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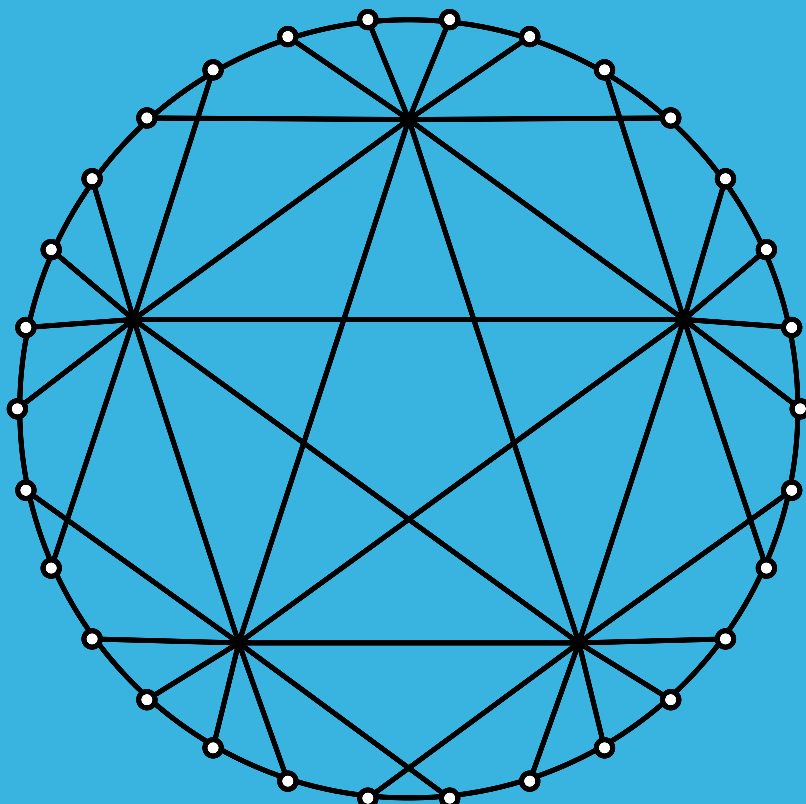

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Self-colorings of graphs

Joseph Briggs, Dean Hoffman, Sarah Heuss Holliday* and Peter Johnson

Abstract. Suppose that G and H are finite simple graphs on the same vertex set V. We will say that H is G-colorable if H is properly colorable from the list assignment N_G, i.e., the assignment of $N_G(v)$ as a list of colors to each $v \in V$. If G itself is G-colorable, we will say that G is self-colorable.

It is shown that every graph G with no isolated vertices is self-colorable. A necessary and sufficient condition on G for its complement to be G-colorable is proven. Multicolorings from the N_G list assignment are considered and questions are posed.

We owe this inquiry to one of Steve Hedetniemi's seminal questions [1].

1 Introduction

All graphs here are finite and simple. G is a graph, its vertex set will be $V(G)$ and its edge set is $E(G)$. If $v \in V(G)$ then $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ is called the *open neighborhood* (or open neighborhood set) of v in G. The corresponding closed neighborhood is $N_G[v] = \{v\} \cup N_G(v)$. If $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$.

At the Fall 2017 Clemson Mini-Conference, Steve Hedetniemi asked: for which graphs G does the indexed collection $\mathcal{N}(G) = [N_G(v) | v \in V(G)]$ of open neighborhoods have a *system of distinct representatives* (SDR). An SDR for $\mathcal{N}(G)$ is a one-to-one function $\phi : V(G) \to V(G)$ such that $\phi(v) \in N_G(v)$ for all $v \in V(G)$. Necessary and sufficient conditions were found in [1].

*Corresponding author: sarah.holliday@gmail.edu
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Theorem 1.1 (Hedetniemi, Holliday and Johnson [1]).
Suppose that G is a graph. $\mathcal{N}(G)$ has an SDR if and only if G has a spanning subgraph the components of which are either single edges (K_2) or cycles.

Proof. Clearly, if G has a spanning subgraph X such that $\mathcal{N}(X)$ has an SDR, then $\mathcal{N}(G)$ has an SDR. It is easy to see that if every component of X is either a single edge or a cycle, then $\mathcal{N}(X)$ has an SDR. So the "if" claim is settled.

Suppose that $\mathcal{N}(G)$ has an SDR $\phi : V \to V$, $V = V(G)$. Then ϕ is a permutation of V with no fixed points. Therefore ϕ factors into "cyclic" permutations on pairwise disjoint subsets of V, with all cycle lengths ≥ 2. Because $v\phi(v) \in E(G)$ for all $v \in V$, these cyclic permutations are associated with single edges, when the permutation is a transposition, and graph cycles, when the permutation cycle length is > 2. $\square$

Corollary 1.2. *If G is bipartite then $\mathcal{N}(G)$ has an SDR if and only if G has a perfect matching.*

Proof. Every even cycle has a perfect matching. $\square$

A spanning subgraph X of a graph G such that every component of X is either an edge or a cycle will be called a *(1,2)-factor* of G. This terminology is due to Tutte [4], but he means by it something a little different: in Tutte's definition, a (1,2)-factor of a graph is a spanning subgraph in which every vertex has degree either 1 or 2. Therefore, our (1,2)-factors are all Tutte (1,2)-factors, but the reverse inclusion does not hold. For instance, P_3, the path on 3 vertices, is a Tutte (1,2)-factor of itself, but does not have a (1,2)-factor in our more restrictive sense.

Obviously Theorem 1.1 can be restated.

Theorem 1.1′. *Suppose that G is a graph. $\mathcal{N}(G)$ has an SDR if and only if G has a (1,2)-factor.*

If it was not clear from the example of $[N_G(v)|v \in V(G)]$, an SDR of an indexed collection $[A_i; i \in I]$ of sets is an injective function $\phi : I \to \bigcup_{i \in I} A_i$ such that $\phi(i) \in A_i$ for all $i \in I$. There is a time-hallowed connection between SDRs and *list colorings* of graphs.

Let us explain the latter. For a set C (of *colors*) let $\mathcal{F}(C)$ denote the collection of finite subsets of C (so $\mathcal{F}(C) = 2^C$ if C itself is finite).

A *list assignment* to the vertices of a graph G is a function $L : V(G) \to \mathcal{F}(C)$, for some set C. For such an L, a *proper L-coloring* of G is a function $\phi : V(G) \to C$ such that, for all $u, v \in V(G)$,

1. $\phi(v) \in L(v)$, and
2. if $uv \in E(G)$, then $\phi(u) \neq \phi(v)$.

Lemma 1.3. *If $[A_i; i \in I]$ is an indexed collection of sets, then an SDR for this collection is the same as a proper L-coloring of $K(I)$, the complete graph on vertex set I, if L is defined by $L(i) = A_i$.*

Letting $V = V(G)$ and $N_G : V \to 2^V$ denote the list assignment that assigns $N_G(v)$ to each $v \in V$, we see that Theorem 1.1 gives a necessary and sufficient condition on a graph G for the complete graph $K(V)$ on V to be properly N_G-colorable. This perspective immediately opens a general question: Given G, for which graphs H with $V(H) = V(G)$ is there a proper N_G-coloring of H?

2 G-colorability

Throughout, G and H will be graphs on the same vertex set V. We will say that H is *G-colorable* if there is a proper N_G-coloring of H. In other words, H is G-colorable if is a function $\phi : V \to V$ such that, for all $u, v \in V$

1. $\phi(v) \in N_G(v)$, and
2. if $uv \in E(H)$ then $\phi(u) \neq \phi(v)$.

The following proposition gives some straightforward basics about G-colorability. Proofs are omitted.

Proposition 2.1. *Suppose that G, H, and X are graphs on the same vertex set V.*

(1) *If G has an isolated vertex, then no graph on V is G-colorable.*

(2) *If H is G-colorable and X is a spanning subgraph of H, then X is G-colorable.*

(3) *If H is G-colorable and G is a spanning subgraph of X, then H is X-colorable.*

Two types of extremal problems arise from Proposition 2.1 (2) and (3).

1. Given G, find the maximal graph or graphs H which are G-colorable.

2. Given H, find the minimal graph or graphs G such that H is G-colorable.

In these problems, "maximal" and "minimal" refer to the partial order on these graphs (with the same vertex set) of inclusion of edge sets.

Example 2.2. *Let $G = K_{1,3}$, the "claw", depicted with vertices labeled in Figure 1.*

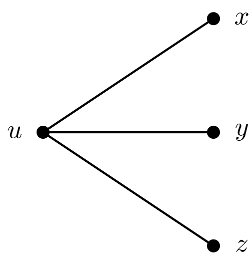

Figure 1: The claw

G is G-colorable: set $u = \phi(x) = \phi(y) = \phi(z)$ and let $\phi(u) \in \{x, y, z\}$. Further, G is the unique maximal graph on $V = \{u, x, y, z\}$ among the G-colorable graphs. (Verification left for the pleasure of the reader.) However, although G is minimal in $Q(G) = \{X | X$ is a graph on V and G is X-colorable$\}$, because deleting any edge of G creates an isolated vertex, G is not the unique minimal graph in $Q(G)$; the 3 different perfect matchings on V are also minimal in $Q(G)$.

Intriguing as we find these questions of maximality and minimality, we postpone further inspection of them for now.

If a graph G is G-colorable, we will say that G is self-colorable.

Theorem 2.3. *A graph G is self-colorable if and only if G has no isolated vertices.*

Proof. The "only if" claim follows from Proposition 2.1.

Suppose that G has no isolated vertices. We can assume that $n = |V(G)| \geq 3$ and proceed by induction on n. Therefore, we can assume that G is connected.

If $v \in V(G)$ has degree 1 then its neighbor w has degree ≥ 2, as G is connected on at least 3 vertices. Then $G - v$ has no isolated vertices. By the induction hypothesis, $G - v$ has a self-coloring. Extend this coloring to G by coloring v with w, to obtain a self-coloring of G.

So now we can assume that all vertices of G have degree at least 2 in G. Take any $v \in V(G)$. Then $G - v$ has no isolated vertex, and is therefore self-colorable, by the induction hypothesis. Consider any such coloring of $G - v$. If any $w \in N_G(v)$ does not appear as a color on any other vertex of $N_G(v)$, then color v with w. Otherwise, every $w \in N_G(v)$ appears as a color on some other $x \in N_G(v)$ in our supposed self-coloring of $G - v$. Therefore, this coloring restricted to $N_G(v)$ is a permutation of $N_G(v)$. Take any $w \in N_G(v)$, recolor w with v and color v with the color formerly on w to obtain a G-coloring of G. $\square$

Now we know which graphs G are self-colorable, and for which G the complete graph on $V(G)$ is G-colorable. A natural next question is: for which G is its complement G-colorable? We do not know the answer to this question, but we do have a necessary condition.

Let the complement of a graph G be denoted $\overline{G}$.

Lemma 2.4. *If $\overline{G}$ is G-colorable, then for every $U \subseteq V = V(G)$ which is independent in G, $|N_G(U)| \geq |U|$.*

Proof. Let $\phi : V \to V$ be a G-coloring of $\overline{G}$. If $U \subseteq V$ is independent in G, then $\overline{G}[U]$, the subgraph of $\overline{G}$ induced by U, is a complete graph. Therefore ϕ restricted to U is a one-to-one function from U into $N_G[U]$. $\square$

A matching M in a graph G *saturates* a set $U \subseteq V(G)$ if each vertex of U is incident to an edge of M.

Lemma 2.5. *If B is a bipartite graph with bipartition U, W, then there is a matching in B which saturates U if and only if for all $S \subseteq U$, $|N_B(S)| \geq |S|$.*

This is the well known "matchings in bipartite graphs" version of Hall's Theorem on SDRs; see [4].

For the next lemma, and therefore for the hard part of the proof of the theorem following it, we are indebted to L. Levine [2].

Lemma 2.6. *Suppose that G is a graph on vertex set V and B is the bipartite graph with bipartition $V, V' = \{v' | v \in V\}$, with $E(B) = \{uv' | uv \in E(G)\}$. Then G has a (1,2)-factor if and only if B has a perfect matching.*

Proof. Suppose that B has a perfect matching M. Because $|M| = |V'|$, M saturates both V and V'. By the way adjacency is defined in B, if $uv' \in E(B)$, then $u \neq v$.

Define $f : V \to V$ by: for $u \in V$, let $f(u) = v \iff uv' \in M$. Because M is a perfect matching, f is well-defined and injective. Therefore, since V is finite, f is a permutation of V with no fixed points. Also, $vf(v) \in E(G)$ for all $v \in V$. Therefore, as in the proof of Theorem 1.1, the disjoint permutation cycles that f the permutation factors into correspond to vertex disjoint edges and cycles in G, and it follows that G has a (1,2)-factor – a spanning subgraph whose components are single edges or cycles.

Now suppose that G has a (1,2)-factor F. We form a perfect matching M in B as follows: for each single-edge component uv of F, put both uv' and vu' in M. For each component of F which is a cycle, C, say $C = u_1u_2 \dots u_ku_1$, put the edges u_1u_2', u_2u_3', …, u_ku_1' in M. (Or, you could add edges to M by going around the cycle the other way, but don't do both!) It is straightforward to see that if every component of F is processed per these instructions, the resulting M will be a perfect matching in B. □

Theorem 2.7. *Suppose that G is a graph on finite vertex set V and $K(V)$ is the complete graph on V. The following are equivalent:*

(a) *G has a (1,2)-factor;*

(b) *$K(V)$ is G-colorable;*

(c) *$\overline{G}$ is G-colorable.*

Proof. Since $\mathcal{N}(G)$ having an SDR is, as explained earlier, equivalent to $K(V)$ being G-colorable, we have, by Theorem 1.1, that (a) and (b) are equivalent. Also, by Proposition 2.1 part (2), (b) implies that all graphs on V are G-colorable, and that implies (c). Therefore, to finish the proof it suffices to prove that (c)$\Rightarrow$(a).

Let B be the bipartite graph with bipartitition V, V' defined with reference to G as in Lemma 2.6. By that lemma, to prove (a), assuming (c), it suffices to show that B has a perfect matching. Because $|V| = |V'|$, it suffices to show that B has a matching saturating V, and therefore, by Lemma 2.5, it suffices to show that $|N_B(S)| \geq |S|$ for all $S \subseteq V$.

Suppose that $S \subseteq V$. Let $I = \{v | v \text{ is an isolated vertex in } G[S]\}$ and let $Q = S \backslash I$. For any subset $U \subseteq V$, let $U' = \{u' | u \in U\} \subseteq V'$. Note that $|U'| = |U|$ for each such U. Assuming (c), by Lemma 2.4 we have $|N_G(I)| \geq |I|$. Since I is the set of isolated vertices in $G[S]$, $N_G(I) \cap Q = \emptyset$ and $Q \subseteq N_G(Q)$. Thus, $|N_B(S)| = |(N_G(I) \cup N_G(Q))'| = |N_G(I) \cup N_G(Q)|$ $\geq |N_G(I) \cup Q| = |N_G(I)| + |Q| \geq |I| + |Q| = S$. □

Corollary 2.8. *Let G and V be as in Theorem 2.7. Then* (a)- (c) *in Theorem 2.7 are also equivalent to each of: for every set $U \subseteq V$ independent in G,*

(d) $|N_G(U)| \geq |U|$*;*

(e) *there is a matching in G which saturates U.*

Proof. This is a corollary of Theorem 2.7 and of its proof. But first, note that if $U \subseteq V$ is independent in G, then no edge of G is incident to more than one $u \in U$. Therefore (e) $\Rightarrow$ (d). For the reverse implication, observe that if $U \subseteq V$ is independent in G then so is each set $S \subseteq U$. Thus (d) applied to the bipartite graph with bipartition $U, N_G(U)$, induced by the $U - N_G(U)$ edges of G, implies (e), by Lemma 2.5. Thus, (d) and (e) are equivalent.

Lemma 2.4 says that (c) implies (d), and the proof of Theorem 2.7 shows that (d) implies (a). □

The surprising fact that the G-colorability of $\overline{G}$ implies the G-colorability of $K(V) = G \cup \overline{G}$ leads us to:

Conjecture 2.9. *If G and H are graphs on the same vertex set, and H is G-colorable, then $G \cup H$ is G-colorable.*

3 Graph-referential multicolorings

G, H, and V will be as before and we add to the mix functions $f : V \to \mathbb{N} = \{0, 1, 2, \ldots\}$. For such a function f, a (G, f)-coloring of H is a function $\phi : V \to 2^V$ such that, for all $u, v \in V$,

1. $\phi(v) \subseteq N_G(v)$;
2. $|\phi(v)| = f(v)$; and
3. if $uv \in E(H)$ then $\phi(u) \cap \phi(v) = \emptyset$.

When f is a constant function, say $f(v) = k$ for all $v \in V$, we will write (G, k) instead of (G, f). Clearly, a $(G, 1)$-coloring is a G-coloring. A (G, k)-coloring of G will be called a k-self-coloring of G. For instance, every cycle C_n on $n > 3$ vertices is 2-self-colorable; further, there is exactly one $(C_n, 2)$-coloring of C_n.

There is an obvious generalization of Proposition 2.1 to (G, k)-colorings. Let $\delta(G)$ denote the minimum degree in G.

Proposition 3.1. *Suppose that G, H, and X are graphs on the same vertex set V, and k is a non-negative integer.*

1. *If there is a (G, k)-coloring of H, then $\delta(G) \geq k$.*
2. *If there is a (G, k)-coloring of H, and X is a subgraph of H, then there is a (G, k)-coloring of X.*
3. *If there is a (G, k)-coloring of H, and G is a subgraph of X, then there is an (X, k)-coloring of H.*
4. *If $k > 0$ and there is a (G, k)-coloring of H, then there is a $(G, k-1)$-coloring of H.*

The question of k-self-colorability is just one of the many that arise from the definition of (G, k)- (more generally, of (G, f)-) coloring, but it is the one that we will primarily focus on for the remainder of this paper. Noting that for every graph G, every simple graph on $V(G)$ is $(G, 0)$-colorable (color each vertex with $\emptyset$), we define $hh(G) = max[k \in \mathbb{N}; G \text{ is } k\text{-self-colorable}]$.

Theorem 2.3 can be restated; $hh(G) \geq 1$ if and only if G has no isolated vertices.

For a graph H, $\omega(H)$ denotes the *clique number* of H, the greatest order of a complete subgraph of H, and $\alpha(H)$ is the *vertex independence number* of H, $\alpha(H) = \omega(\overline{H})$.

Proposition 3.2. *Suppose that G and H are graphs on a vertex set V, k is a positive integer, and H is (G,k)-colorable. Then $\omega(H) \leq \frac{|V|}{k}$.*

Proof. This proof will resemble that of Lemma 2.4. If $\phi : V \to 2^V$ is a (G,k)-coloring of H, and $U \subseteq V$ induces a complete graph in H, then the sets $\phi(u)$, $u \in U$, are pairwise disjoint k-subsets of V. Therefore, $k|U| \leq |V|$. ☐

Corollary 3.3. *If $k > 1$ is an integer, then for no graph G is the complete graph $K(V)$ on $V = V(G)$ (G,k)-colorable.*

Corollary 3.4. *Suppose G is a graph on a vertex set V, and k is a positive integer. If G is k-self-colorable then $\omega(G) \leq \lfloor \frac{|V|}{k} \rfloor$. If $\overline{G}$ is (G,k)-colorable, then $\alpha(G) \leq \lfloor \frac{|V|}{k} \rfloor$.*

Corollary 3.5. *For any graph G, $hh(G) \leq min[\delta(G), \lfloor |V(G)|/\omega(G) \rfloor]$.*

Example 3.6. *The graph G in Figure 2 has $hh(G) = 1$, while $\delta(G) = |V(G)|/\omega(G) = 2$.*

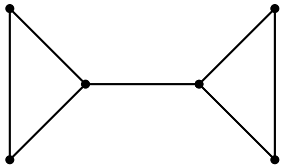

Figure 2: A graph for which the inequality in Corollary 3.5 is strict.

If we add 2 edges in Figure 2, we can get the “prism,” depicted in Figure 3.

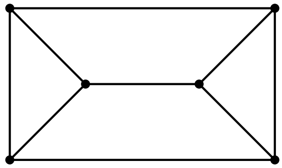

Figure 3: The prism

It is easy to see that the prism is 2-self-colorable, but it is not 3-self-colorable, by Corollary 3.5. This raises a question: is the inequality in Corollary 3.5 always equality when G is regular and connected? Turns out the answer is no.

Proposition 3.7.

(1) *If G is triangle-free, then G is $\delta(G)$-self-colorable.*

(2) *If G is r-regular then G is r-self-colorable if and only if G is triangle-free.*

We leave the proofs to the reader.

By Proposition 3.7 (2), to find a counterexample to the conjecture that for every r-regular connected graph, equality in Corollary 3.5 holds, it will suffice to find a connected r-regular graph G with $|V(G)|/\omega(G) \geq r$, with $\omega(G) \geq 3$. Possibly there are many such examples, but here is an entire family of them, one for each $r \geq 4$.

Example 3.8. *For $r \geq 3$, let the vertices of C_r be replaced by K_{r-1}'s and let K_{r-1}'s corresponding to adjacent vertices on the C_r be joined by a matching consisting of $r-1$ edges.*

The resulting graph G is r-regular, with $\omega(G) = r - 1$. G is triangle-free $\iff r = 3$. In all cases, $|V(G)|/\omega(G) = r(r-1)/(r-1) = r$.

Therefore, for $r \geq 4$, G is not triangle-free, whence $hh(G) < r = \delta(G) = |V(G)|/\omega(G)$. .

So, what is $hh(G)$? The reader should note that what we are calling G here could be any one of a number of non-isomorphic graphs, depending on how those matchings are installed. We leave as a recreation the verification that if $r > 3$, then $hh(G) = 3$ for every such G. When $r = 3$ then there are exactly two non-isomorphic choices for G, one with triangles, the other without. When G is the triangle-free one, $hh(G) = 3$, by Proposition 3.7. Also by Proposition 3.7, when G is the one with triangles, $hh(G) < 3$. We leave it as a puzzle whether $hh(G)$ =1 or 2 in this case.

Here is another puzzle: can an r-regular, connected graph G be found such that

$$hh(G) < \left\lfloor \frac{|V(G)|}{\omega(G)} \right\rfloor < r?$$

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Joseph Briggs
Auburn University, Auburn, AL, USA
jgb0059@auburn.edu

Dean Hoffman
Auburn University, Auburn, AL, USA
hoffmdg@auburn.edu

Sarah Heuss Holliday
Kennesaw State University, Kennesaw, GA, USA
sarah.holliday@gmail.com

Peter Johnson
Auburn University, Auburn, AL, USA
johnspd@auburn.edu