Colored Simultaneous Geometric Embeddings and Universal Pointsets

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Abstract

A set of n points in the plane is a universal pointset for a given class of graphs, if any n-vertex graph in that class can be embedded in the plane so that vertices are mapped to points, edges are drawn with straight lines, and there are no crossings. A set of graphs defined on the same n vertices, which are partitioned into k colors, has a *colored simultaneous geometric embedding* if there exists a set of k-colored points in the plane such that each vertex can be mapped to a point of the same color, resulting in a straight-line plane drawing of each graph. We consider classes of trees and show that there exist universal or near universal pointsets for 3-colored caterpillars, 3-colored radius-2 stars, and 2-colored spiders.

1 Introduction

Simultaneously visualizing a set of related graphs often arises naturally in bioinformatics (e.g., evolutionary trees), social sciences (e.g., friendship networks), and other research areas. These interactions are often complex and difficult to visualize in a single diagram and so users need to extract information from multiple diagrams of different types. In order to navigate such a system, preserving the mental map of related structures that serve as landmarks in the different diagrams can be vital. To facilitate easier reconstruction of the mental map, corresponding nodes and common edges can be drawn in the same way, leading to the notion of *simultaneous embeddings* [3].

In this paper, we only consider geometric drawings with straight-line edges. We omit the "geometric" clarification henceforth. There are two variations of simultaneous embeddings: *with* and *without mapping*. In the first, a 1-1 mapping between vertices of different layers is part of the input. Two corresponding vertices in different graphs are then drawn on the same point. In the latter, each vertex of a layer can be placed at any one of the points in the pointset, regardless of the placement of the vertices in other layers.

The problem of *colored simultaneous embedding* [2] generalizes these two extremes: The input is a set of planar graphs $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, $\ldots$, $G_r = (V, E_r)$ all on the *same* set of vertices $|V| = n$ strictly partitioned into k *colors*. That is to say $V = V_1 \cup V_2 \cup \cdots \cup V_k$ where $V_i \cap V_j = \varnothing$ for $1 \leq i < j \leq k$ in which the vertices of V_i have color c_i for $i \in [1..k]$ in each graph G_j for $j \in [1..r]$.

The output is a set of points $|P| \geq n$ that are also strictly partitioned into k colors (in which each color class of P is at least the size of the corresponding color class of V) with a fixed embedding in the Euclidean plane such that each graph G_j for $j \in [1..r]$ has a straight-line planar drawing where each vertex of color c_i for $i \in [1..k]$ is placed on exactly one point of P also of color c_i. Unless specified otherwise, $|P| = n$.

The partitioning of V into k colors gives a partial mapping between graphs. If all $k = n$ colors are used, then it is a 1-1 mapping in which each vertex of V is mapped to precisely one point of P. If only $k = 1$ color is used, then there is no mapping, and each vertex of V can be freely placed on any one point of P. This corresponds to the problems of simultaneous embedding with and without mapping.

1.1 Previous Work

Brass *et al.* [3] showed that some planar pairs, such as pairs of cycles and pairs of caterpillars always admit a simultaneous embedding with mapping. They also showed that this is not always possible even for pairs of outerplanar graphs and triples of paths. Kaufmann *et al.* [8] showed that simultaneous embedding with mapping do not always exist for pairs of trees. If no mapping is given, Brass *et al.* [3] also showed that a planar graph and any number of outerplanar graphs share a simultaneous embedding. It is not known whether this is possible for an arbitrary pair of planar graphs.

Simultaneous embedding is also related to *universal pointsets* problems, where the goal is to find a pointset P in the Euclidean plane that allows for any number of graphs of a given class to be drawn with straight-line edges and no crossings on P. Rosenstiehl and Tarjan [9] posed the question of whether there exists a universal pointset of size n for all n-vertex planar graphs. The question was answered in the negative by de Fraysseix *et al.* [4] who presented a set of n-vertex planar graphs that requires a pointset of size $\Omega(n+\sqrt{n})$. For restricted

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classes of n-vertex planar graphs universal pointsets of size n have been found. Gritzman *et al.* [7] showed that a set of n points in general position is a universal pointset for trees and outerplanar graphs for which Bose [1] gives efficient drawing algorithms.

For the problem of colored simultaneous embedding, Brandes *et al.* [2] proved that any set of 2-colored n points in general position separable by a line in the plane forms a universal pointset of size n for any number of 2-colored paths. This allows the simultaneous embedding of a tree or an outerplanar graph with any number of paths on 2 colors. Brandes *et al.* also showed that there also exists a universal pointset of size n for 3-colored paths, and provided several negative results in showing that five 5-colored paths, four 6-colored paths, three 6-colored cycles, and three 9-colored paths do not always share a simultaneous embedding.

1.2 Our Contribution

We provide universal and near-universal pointsets[1] for three classes of trees: (1) 3-colored caterpillars (graphs in which the removal of all degree-1 vertices leaves a path), (2) 3-colored radius-2 stars (stars, $K_{1,k}$, in which each edge is subdivided at most once), and (3) 2-colored spiders (stars in which each edge is subdivided arbitrarily multiple times). Due to space limitations some proofs are only sketched or omitted altogether; a complete version of this paper is available [5].

2 Universal Pointsets for Trees

We present universal pointsets of cardinality $n+c$, where $c \in \{0, 1, 3\}$ for three classes of n-vertex trees. The first pointset for 3-colored caterpillars has size n. The second pointset for 3-colored radius-2 stars has size $n+3$, in general, and size $n+1$ in a restricted case, so that not every point is used in every radius-2 star. The last pointset for 2-colored spiders is of size n.

2.1 Caterpillars on Three Colors

Recall that *caterpillars* are trees in which the removal of degree-1 vertices leaves a path. Here we show that there exists a universal pointset for 3-colored caterpillars.

Theorem 1 *There exists a universal pointset P of size n on which any number of n-vertex 3-colored caterpillars can be simultaneously embedded.*

Proof. Let T be any 3-colored caterpillar on colors c_1, c_2, and c_3, where $|c_1|+|c_2|+|c_3| = n$. Let ℓ_1, ℓ_2 and ℓ_3 be three line segments each with endpoint O at the origin meeting at 120° angles. Start by placing $|c_i|$ points along ℓ_i so that the first point of each color c_i is at a distance of 1 from O, the last point is at a distance of 3 from O, and the remaining points are uniformly distributed in between. Perturb the points of each ℓ_i in a clockwise direction so that they lie along a common circular arc with no point perturbed by more than a distance ε. For sufficiently small ε, each point has a line of sight to any other point without intersecting any circular arc; see Fig 1.

[1]While a *universal* pointset has exactly as many points as vertices in the graph, a *near universal* pointset has a few extra points.

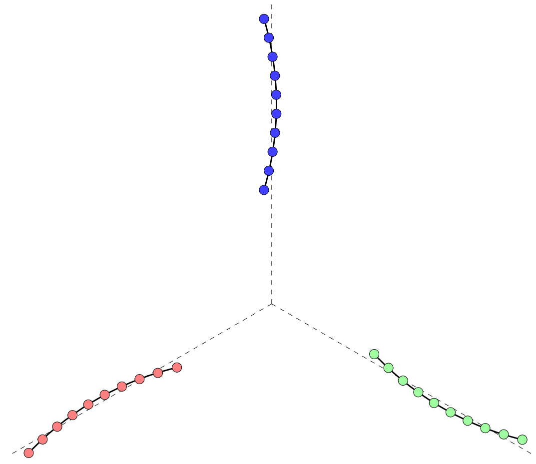

Figure 1: The universal pointset for 3-colored caterpillars.

Let S be the spine (the path after all leaves are removed) of the caterpillar T. Starting from an endpoint of S, draw all incident legs before drawing the next edge of the spine. We repeat this process for each spine vertex and its incident legs until the whole caterpillar is drawn. In doing so, pick the point of the corresponding color that is closest to the origin not already taken. Since every point has a line of sight to any other point and for a given p of T, the previously drawn edges only blocks line of sight to the points already taken, following this process results in a plane drawing. □

2.2 Radius-2 Stars on Three Colors

A *radius-2 star* is the result of subdividing each edge of a star, $K_{1,k}$, at most once. It is a tree consisting of any number of paths, or *legs*, of length at most 2 sharing a common endpoint r, the *root*. We show that there exists a near-universal pointset of size $n+3$ for radius-2 stars. If the color classes are of the same size and every edge of the star is subdivided, then $n+1$ points suffices.

Theorem 2 *There exists a near-universal pointset P of size $n+3$ on which any number of n-vertex 3-colored radius-2 stars can be simultaneously embedded.*

Proof. Let T be any 3-colored radius-2 star on colors c_1, c_2, and c_3, where $|c_1|+|c_2|+|c_3| = n$. Place one point p_i of each color c_i at $(i-2, 0)$ so that p_2 lies at the

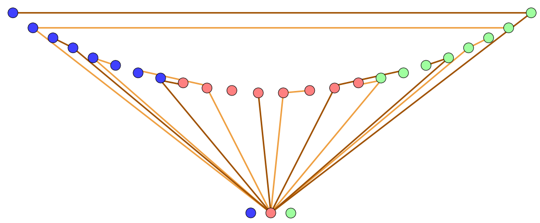

Figure 2: Near universal pointset for 3-colored radius-2 stars.

origin O, p_1 lies one unit left of O, and p_3 lies one unit to the right of O. Let A be a concave circular arc centered above O that is visible in its entirety from each point p_i. Place $|c_1|$ points of color c_1 along the leftmost part of the arc A, followed by $|c_2|$ points of color c_2 along the central part of A, and then $|c_3|$ points of color c_3 along the rightmost part of A; see Fig. 2.

We show that these $n+3$ points comprise a universal pointset for T. Place the root r of T on the corresponding point p_i of its color c_i, and call this point X. Each leg of the radius-2 star will be drawn from X to a point Y along the arc A with the line segment $\overline{XY}$, and if the leg has length 2, from Y to Z along A with the additional line segment $\overline{YZ}$. We need to ensure that the appropriate points Y and Z are selected for each leg of length 2 in order to avoid crossings.

We denote the legs of T of length 1 as $r{-}s$ and the legs of length 2 as $r{-}s{-}t$ where vertices s and t have colors c_j and c_k, respectively. We start by drawing each leg $r{-}s$ of length 1 in which we pick the free point nearest to X of the color c_j to be the point Y. We next draw the legs $r{-}s{-}t$ in which c_j and c_k differ. If neither c_j nor c_k is c_2, we pick the free points furthest from X of colors c_j and c_k, respectively, to be the points Y and Z. Then the triangle $\triangle XYZ$ contains all remaining points.

Otherwise, either c_j or c_k is c_2, i.e. $j = 2$ or $k = 2$. We take the furtherest free point of color c_2 to be point Y. We take the nearest free point of color c_1 or c_3 to be point Z. That point has color c_1 or c_3 depending on whether $j \in \{1, 3\}$ or $k \in \{1, 3\}$. In this case, the triangle $\triangle XYZ$ only contains previously used points of colors c_j and c_k. Finally, we draw the legs $r{-}s{-}t$ in which s and t share the same color c_j. We pick the remaining two points nearest to X of color c_j to be the points Y and Z. Given that any points between Y and Z along A must also be of color c_j, the triangle $\triangle XYZ$ contains no unused points.

For each leg, all remaining free points either lie inside or outside of the respective triangle $\triangle XYZ$. Hence, the line segment $\overline{YZ}$ cannot cross any previous line segment, giving a plane drawing of T. $\square$

For some colorings of a radius-2 star, we can reduce the size of the pointset by always using the points p_1 and p_3. Due to space constraints we omit the proof of the following Theorem:

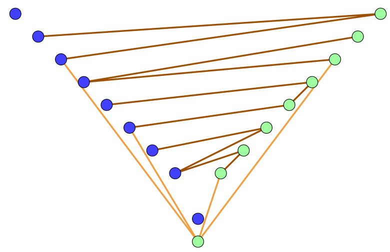

Figure 3: Near universal pointset for 2-colored spiders.

Theorem 3 *There exists a near-universal pointset P of size $n+1$ on which any number of n-vertex 3-colored radius-2 stars can be simultaneously embedded provided that each color class has the size $n/3$ and each leg has length 2.*

2.3 Spiders on Two Colors

A *spider* is an arbitrarily subdivided star, $K_{1,k}$. It is a tree consisting of any number of paths, or *legs*, sharing a common endpoint r, the *root*. We present a universal pointset of size n for n-vertex spiders. We first show how to embed any spider on a pointset of size $n+2$ and then show how to eliminate the two extra points.

Lemma 4 *There exists a near-universal pointset P of size $n+2$ on which any number of n-vertex 2-colored spiders can be simultaneously embedded.*

Proof. Let T be any 2-colored spider on colors c_1 and c_2, where $|c_1| + |c_2| = n$. Let ℓ_1 and ℓ_2 be two rays with a common endpoint O at the origin meeting at a $90°$ angle; see Fig. 3. Start by placing $|c_i|$ points along ℓ_i so that the first point of each color ℓ_i is at the nearest integer grid position (i.e., $(1, 1)$ and $(-1, 1)$). The remaining points are consecutive integer grid points along the lines ℓ_i. Next we place one point p_i of each color c_i directly below the origin at positions $(0, -i)$, i.e., $(0, -1)$ and $(0, -2)$.

We show that this pointset is universal for T. Place the root r on the point of the correct color below the origin. For each leg (a 2-colored path from r to an endpoint), place its vertices at the correctly colored free point nearest to the origin O. The first path is drawn without crossings, as consecutive points along the path are either of the same color in which case there clearly are no crossings, or are of different color which leads to the path zig-zagging between points on ℓ_1 and ℓ_2. In the latter case, there are no crossings as the path always takes vertices farther away from the origin.

For the k^{th} leg in the spider, the previous paths have taken a set of consecutive points of each color, defining a triangle $\triangle_{k-1}$, determined by r and the last two taken points, farthest from the origin along l_1 and l_2. The k^{th} leg is embedded as before, by placing its vertices at the correctly colored free point nearest to the origin O. As before, the edges of the k^{th} leg do not cross each other as the path zig-zags farther and farther away from O. The path does not enter the triangle $\triangle_{k-1}$ as all but the first edge of the k^{th} leg are above $\triangle_{k-1}$, and the first edge stays clear of $\triangle_{k-1}$ as it goes from r to a point either strictly to the left or to the right of $\triangle_{k-1}$. □

Next, we consider the two colorings of the root of the spider to show that we can always use the other point below the origin, thereby reducing the pointset to size n.

Theorem 5 *There exists a universal pointset P of size n on which any number of n-vertex 2-colored spiders can be simultaneously embedded.*

Proof. We start with the pointset given in Lemma 4 and instead of placing $|c_i|$ points along ℓ_i, we only place $|c_i| - 1$ points. The resulting set of points is of the desired size n and, as we show below, is universal for 2-colored spiders.

First, we consider spiders in which the root, r, has color c_1. As before, we place r on the point p_1 at $(0, -1)$. We need to show that it always possible to use the point p_2 at $(0, -2)$ of color c_2 for some leg of the spider. We use the next available correctly colored point that is nearest to O with one notable exception: We ignore the point p_2 below r until we encounter the last vertex of color c_2 in drawing T in which case we use p_2. This prevents the leg from self-intersecting when drawn—any edges incident to p_2 must lie along the convex hull of the points used to draw the leg.

Next, we consider spiders in which the root is colored c_2. We place r on the point p_2 at $(0, -2)$. We attempt to follow the previous strategy from Lemma 4. We draw each leg as before starting with the next available point that is nearest to O of the correct color, but this time we use the point p_1 as the first free point for the color c_1. However, this may not always work if the vertices of the first leg, r–a–b–c–$\cdots$, strictly alternate between the colors c_1 and c_2 in which a and c have color c_1 and b has color c_2. In this case, edge (r, a) crosses edge (b, c). To avoid this, we revise our strategy by reversing the order in which we draw each leg. We start with the next available correctly colored point that is the *furthest* from O, instead of the one nearest to O. Point p_1 is the point of color c_1 nearest to O, and hence, p_1 will be chosen as the last point for color c_1 instead of being chosen first.

The $(k+1)^{\text{th}}$ leg is fully contained inside the triangle $\triangle_k$ formed by r and the two taken points along ℓ_1 and ℓ_2, nearest to the origin O, along the k^{th} leg. These triangles nest as in Lemma 4, so no crossings are introduced between legs. Since we avoid crossings as we draw each leg we get a plane drawing. □

3 Conclusions and Open Problems

The only previously known universal pointset for 3-colored graphs was that for paths by Brandes *et al.* [2]. The three types of graphs for which we provide universal pointsets are considerably larger and more involved than paths. Furthermore, taken together, caterpillars, radius-2 stars and degree-3 spiders form the unlabeled level planar (ULP) trees, which play a role in level planarity [6].

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