Every four-colorable graph is isomorphic to a subgraph of the Visibility Graph of the Integer Lattice

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Abstract

We prove that a graph is 4-colorable if and only if it can be drawn with vertices in the integer lattice, using as edges only line segments not containing a third point of the lattice.

1 Introduction

A *lattice point* is a point of the plane having integer coordinates. The *Integer Lattice,* $\mathbb{Z}^2$, is the set of lattice points. Two distinct lattice points are *visible* if the line segment joining them contains no third point of $\mathbb{Z}^2$. We say that a line segment is *primitive* if its endpoints are two visible lattice points.

The *Visibility Graph* of the integer lattice, $\mathcal{V}(\mathbb{Z}^2)$, is the graph having as vertex set the integer lattice, and where two points are connected by an edge if they are visible.

A *k-coloring* of a graph $G = (V, E)$ is a function $f : V \to C$, for some set C of k *colors*; such that $f(u) \neq f(v)$ for every edge uv of G. If such a k-coloring of G exists, then we say that G is *k-colorable.* The *chromatic number* of G, $\chi(G)$, is the least k such that G is k-colorable.

Kára et al. show that the chromatic number of $\mathcal{V}(\mathbb{Z}^2)$ is four [4]. They do it by noting that $\{(0,0),(0,1),(1,0),(1,1)\}$ induces a subgraph whose chromatic number is four; and using the following Proposition, whose short proof we reproduce for self-containment of this note.

Proposition 1 *The function*

$$c((i,j)) \stackrel{def}{=} (i \ mod \ 2, j \ mod \ 2) \qquad (1)$$

is a four-coloring of $\mathcal{V}(\mathbb{Z}^2)$.

Proof. For any two points $p_1 = (a, b)$ and $p_2 = (c, d)$ in $\mathbb{Z}^2$ for which $c(p_1) = c(p_2)$, both $a+c$ and $b+d$ are even, which implies that the midpoint of the segment $\overline{p_1p_2}$ is a lattice point; thus these two equal-colored points are not visible and therefore they are not adjacent in $\mathcal{V}(\mathbb{Z}^2)$. □

The fact that $\chi(\mathcal{V}(\mathbb{Z}^2))$ is four immediately implies that every graph drawn in the plane using only primitive segments as edges is four-colorable. It is then natural to ask if the converse is true: *Can every 4-colorable graph be drawn in the plane using only primitive segments as edges?* In this note we give an affirmative answer to this question.

To do so, in Section 2 we give an explicit construction of a family of point sets, each of them inducing complete four-partite graphs in the visibility graph of the integer lattice. The four color classes of those graphs can then be used for placing the vertices of each color class of any 4-colorable graph $G = (V, E)$, in such a way that the resulting arrangement is a drawing of G using only primitive segments as edges.

Since our construction is very simple to carry out, the position of the vertices in such drawing can be computed in time $O(|V|)$. Our construction has also the nice property of being compact, in the sense that the area and the perimeter of the enclosing box of the construction grows linearly with respect to the number of vertices of the constructed graph.

We close this note in section 3 presenting some open questions and two conjectures. It is the purpose of this note to stimulate research on the consequences of our main theorem.

2 Main result

Since any 4-colorable graph with n vertices is a subgraph of the complete four-partite graph $K_{n,n,n,n}$, then the problem of whether any four-colorable graph can be drawn in the plane using only primitive segments as edges, can be restated in graph-theoretic language as follows. *For every positive integer* n*, does the visibility graph of the integer lattice have a subgraph isomorphic to the complete 4-partite graph* $K_{n,n,n,n}$*?*

To show that this is true, we give an explicit construction of such a subgraph for every positive integer n. Let $[n]$ denote the set of integers $\{1, 2, \ldots, n\}$. For any given positive integer n, consider the following four sets of lattice points.

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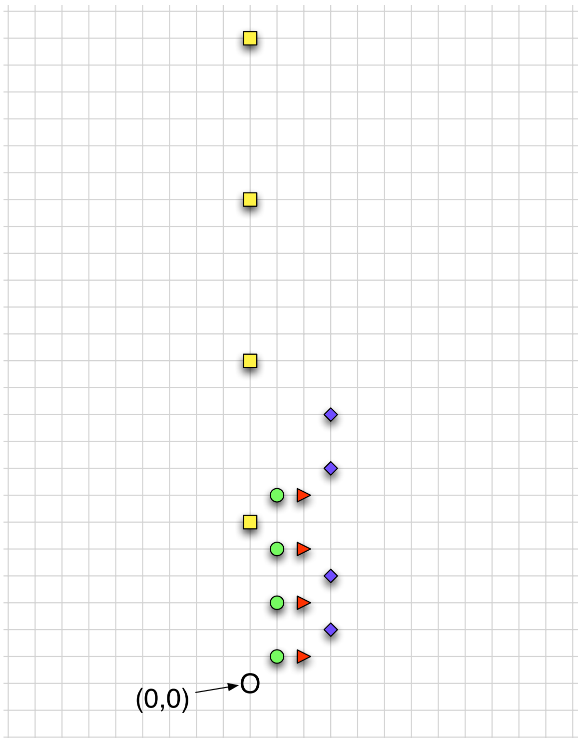

Figure 1: The set $P(4)$.

$$
\begin{aligned}
P_0(n) &\stackrel{def}{=} \{(0,6i) | i \in [n]\}, \\
P_1(n) &\stackrel{def}{=} \{(1,2i-1) | i \in [n]\}, \\
P_2(n) &\stackrel{def}{=} \{(2,2i-1) | i \in [n]\}, \\
P_3(n) &\stackrel{def}{=} \left\{ \left(3, 2\left[i-1+\lceil\frac{i}{2}\rceil\right]\right) | i \in [n] \right\}.
\end{aligned}
$$

Note that the set of y-coordinates of $P_3(n)$ is the set of the first n even positive integers that are not divisible by the number three.

Now set

$$P(n) \stackrel{def}{=} P_0(n) \cup P_1(n) \cup P_2(n) \cup P_3(n).$$

See Figure 1 for a small example of this construction.

Before stating our main result, we present the following technical lemma, whose easy proof is left to the reader.

Lemma 2 *Let (a,b) and (c,d) be two different lattice points. The segment $\overline{(a,b)(c,d)}$ is primitive if and only if* $\gcd(|a-c|,|b-d|) = 1$.

We may now claim a special property of $P(n)$.

Theorem 3 *The graph induced by $P(n)$ in the visibility graph of the integer lattice is isomorphic to $K_{n,n,n,n}$.*

Proof. We use the characterization of primitive segments of Lemma 2 for discussing whether two points are adjacent or not in $\mathcal{V}(\mathbb{Z}^2)$.

Since all of the sets $P_0(n)$, $P_1(n)$, $P_2(n)$, and $P_3(n)$ lie each in exactly one column of $\mathbb{Z}^2$, and since each of them has any two points at distance at least two, then each of these sets is independent.

We now show that any two points in two different sets $P_i(n), P_j(n)$ are adjacent in $\mathcal{V}(\mathbb{Z}^2)$.

1. For each $i \in \{0,1,2\}$, every point (i,b) of P_i is adjacent to every point $(i+1,d)$ of P_{i+1} because

$$\gcd(|(i+1)-i|,|d-b|) \quad = \quad \gcd(1,|d-b|) = 1.$$

2. Every point $(0,b)$ of P_0 is adjacent to every point $(2,d)$ of P_2 because $d-b$ is odd, which implies

$$\gcd(2-0,|d-b|) = 1.$$

3. Every point $(1,b)$ of P_1 is adjacent to every point $(3,d)$ of P_3 because $d-b$ is odd, which implies

$$\gcd(3-1,|d-b|) = 1.$$

4. Every point $(0,b)$ of P_0 is adjacent to every point $(3,d)$ of P_3 because $d-b$ *is not* a multiple of 3, which implies

$$\gcd(3-0,|d-b|) = 1.$$

□

Our main result follows immediately.

Theorem 4 *A graph is 4-colorable if and only if it can be drawn with vertices in the integer lattice, using only primitive segments for the edges.*

3 Conclusions

We have proved that any 4-colorable graph is isomorphic to a subgraph of the visibility graph of the integer lattice. We have done so by giving, for each positive integer n, an explicit construction of an induced subgraph of $\mathcal{V}(\mathbb{Z}^2)$ isomorphic to $K_{n,n,n,n}$. Our construction has the advantage of being very simple to carry out and of using only a small portion of the integer lattice.

Further research related to this work includes the problem of determining whether any planar graph is isomorphic to a *plane* subgraph of $\mathcal{V}(\mathbb{Z}^2)$. An affirmative answer would yield an alternate proof of the Four Color Theorem [1], which states that every planar graph is 4-colorable. We believe this is the case and so we state the following conjecture.

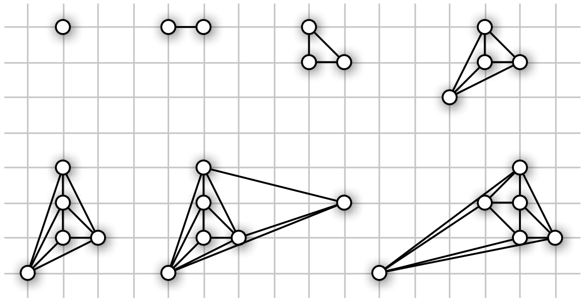

Figure 2: Maximal planar graphs with up to six points, as plane subgraphs of $\mathcal{V}(\mathbb{Z}^2)$.

Conjecture 1 *Any planar graph is isomorphic to a plane subgraph of the visibility graph of the integer lattice.*

We have verified that this is the case for all maximal planar graphs with up to six vertices. See Figure 2 for an illustration.

Note that every planar graph has a planar drawing in the integer lattice that uses only line segments for the edges; nevertheless, some of those segments may connect two points that are not visible. De Fraysseix, Pach, and Pollack [3], Schnyder [5], and Chrobak and Nakano [2] proved that one such drawing of a graph with n vertices can be realized in grids of sizes $(2n-4)\times(n-2)$, $(n-2)\times(n-2)$, and $\lfloor 2(n-1)/3\rfloor \times (4\lfloor 2(n-1)/3\rfloor - 1)$, respectively. We believe that if Conjecture 1 is true, then the plane drawing of G using only primitive segments can be realized in a grid of similar $O(n)\times O(n)$ bounded size.

A stronger conjecture is the following, where $c((i,j))$ is the four coloring of $\mathcal{V}(\mathbb{Z}^2)$ described in Equation 1 on page 1.

Conjecture 2 *Any planar graph G is isomorphic to a plane subgraph H of the visibility graph of the integer lattice, in such a way that*

$$c((i,j)) \stackrel{def}{=} (i \ mod\ 2, j \ mod\ 2)$$

is a coloring of H that uses exactly $\chi(G)$ colors.

Note that our construction of the sets $P(n)$ shows that any 4-colorable graph G is isomorphic to a subgraph H of $\mathcal{V}(\mathbb{Z}^2)$ in such a way that $c(i,j)$ is a coloring of H that uses exactly $\chi(G)$ colors. This is because

$$\begin{aligned}
\{c(p) | p \in P_0(n)\} &= \{(0,0)\},\\
\{c(p) | p \in P_1(n)\} &= \{(1,1)\},\\
\{c(p) | p \in P_2(n)\} &= \{(0,1)\},\\
\{c(p) | p \in P_3(n)\} &= \{(1,0)\}.
\end{aligned}$$

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