n}\sum_{d \mid n,\, d\, odd} \mu(d)\, 2^{n/d}.$$

Here, the sum is over the odd divisors d of n, and $\mu(d)$ is the so-called Möbius function of elementary number theory. That is, $\mu(d)$ is 0 if d is divisible by a perfect square, and $\mu(d) = (-1)^k$ if d is the product of k distinct prime factors. Thus, for n = 2, 3, 4, 5, 6, the formula gives $\frac{1}{4}(2^2)=1$, $\frac{1}{6}(2^3-2)=1$, $\frac{1}{8}(2^4)=2$, $\frac{1}{10}(2^5-2)=3$, and $\frac{1}{12}(2^6-2^2)=5$ cyclic Gilbreath permutations. For example, the three values of c for $n = 5$ give:

2	1	5	4	3
0	–1.9854	1.9564	1.8424	1.40900

3	1	5	4	2
0	–1.8607	1.6017	0.7047	–1.3640

4	1	5	3	2
0	–1.6254	1.0165	–0.5920	–1.2749

These lead, respectively, to the two-line arrays:

1	2	3	4	5
2	3	4	5	1

1	2	3	4	5
3	4	2	5	1

1	2	3	4	5
4	3	5	2	1

where the second rows are Gilbreath permutations.

Recall that there are exactly 2^{n-1} Gilbreath shuffles. The formula above shows that there are approximately $\frac{2^{n-1}}{n}$ Gilbreath n-cycles. Jason Fulman gives a formula for the number of unimodal permutations with a given cycle structure.[8]

To conclude, let us try to explain in what sense the Mandelbrot set is universal. For fixed c, the square and add operation changes x to $x^2 + c$. As c varies, we have a family of different iteration schemes. Curt McMullen showed that *any* family of functions of the plane to itself has all the complexity of the Mandelbrot set complete with its holes, fractal dimensions, and infinite subtlety. Of course, this also means it contains all the Gilbreath permutations described above. A more careful version of McMullen's theorem strains the confines of this page.[9]

We know of two applications of Gilbreath's principles outside of magic. The mathematician N. G. de Bruijn (whom we met in chapters 2, 3, and 4) published "A Riffle Shuffle Card Trick and Its Relation to Quasicrystal Theory" in 1987.[10] The quasicrystals referred to are Penrose tiles. These are two shapes of tiles that can be used to tile the plane but only in a nonperiodic way (see figures 8 and 9). They have a fascinating story, which is detailed in Marjorie Senechal's book *Quasicrystals and Geometry*, or the more accessible *Miles of Tiles*, written by the mathematician Charles Radin. Most accessible of all is Martin Gardner's treatment in *Penrose Tiles to Trapdoor Ciphers*.[11]

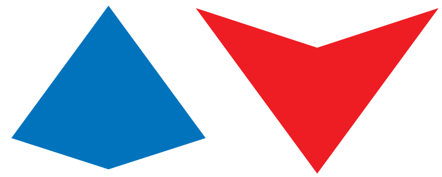

Figure 8. The two Penrose tiles

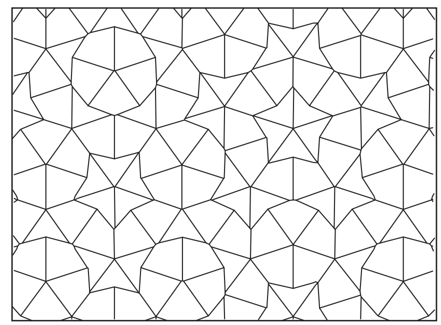

Figure 9. Tiling with Penrose tiles

A BIT OF MAGIC

De Bruijn shows that the Gilbreath Principle leads to understanding useful facts about the properties of Penrose tilings. Along the way, de Bruijn worked the following extension of Gilbreath's First Principle. Before starting, separate the cards so you have all clubs together, hearts together, spades together, and diamonds together. Form one twenty-six-card pile with spades and diamonds alternating (SDSD . . .). Then form another twenty-six-card pile with hearts and clubs alternating (HCHC . . .). If the two piles are riffle shuffled together, we know from before that each consecutive pair will consist of one red and one black. However, if the two piles are put together and the deck of fifty-two is cut freely, this need not work out. De Bruijn's "extension" allows a free cut. He proves that *either* each consecutive pair contains one red and one black throughout, *or* each consecutive pair contains one major suit (i.e., a heart or spade) and one minor suit (i.e., a club or diamond). With suitable arrangement, major/minor may be replaced by odd/even or high/low, which might be more suited to a magic trick.

De Bruijn's extension goes beyond the original Gilbreath. In light of our theorem, how can this be? De Bruijn adds an extra restriction (the packet cut off is not of a freely chosen size), but he gets his freedom at the end—a free cut. We have tried to marry de Bruijn's extension with our ultimate principle. The mix makes a nice example of how progress occurs.

The second application outside card magic comes in the world of designing sorting algorithms for computers. Huge memory files are often stored on external discs. Several discs can be read at once.

Stanford computer scientist Donald Knuth used the Gilbreath Principle to give an "improved superblock striping" technique that allows two or more files, distributed on discs, to be merged without possible conflict (in other words, the need to read two blocks from the same disc at the same time). This is explained in Knuth's monumental book series *The Art of Computer Programming*. A videotape of Knuth's talk on this superblock striping technique is available from the Stanford University Computer Science Department.

Some Proofs. We continue by providing the promised proofs for some of the theorems above. Why are there 2^{N-1} Gilbreath shuffles of N cards? Let us select some arbitrary subset $S = \{s_1, s_2, \ldots, s_j\}$ of $\{2, 3, \ldots, N\}$. Now form the Gilbreath permutation by placing j in position 1, then $j-1$ in position s_1, $j-2$ in position s_2, etc., and placing the numbers greater than j in *increasing order* in the positions *not* in S. It is clear that all Gilbreath permutations can be uniquely built this way. Since the number of ways of choosing S is just 2^{N-1}, then we have the desired result.

Here is a proof that the four properties (1), (2), (3), and (4) listed in our Ultimate Gilbreath theorem are all equivalent. The arguments are elementary but not so easy to discover. They make nice examples of how card tricks can lead to mathematics.

PROOF. After a Gilbreath shuffle, the top j cards form an interval $\{a, a+1, \ldots a+j-1\}$ or $\{a, a-1, \ldots, a-(j-1)\}$ for some value of a. As such, they consist of distinct values modulo j. Thus, (1) implies (2). If π satisfies (2) for each j then π satisfies (3). To see this, consider π satisfying (2). Clearly, the entries in the first block are distinct. But the top $2j$ are also distinct modulo $2j$ and consist of exactly two of each value modulo j. Since the top j are one of each value modulo j, it must be that $\pi(j+1), \pi(j+2), \ldots, \pi(2j)$ are distinct modulo j. This in turn implies that $\pi(2j+1), \pi(2j+2), \ldots, \pi(3j)$ are distinct, and so on. Clearly, (3) implies (2), so (2) and (3) are equivalent.

To see that (2) implies (1), observe that (2) implies that the top j cards form an interval of values. Suppose the top card ($\pi(1)$) is k. The next card must then be $k+1$ or $k-1$, since if it is $k \pm d$ for some $d>1$, then the top d cards would not be distinct modulo d. Suppose the top $j+1$ cards were $a, a+1, \ldots, a+j$. If the next was not $a-1$ or $a+j+1$, but $a+j+d$ for some $d>1$, then, again, modulo d, things would repeat.

Finally, this "interval" property of π implies it can be decomposed into two chains $k+1, k+2, \ldots, n$ and $k, k-1, \ldots, 1$. For this, proceed sequentially. If the top card is k, the next must be $k+1$ or $k-1$. Each value that increases the top of the interval is put in one chain, and each value that decreases the bottom is put in the second chain. Since, for such intervals, increasing values occur further down in π, the two chains formed do the job. This finishes the proof (whew!).

FURTHER REMARKS

1. The decomposition into two chains is not unique. If we deal off k cards and, in the shuffle, $k+1$ is left above k, it is impossible to distinguish this from $k+1$ being dealt off.

2. Instead of dealing, we can cut off and turn a packet of k face-up, then shuffle the two packets together.

3. As a last mathematical detail: At the start of this chapter we gave a heuristic calculation of the chance that a well-shuffled deck of $2N$ cards has one red and one black card in each consecutive pair. Naïve heuristics suggest that when N is large, the pair choices are roughly independent and each one has the $\frac{1}{2}$ chance of coming up red/black in some order. This would result in a probability of $\frac{1}{2^N}$ of happening. However, the events we are considering are *not* independent. In particular, if we start with a shuffled deck of N red and N black cards, the chance that, after the first card is selected, the *next* card selected has a different color from the first card is slightly *greater* than $\frac{1}{2}$. After all, there are only $N-1$ cards with the first card's color left in the deck, while there are still N cards with the opposite color. This imbalance happens for each of the pairs selected, and becomes greater as the number of cards gets smaller. For example, for a four-card deck (i.e., $N=2$), the chance that the first two cards form a red/black pair is $\frac{2}{3}$. The result of multiplying all these "imbalances" together is that the probability that our well-shuffled deck will have the desired property is exactly $\frac{2^N}{\binom{2N}{N}}$, which is approximately equal to $\frac{\sqrt{\pi N}}{2^N}$, using the Stirling approximation again. For $N=26$, this is $1.353\ldots \times 10^{-7}$, which amounts to less than one chance in seven million.

SOME HISTORY

Gilbreath's First Principle originally appeared as the trick Magnetic Colors in the magic magazine the *Linking Ring* in July 1958. The *Linking Ring* is the official publication of one of the two largest American magic organizations, the International Brotherhood of Magicians, or IBM. (The other is the Society of American Magicians, or SAM.) The *Linking Ring* has been published monthly since 1923. A typical issue contains advertisements from magic dealers, historical articles, editorials denouncing magical exposés, and a large section of tricks contributed by IBM members. You cannot find it in libraries. As with most magical information, it is for magicians only.

Back in 1958, young Norman Gilbreath introduced himself in the magazine as follows: "I have been interested in magic for 10 years. I am a math major at the University of California in Los Angeles (UCLA). Being a supporter of the art of magic, I have created over 150 good tricks and many others not so good. Here are a couple I hope you can use." He then provided a brief description of what is now called Gilbreath's First Principle, in which he dealt the deck into two piles, following the shuffle, and revealed that the cards in each pair have opposite colors.

Gilbreath's trick was picked up and varied almost immediately. In the January 1959 issue of the *Linking Ring*, card experts Charles Hudson and Edward Marlo wrote, "It is not often one runs across a new principle in card magic. . . . Norman Gilbreath's 'Magnetic Colors' has proven the most popular card effect to appear in the parade for a long time." Gilbreath weighed in eight years later by introducing his second principle in the June 1966 issue of the *Linking Ring*. By this time, Gilbreath was a professional mathematician working for the Rand Corporation. He held this job for his entire career. The second principle was featured in this special issue of the magazine devoted to Gilbreath's magic. It included new uses for the first principle and many noncard tricks. Gilbreath published later variations that involved mixing red decks with blue decks and face-up cards with face-down cards (with some effort, you can find these in the magic magazine *Genii*).[13]

The nonmagical public heard about the Gilbreath Principle in Martin Gardner's *Scientific American* column in August 1960. He expanded this into a chapter in his third book, *New Mathematical Diversions from "Scientific American."* New presentations and applications have regularly appeared in magic journals. A booklet titled “Gilbreath’s Principles,” written by mathematics teacher and magician Reinhard Muller, appeared in 1979. Chapter 6 of Justin Branch’s *Cards in Confidence*, vol. 1, is filled with many variations.[14] While our Ultimate Gilbreath Principle shows there can be no really new *principle*, the variations make for good magic.