On Graph Thickness, Geometric Thickness, and Separator Theorems

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Abstract

We investigate the relationship between geometric thickness and the thickness, outerthickness, and arboricity of graphs. In particular, we prove that all graphs with arboricity two or outerthickness two have geometric thickness $O(\log n)$. The technique used can be extended to other classes of graphs so long as a standard separator theorem exists. For example, we can apply it to show the known bound that thickness two graphs have geometric thickness $O(\sqrt{n})$, yielding a simple construction in the process.

1 Introduction

In many applications of graph visualization and graph theory, it is often useful to draw the edges of a graph with multiple colors or in multiple layers. The general class of thickness problems deals with determining the minimum number of these colors needed under various conditions. Traditionally, the *(graph) thickness* of a graph G is defined to be the minimal number of planar subgraphs whose union forms G. From an edge coloring perspective, we can also define the thickness to be the smallest number of edge colors needed so that we can draw the graph in the plane with no intersections between two edges having the same color. In this variant, the only constraint on how a graph's edges are drawn is continuity. Essentially, the thickness of a graph is a minimum edge coloring of the graph such that each color represents a planar graph. Initially, the notion of thickness derived from early work on *biplanar* graphs, graphs with thickness two [18, 20, 28]. Generalized by Tutte [29], the research in graph thickness problems is too rich to summarize here. The interested reader is referred to the survey by Mutzel *et al.* [24].

By adding the constraint that all edges must be represented by straight-line segments, we arrive at the *geometric thickness* problem [10, 21]. If we further constrain the problem such that the vertices must lie in convex position, the edge coloring number is known as the *book thickness* [8]. Although the values of graph, geometric, and book thickness are related, there are graphs that have different values for each [13, 14].

We can also constrain the graph thickness in other ways. For example, the *arboricity* of a graph G is defined to be the smallest number of forests whose union is G [25, 26, 27]. In other words, the minimum number of edge colors such that the subgraph induced by each color is a forest of trees. In *linear arboricity* the resulting colored subgraphs must be collections of paths [2, 3]. In *outerthickness*, the resulting colored subgraphs must be outerplanar graphs [16, 17]. Outerplanar graphs are analogous to graphs with book thickness one; however, outerthickness and book thickness are not identical.

Another way to look at the problem is to divide the edges of the graphs into different layers and draw the layers independently as planar graphs such that the vertex positions are identical in each layer. In this case, the problem is to minimize the number of layers. A common related application of this problem is in VLSI design where wires are placed in various layers to avoid crossings, see for example [1]. Minimizing the number of layers reduces cost and improves performance of the created circuits.

Another related problem is the area of simultaneous embeddings. In simultaneous embedding problems, the edges are already assigned to various layers, and one must determine a placement of vertices to realize each drawing without crossings, if possible [9].

1.1 Related Work

Our work is motivated by and related to recent results in characterizing the geometric thickness of graphs. In particular, Eppstein [14] characterizes a class of graphs having arboricity three and thickness three, for which the geometric thickness grows as a function of n, the number of vertices. The proof relies on Ramsey theory and so the lower bound on the geometric thickness is a very slow growing function of n. Eppstein [13] also characterizes a class of graphs with arboricity two and geometric thickness two having a book thickness that grows as another slow function of n.

Using results on the simultaneous embedding of two paths [9], Duncan *et al.* [12] proved that graphs with linear arboricity two have geometric thickness two. Arboricity-two graphs are not as simple. Geyer *et al.* [15] show two trees which cannot be embedded simultaneously. In the context of geometric thickness, this would imply that one cannot simply take a graph of arboricity two, decompose it into two forests arbitrarily, and then embed the two graphs simultaneously. However, because the *union* of the two trees described

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does have geometric thickness two, it is still in fact open as to whether arboricity two implies geometric thickness two.

In [23], Malitz characterizes the book thickness of graphs with E edges to be $O(\sqrt{E})$. This result immediately implies that thickness-t graphs have geometric thickness $O(\sqrt{tn})$. There has also been work on characterizing the geometric thickness of a graph in terms of its degree. In particular, graphs with degree less than two trivially have geometric thickness one, graphs with degree three and four have geometric thickness two [12], and there exist graphs with degree $\delta \geq 9$ having geometric thickness at least $c\sqrt{\delta}n^{1/2-4/\delta-\epsilon}$ for sufficiently large n and constants c and $\epsilon > 0$ [7].

Dujmović and Wood [11] discuss the relationship between geometric thickness and graph treewidth. In particular, they show that graphs with treewidth k have geometric thickness at most $\lceil k/2 \rceil$. This is complementary to our work as even planar graphs can have arbitrarily large treewidth. For example, the $n \times n$ grid graph has treewidth n. However, the treewidth is known for many types of graphs.

1.2 Our Results

In this paper, we provide further analysis into the relationship between geometric thickness and thickness. In particular, we show that graphs with arboricity two and outerthickness two have a geometric thickness of $O(\log n)$. We show these by providing a more generalized solution for graphs that can be decomposed into two subgraphs having some separation property. This allows for further relations between the two thickness measures. For example, we can also show that graphs with thickness two have a geometric thickness of $O(\sqrt{n})$, which is also immediately implied by the results from [23]. Additionally, if the graph can be decomposed into two K_h-minor free graphs, then the graph has geometric thickness $O(h^{3/2}\sqrt{n})$.

2 Using a Separator Theorem

A *cut set* for a graph $G = (V, E)$ is a set of vertices $C \subset V(G)$ such that the subgraph of G induced by the removal of the vertices in C consists of at least two connected components G_1 and G_2. Note, if the cut set produces more than two connected components, we can treat multiple components as one subgraph. Therefore, we assume that the removal of the cut set creates two subgraphs G_1 and G_2 such that there are no edges in $E(G)$ between vertices in G_1 and G_2. We refer to G_C as $G - G_1 - G_2$, which consequently is the set of all edges having at least one endpoint in C. For convenience, we also let $V_1 = V(G_1)$, $V_2 = V(G_2)$.

For two functions f and g and constant c, let $\mathcal{G}$ be a class of graphs having a separator property on f and g that states for any graph $G \in \mathcal{G}$ with $n = |V(G)| > c$, there exists a cut set C such that $G_1, G_2 \in \mathcal{G}$ and $|V_i| \leq f(n)$ for $i = 1, 2$ and $|C| \leq g(n)$. Our primary result uses the following key lemma:

Lemma 1 *Let G be a graph in $\mathcal{G}$ with n vertices. There exists an assignment of colors to $e \in E(G)$ in the range 1 to $\mathcal{F}(n)$, and unique x-coordinate values to $v \in V(G)$, in the range 1 to n, such that for any assignment of y-coordinates to v, with all vertex points being in general position, no two edges with the same color assignment intersect, except at common endpoints, when drawn as straight-line segments from their respective endpoints. Here $\mathcal{F}(n)$ is defined by the following recurrence relation:*

$$\mathcal{F}(n) = \begin{cases} c & \text{if } n \leq c \\ \mathcal{F}(f(n)) + g(n) & \text{otherwise.} \end{cases}$$

Proof. In our arguments, we shall also color each vertex as $c(v)$ so that an edge $e = (u, v) \in E(G)$ has color either $c(u)$ or $c(v)$.

We prove this lemma inductively. If $n \leq c$, we simply (arbitrarily) assign the vertices $v \in V(G)$ with increasing x-coordinates from 1 to n and the colors $c(v)$ as 1 to n. We then assign each edge $(u, v) \in G$ with $u, v \in V(G)$ as color $c(u)$, the choice of u or v is arbitrary. This process requires the assignment of at most c colors. Since edges with similar colors also share a common endpoint, the only color crossings possible are between adjacent edges but if the points are in general position, the only intersection is at the endpoints. Therefore, our lemma holds for the base case.

Assume now that the lemma holds for all graphs with size less than n. Let G have size n. Since G belongs to the class $\mathcal{G}$ with the separator property, we can partition $V(G)$ into the three sets V_1, V_2, C. Let $n_1 = |V_1|$, $n_2 = |V_2|$, $n_c = |C|$.

We then compute color and x-coordinate assignments separately for G_1 and G_2. From our inductive assumption, both G_1 and G_2 can be assigned values independently without any invalid crossings in their respective graphs.

To combine the two assignments and provide assignments for the remaining vertices and edges, we proceed as follows. Let V_1 have x-coordinate assignments ranging from 1 to n_1 and V_2 have x-coordinate assignments ranging from 1 to n_2. We assign the x-coordinates of C in (arbitrary) order from $n_1 + 1$ to $n_1 + n_c$. We shift the x-coordinates of V_2 over by $n_1 + n_c$. Notice that shifting the values of V_2 does not affect the intersection properties of G_2, as the entire graph is moved. Let c_1 and c_2 be the number of colors assigned to G_1 and G_2. Let $c' = \max(c_1, c_2)$. We color the vertices of G_C with n_c distinct colors ranging from $c' + 1$ to $c' + n_c$. We then color the edges in E as follows. If $e \in E(G_1 \cup G_2)$, we use the color assigned during the construction of G_1 and

G_2. Otherwise, $e \in E(G_C)$ and let $v \in C$ be an endpoint of e, because the separation property guarantees that there are no edges between V_1 and V_2. We then color the edge e with the value $c(v)$. If both endpoints are in C, the choice of v is arbitrary.

This assignment process guarantees that the vertices have x-coordinates in the range of 1 to n. To see that there are no crossing violations, observe that from our inductive assumption there are no crossing violations between edges in G_1 or between edges in G_2. In addition, because of the placement of the vertices for V_1 and V_2, there can be no edge crossings between an edge in G_1 and an edge in G_2. Therefore, any crossing violations must involve at least one edge in G_C. Since edges in G_C are colored differently than edges in G_1 or G_2, the other edge must also be in G_C. However, two edges in G_C with the same color must also have a common endpoint in C and so cannot intersect if the vertices are in general position. Therefore, there can be no crossing violations.

To complete the proof, recall that $n_c \leq g(n)$ and $n_1, n_2 \leq f(n)$. The number of colors used is consequently bounded by $c' + n_c \leq \mathcal{F}(f(n)) + g(n)$. □

We now use this lemma to prove our main theorem.

Theorem 2 *Assume we have a graph H whose edges can be colored into two layers H_1 and H_2 such that $H_1, H_2 \in \mathcal{G}$. Then H has geometric thickness $O(\mathcal{F}(|H|))$ where $\mathcal{F}$ is defined as in the preceding lemma.*

Proof. From Lemma 1, we know that there is an assignment of colors and x-coordinates for both H_1 and H_2 separately. For H_2 we simply transpose the x and y coordinates. Therefore, each vertex $v \in V(H)$ has x-coordinate defined by H_1's assignment and y-coordinate defined by H_2's assignment. From Lemma 1, we know that the choice of y-coordinate does not affect H_1 and symmetrically for H_2. The only caveat is that the vertices may not be in general position, which could cause overlap. A simple solution is to perturb the positions slightly resulting in no new crossings and eliminating any overlapping edges.

The colors between the two assignments are kept distinct. That is, we color the edges in H_1 with a different color set from any edges in H_2, thereby avoiding any new crossing violations. □

3 Specific Examples

In this section, we show specific examples of graphs with varying thickness values. From [5, 22], we know that every planar graph has a separator property with $f(n) = 2n/3$ and $g(n) = 3\sqrt{n}/2$. Solving for $\mathcal{F}(n)$ yields the following (known) corollary:

Corollary 3 *Any graph with (graph) thickness two has geometric thickness $O(\sqrt{n})$.*

It is also well known that trees have centroid vertices yielding a separator property with $f(n) = 2n/3$ and $g(n) = 1$. Solving for $\mathcal{F}(n)$ yields the following corollary:

Corollary 4 *Any graph with arboricity two has geometric thickness $O(\log n)$.*

We can easily extend this tree property to outerplanar graphs.

Corollary 5 *Any graph with outerthickness two has geometric thickness $O(\log n)$.*

In fact, since k-outerplanar graphs have $2k$ separators [6], we also get the following:

Corollary 6 *Any graph that can be decomposed into two k-outerplanar graphs has geometric thickness $O(k \log n)$.*

We can also use more general separator theorems. For example, Alon *et al.* [4] show that any graph with n vertices and no K_h-minor has a separator with $f(n) = 2n/3$ and $g(n) = h^{3/2}n^{1/2}$. This yields the following:

Corollary 7 *For any $h > 0$, any graph that can be decomposed into two K_h-minor-free graphs has geometric thickness $O(h^{3/2}n^{1/2})$.*

K_4-minor-free graphs including series-parallel graphs have a separator of size 2. [19]

Corollary 8 *Any graph that can be decomposed into two K_4-minor-free graphs has geometric thickness $O(\log n)$.*

4 Closing Remarks

We have shown that for certain classes of graphs the geometric thickness can be bounded by a non-trivial function of the number of vertices in n. In particular, we have related arboricity two and outerthickness two to geometric thickness $O(\log n)$. Given that some arboricity three, and hence thickness three, graphs have been shown to have $\omega(1)$ geometric thickness, albeit using a very slow-growing function, it would be interesting to show a lower bound for thickness two graphs. Also, it remains open to show that the geometric thickness bound for arboricity two graphs is tight.

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