Weak conflict-free colorings of point sets and simple regions

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Abstract

In this paper we consider the *weak conflict-free colorings* of regions and points. This is a natural relaxation of *conflict-free coloring* [ELRS03]. One of the most interesting type of regions to consider for this problem is that of the *axis-parallel rectangles*. We completely solve the problem for a special case of them, for *bottomless rectangles*. We also give complete answer for *half-planes* and pose several open problems. Moreover we give efficient algorithms for coloring with the needed number of colors. For space limitations we do not give the proofs in this version of the paper, to represent the proof techniques we give one proof in the Appendix.

1 Preliminaries

Motivated by a frequency assignment problem in cellular telephone networks, Even, Lotker, Ron and Smorodinsky [ELRS03] studied the following problem. Cellular networks facilitate communication between fixed *base stations* and moving *clients*. Fixed frequencies are assigned to base-stations to enable links to clients. Each client continuously scans frequencies in search of a base-station within its range with good reception. The fundamental problem of frequency assignment in cellular networks is to assign frequencies to base-stations such that every client is served by some base-station, i.e. it lies within the range of the station and no other station within its reception range has the same frequency. Given a fixed set of base-stations we want to minimize the number of assigned frequencies.
First we assume that the ranges are determined by the clients, i.e. if a base-station is in the range of some client, then they can communicate. Let P be the set of base-stations and $\mathcal{F}$ the set of all possible ranges of any client. Given some set $\mathcal{F}$ of planar regions and a finite set of points P we define $cf(\mathcal{F}, P)$ as the smallest number of colors which are enough to color the points of P such that in every region of $\mathcal{F}$ containing at least one point, there is a point whose color is unique among the points in that region. The maximum over all point sets of size n is the so called **conflict-free coloring number** (cf-coloring in short), denoted by $cf(\mathcal{F}, n)$.

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Determining the cf-coloring number for different types of regions $\mathcal{F}$ is the main aim in this topic. Regions for which the problems has been studied include circles ([ELRS03], [PTT07], [Sm06], etc.) and axis-parallel rectangles ([ChSzPT], [PT03], [AEGR07], etc).
It is also a natural case to assume that the ranges are determined by the base-stations, i.e. if a client is in the range of some base-station, then they can communicate. For a *finite* set of planar regions $\mathcal{F}$ we define $\overline{cf}(\mathcal{F})$ as the smallest number of colors which is enough for coloring the regions of $\mathcal{F}$ such that for every point in $\cup\mathcal{F}$ there is a region whose color is unique among the colors of the regions covering it. For a (not necessarily finite) set $\mathcal{F}$ of planar regions let $\overline{cf}(\mathcal{F}, n)$, the **conflict-free region-coloring number of** $\mathcal{F}$ be the maximum of $\overline{cf}(\mathcal{F}')$ for $\mathcal{F}' \subseteq \mathcal{F}$, $|\mathcal{F}'| = n$.

Modifying the definition of a conflict-free coloring, $wcf(\mathcal{F}, P)$ equals to the minimum number of colors needed to color the points of P such that whenever a region covers at least 2 of them, then there are 2 points with different colors covered by it. The maximum over all n element point set of size n is the **weak conflict-free coloring number** (wcf-coloring in short), denoted by $wcf(\mathcal{F}, n)$.

Observation 1 $cf(\mathcal{F}, n) \geq wcf(\mathcal{F}, n)$.

Further relaxing our definition we can define $wcf_k(\mathcal{F}, P)$ as the minimum number of colors needed to color the points of P such that whenever a region covers at least k of them, then there are 2 points with different colors covered by it. The maximum of this value over all point sets of size n is denoted by $wcf_k(\mathcal{F}, n)$.

Observation 2 $wcf(\mathcal{F}, n) = wcf_2(\mathcal{F}, n)$.
$wcf_k(\mathcal{F}, n) \leq wcf_l(\mathcal{F}, n)$ *if* $k \geq l$.

A simple corollary of a theorem of [ELRS03] shows that the weak conflict-free coloring number gives a good upper bound to the conflict-free coloring number. More precisely, they present the following algorithm and prove that it gives a conflict-free coloring. In each step take a biggest color class in a weak conflict-free coloring of the point set. After coloring it to a new color, delete it and do the same for the new (smaller) point set. This algorithm gives the following bounds, stated in [HS05] in a slightly different way.

Lemma 1 *[HS05]*

(i) If $wcf(\mathcal{F}, n) \leq c$ *for some constant* c*, then* $cf(\mathcal{F}, n) \leq \frac{\log n}{\log(c/(c-1))} = O(\log n)$,

(ii) if $wcf(\mathcal{F}, n) = O(n^\epsilon)$ *for some* $\epsilon > 0$*, then* $cf(\mathcal{F}, n) = O(n^\epsilon)$.

Observation 1 and Lemma 1 show that wcf and cf are usually close to each other. Often the best known bound for cf is obtained from Lemma 1. This is the main motivation why we want to determine the weak conflict-free coloring number for different types of regions.

Again we can define the dual version. For a finite $\mathcal{F}$, $\overline{wcf}(\mathcal{F})$ equals to the smallest number of colors which are enough to assign colors to the regions in $\mathcal{F}$ such that for every point covered by at least 2 regions in $\mathcal{F}$, there are two differently colored regions among the regions covering it. For a not necessarily finite $\mathcal{F}$ the maximum of this value over all n element subsets of $\mathcal{F}$ is the **weak conflict-free region-coloring number**, denoted by $\overline{wcf}(\mathcal{F}, n)$. Finally, we can again define $\overline{wcf_k}(\mathcal{F}, n)$ by restricting the condition only for points covered by at least k regions in $\mathcal{F}$.
The dual version of Lemma 1 holds as well.

For the types of regions we study, the weak conflict-free coloring number can always be bounded from above by a constant not depending on n. Thus, we define $wcf_k(\mathcal{F}) = \max_n wcf_k(\mathcal{F}, n)$ and similarly $\overline{wcf_k}(\mathcal{F}) = \max_n \overline{wcf_k}(\mathcal{F}, n)$ if they exist. Our main aim is to determine these numbers and give coloring algorithms using this minimal number of colors.

2 Results

Our results are about **bottomless rectangles** and **half-planes**. A bottomless rectangle is the set of points (x, y) with $a < x < b$ and $y < c$. We denote the set of these regions by $\mathcal{B}$. Note that for our purposes these regions behave equivalently to axis-parallel rectangles with bottom edges sitting on a common base line, an interesting special case of the set of all axis-parallel rectangles.

We start with our results on bottomless rectangles.

Claim 2 *(folklore)* $wcf_2(\mathcal{B}) = 3$.

The following theorem determines $wcf_k(\mathcal{B})$ for every k as this value decreases as k grows and it cannot go below 2.

Theorem 1

(i) $wcf_3(\mathcal{B}) = 3$.

(ii) $wcf_4(\mathcal{B}) = 2$.

(iii) Such colorings can be found in $O(n \log n)$ *time.*

The first part of the theorem is proved by showing a set of points which cannot be colored with two colors in the desired way.
The proof of the second part of the theorem gives a recursive coloring. We added the proof of this theorem as this illustrates the most simply our general approach used in all the proofs.

Proof. *(ii)* We want to color the points red and blue such that any bottomless rectangle covering at least 4 points covers two differently colored points. From now on a neighbor of some point means the neighbor in the left to right order of the points.
We color the points in upwards order. We define P' to be the set of points already processed. We start with P' being the empty set and reinsert the points in upwards order. First we insert the lowest point of P into P' and color it red. After each step we preserve the following two assumptions on P'. There might be some points left uncolored in P', but *no two neighboring points in P' are left uncolored*, moreover considering only the colored points we never have two neighboring ones having the same color (*the coloring is alternating from left to right*). In each step we insert to P' the next point p of P in upwards order, so p is above all the points of $P' \setminus \{p\}$.
If p has only one neighbor in P', wlog. a right one, then if this neighbor is colored, leave p uncolored. If this right neighbor is uncolored then by assumption this neighbor's right neighbor must be colored, wlog. red. In this case color p red and its right neighbor blue, preserving the assumptions.
If p has a left and a right neighbor too among the already inserted points $P' \setminus \{p\}$, then if both of them are colored, then they have different colors by assumption, we leave p uncolored in this step. If only one of them is colored, for example wlog. we can assume that its left neighbor is colored red, then color p blue and its right neighbor red, i.e. we color these two points to preserve the alternating coloring. There are no other cases and these steps preserve the assumptions.
In the end of this procedure there might be some points still uncolored, we can color them arbitrarily, for example color all of them red.
Computing the order of the points takes $O(n \log n)$ time, the rest of the algorithm has n steps, each computable in $O(\log n)$ time, the final coloring step takes at most linear time, so the whole algorithm runs in $O(n \log n)$ time.
Now we only need to prove that this coloring is good. We need to consider bottomless rectangles covering at least 4 points, so let B be such. Let p be the highest point covered by B. It is not the lowest point of P. From now on we observe the step in the algorithm

when we inserted p to P'. Suppose that we colored p in this step, wlog. red. Thus, by the algorithm all of his neighbors (one or two) are colored blue in this step or before. It is easy to see that B covers at least one of them, and so B covers a red and blue point too, as needed. If p was left uncolored then all of its neighbors in P' were colored already before this step. Suppose p has a left and a right neighbor too and B covers both of them. By the algorithm they have different colors, as needed. Otherwise, B covers only one neighbor of p, suppose the right one, q. As B does not cover points of P' to the left of p, it must cover at least 3 points to the right of p. By the assumption always preserved by the algorithm, at least one of p's second and third neighbor to the right must be colored as well and with a color differing from the color of q. Thus, B covers a red and a blue point, as needed. $\square$

The following theorem about the region-coloring of bottomless rectangles determines $\overline{wcf}_k(\mathcal{B})$ for every k.

Theorem 2

(i) $\overline{wcf_2}(\mathcal{B}) = 3$.

(ii) $\overline{wcf_3}(\mathcal{B}) = 2$.

(iii) Such colorings can be found in $O(n^2)$ *time.*

The first part is proved by using a recursive coloring, which we call the 'divide and color method', roughly speaking we split the problem into some smaller parts, color by induction and put them back together. The proof of the second part is a more advanced application of the same idea. The coloring algorithms we got here are more complicated then in the proof Theorem 1(ii) and their running time is a bit more as well, but still bounded by $O(n^2)$.

Now we state the theorems about half-planes. We denote the set of all half-planes by $\mathcal{H}$.

Theorem 3

(i) $wcf_2(\mathcal{H}) = 4$.

(ii) $wcf_2(\mathcal{H}, P) \leq 3$, *except when* $n = 4$ *and the* 4*th point is inside the triangle determined by the first* 3 *points (see Figure 1), in which case* $wcf_2(\mathcal{H}, P) = 4$.

(iii) $wcf_3(\mathcal{H}) = 2$.

(iv) Such colorings can be found in $O(n \log n)$ *time.*

The proof gives a coloring of the points on the convex hull and of the points inside the hull using a different algorithm.

Theorem 4

(i) $\overline{wcf_2}(\mathcal{H}) = 3$.

(ii) $\overline{wcf_4}(\mathcal{H}) = 2$.

(iii) Such colorings can be found in $O(n \log n)$ *time.*

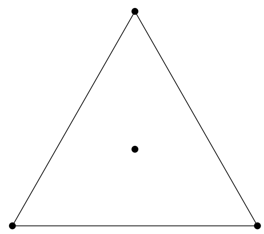

Figure 1: The exceptional case of Theorem 3(ii)

In the proof we dualize the points and half-planes and in this dual version we give an algorithm coloring the points corresponding to the half-planes of the original problem.

Problem 1 *Determine the value of* $\overline{wcf_3}(\mathcal{H})$, *i.e. the lowest number of colors needed to color any finite set of half-planes such that if a point of the plane is covered by at least* 3 *of them then not all of the covering half-planes have the same color.*

The case of axis-parallel rectangles (denoted by $\mathcal{R}$) is still far from being solved, the best bounds are $wcf(\mathcal{R}, n) = \Omega(\frac{\log n}{(\log\log n)^2})$ ([ChSzPT]) from below and recently $wcf(\mathcal{R}, n) = \tilde{O}(n^{.382+\epsilon})$ ([AEGR07]) from above, improving the previous bound $wcf(\mathcal{R}, n) = O(\sqrt{\frac{n \log\log n}{\log n}})$ ([PT03]). So probably one of the most interesting problems is still to give better bounds for $wcf(\mathcal{R}, n)$, i.e. the lowest number of colors needed to color any set of n points, such that if an axis-parallel rectangle covers at least two of them then not all of those covered by it have the same color.

The case of discs (denoted by $\mathcal{D}$) in the plane is only partially solved. One can check that a proper 4-coloring of the Delaunay-triangulation of a point set gives a good coloring for $k = 2$, and from that $wcf_2(\mathcal{D}) = 4$. Further, in [PTT07] it is shown that $wcf_k(\mathcal{D}) > 2$ for any k.

Problem 2 *Is it true for some* k *that* $wcf_k(\mathcal{D}) = 3$*? If yes, find the smallest such* k.

In the dual case, it is known that $\overline{cf}(\mathcal{D}, n) = \Theta(\log n)$ [ELRS03] but until recently it was not known whether $\overline{wcf_k}(\mathcal{D}, n)$ can be bounded from above by a constant for some k, when it was shown that $\overline{wcf_2}(\mathcal{D}, n) = 4$ ([Sm06]).

Problem 3 *Give better bounds for* $\overline{wcf_k}(\mathcal{D}, n)$ *when* $k > 2$.

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