The spanning ratio of the Delaunay triangulation is greater than $\pi/2$

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Abstract

Consider the Delaunay triangulation T of a set P of points in the plane. The *spanning ratio* of T, *i.e.* the maximum ratio between the length of the shortest path between this pair on the graph of the triangulation and their Euclidean distance. It has long been conjectured that the spanning ratio of T can be at most $\pi/2$. We show in this note that there exist point sets in convex position with a spanning ratio > 1.5810 and in general position with a spanning ratio > 1.5846, both of which are strictly larger than $\pi/2 \approx 1.5708$. Furthermore, we show that any set of points drawn independently from the same distribution will, with high probability, have a spanning ratio larger than $\pi/2$.

1 Introduction

For a graph embedded in a Euclidean space, the *dilation* for any pair of points is the ratio of their distance along edges of the graph over their Euclidean distance along a straight line. The concept of dilation is used in computational geometry for the construction of *spanners* [6, 7, 9–11]: a t-spanner is a graph defined on a set of points such that the dilation between any two points is at most t.

One of the first results in computational geometry on spanners was a proof of Chew's conjecture [2] that the Delaunay triangulation is a spanner in the plane. Dobkin, Friedman and Supowit proved that the Delaunay triangulation of any set of points in the plane is a $(1+\sqrt{5})\pi/2$-spanner [5]; the best upper bound known is $t = (4\sqrt{3}/9)\pi \approx 2.418$ by Keil and Gutwin [8]. In this conference, Cui, Kanj and Xia established a new upper bound of 2.33 for points in convex position [3].

However, no point sets are known to actually achieve these ratios. Up until now, the largest ratio known was $\pi/2 - \varepsilon$, for any ε, achieved by sampling points P uniformly on a circle [2]. Mark two antipodal points p and p', and triangulate P by taking all edges almost perpendicular to $\overline{pp'}$, as in Figure 1. Since all points are co-circular, any triangulation of P is a valid Delaunay triangulation; alternatively, we could perturb the points to break the co-circularity, and make this the only triangulation. The shortest path from p to p' via the network follows the boundary of the circle, so its length approaches π. On the other hand, the Euclidean distance is clearly 2, leading to a dilation of $t \to \pi/2$.

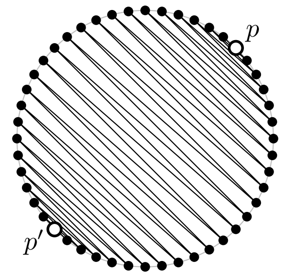

Figure 1: A set of points on a circle has a spanning ratio that approaches $\pi/2$.

In their book on spanners, Narasimhan and Smid mention that "it is widely believed that, for every set of points in R^2, the Delaunay triangulation is a $(\pi/2)$-spanner" [11]. At CCCG 2007, the special case of the spanning ratio of a set of points in convex position was stated as an open problem [4]. We show here that even for points in convex position, there are point sets that achieve a spanning ratio larger than $\pi/2$, by providing a concrete example. Additionally, we can modify the construction and create a point set not in convex position with slightly larger spanning ratio. We do not yet know whether the maximum spanning ratio can be approached by points in convex position.

Finally, we show how to modify the example to prove that for any random set of points drawn from a single distribution, with high probability the spanning ratio of the Delaunay triangulation of that point set is very close to (or even larger than) that of our example.

2 Examples with greater spanning ratio

To form our examples of points with spanning ratio greater than $\pi/2$, we use a simple observation about paths around sectors of unit circles:

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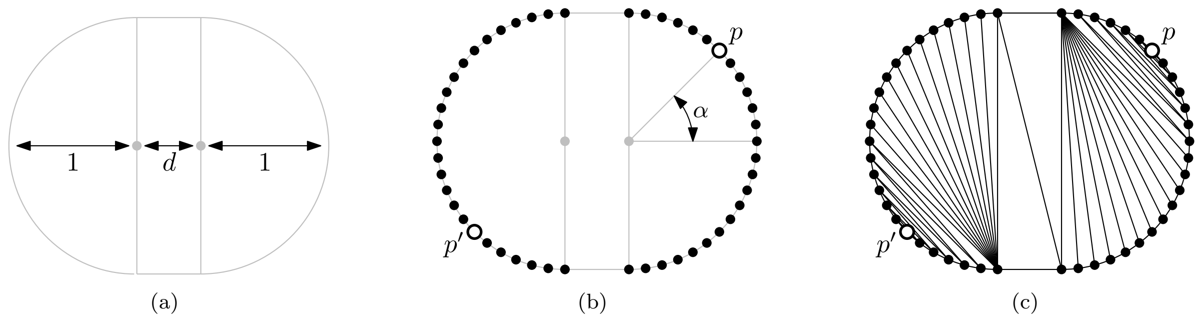

Figure 2: (a) The basic construction consists of two unit semicircles with centers on the x axis separated by d. (b) We evenly sample points on the semicircles, and mark two points p and p' that make an angle of α with the x-axis. (c) We choose the Delaunay triangulation to maximize the shortest path in the triangulation from p to p'.

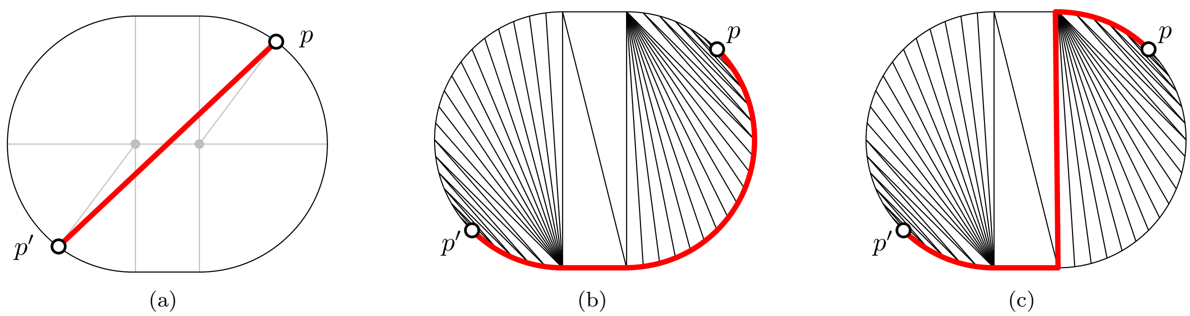

Figure 3: (a) The straight line between p and p'. (b) One locally shortest path simply follows the boundary and has length $\pi + d$. (c) Another locally shortest path crosses the construction once and has length $\pi + 2 - 2\alpha + d$.

Observation 1 *Consider the sector of a unit circle defined by points q, q' that subtend angle θ, and place p between them so that arc pq has angle β. The shortest path from p to q around the boundary of this sector follows the arc if $\beta \leq \theta/2 + \sin(\theta/2)$.*

We are now ready to describe the construction for points in convex position. Form a convex region bounded by two unit semicircles having centers on the x axis separated by distance d, as shown in Figure 2(a). Introduce points on the boundary uniformly; actually, it suffices to introduce points only on the semicircles. Mark two points p and p' at an angle of α from the x-axis, as shown in Figure 2(b). Next, triangulate the semicircle with p by adding chords to the convex hull in a way that ensure that any shortest path from p to the endpoints of the semicircle follow the boundary of the circle. One possibility is shown in Figure 2(c); actually, we can accept any semicircle chord that defines an arc containing p and satisfies Observation 1 ensures that the arc is the shortest path from p to either endpoint.

In our triangulation there are two types of locally optimal paths from p to p', drawn thicker in Figure 3(b) and 3(c). The first type, which follows the perimeter of the region (clockwise or counter-clockwise), has length $\pi + d$, since we walk around one semicircle and bridge the gap of width d. The other type, which crosses over via one of the vertical edges, and has length $2 \cdot (\pi/2 - \alpha)$ for the two circular arcs plus $2 + d$ for the straight parts, so $\pi + 2 - 2\alpha + d$ in total. Observation 1 ensures that any other path will be longer. If we set $\alpha = 1$, these two lengths are equal. Finally, we have to compute the length of the Euclidean distance between p and p', which is easily shown to be $\sqrt{4 + d^2 + 4d\cos 1}$. Thus, the dilation approaches $t = \frac{\pi+d}{\sqrt{4+d^2+4d\cos 1}}$; if we set $d = 0.29$ this gives $t > 1.581 > \frac{\pi}{2}$.

Theorem 1 *There exists a set P of points in convex position in the plane, such that the Delaunay triangulation of P has a spanning ratio of* 1.5810.

If we allow points that are not in convex position, we can modify this construction and increase the spanning ratio slightly. The idea is to bend the two straight segments so their points lie on a common circle. The graph of the triangulation inside the polygon remains unchanged, but we would like to prevent short-cuts by edges that are now needed to complete the triangulation outside.

At each of the four locations where a smaller circle meets C, we add a *shield* point s on the line through the center of the small circle, and draw a ray from s through the center of C. We place points densely on the arcs of the circles, leaving gaps in the angle formed by the ray from s. The position of s on its line is chosen so that the tangents from s to the two circles form a path that is just longer than the path that follows the polygon boundary. The triangulation outside the polygon is completed by fans from each shield point s.

For the best ratio, we use sectors of the unit circles that subtend only 2.2895 radians and place p and p' by Observation 1. Separating the circle centers by 0.29, we obtain a straight line of length 2.4 and a spanning ratio > 1.5846.

Theorem 2 *There exists a set P of points in the plane such that the Delaunay triangulation of P has a spanning ratio of* 1.5846*.*

3 The spanning ratio of random point sets

The example described in the previous section is highly degenerate. However, for any finite number n of points, it is easy to alter the construction slightly such that the points are in general position, while keeping a spanning ratio arbitrarily close to the one of the example above. For points in general position, we know that there exists a δ such that when every point is perturbed by at most δ, the combinatorial structure of the Delaunay triangulation remains the same. See [1] for a proof and an algorithm to compute δ.

With this alteration, we can use the example to show that with high probability a random set of points has a spanning ratio that is close to the one in the example. The spanning ratio of the Delaunay triangulation is a special case of what we call a *standard parameter sequence*. We say that a function L_n from $\left(\mathbb{R}^d\right)^n$ to $[0, \infty)$ is a standard parameter sequence when it complies with the following conditions.

(i) There is an upper limit:

$$\liminf_{n\to\infty} \sup_{x_1,\ldots,x_n} L_n(x_1, \ldots, x_n) = L^* \in (0, \infty).$$

(ii) For each n, L_n is scale and translation invariant, that is, $L_n(ax_1 + b, \ldots, ax_n + b) = L_n(x_1, \ldots, x_n)$ for all $a \neq 0, b \in \mathbb{R}^d$.

(iii) The limiting configurations are local:

$$\liminf_{n\to\infty} \sup_{x_1,\ldots,x_n\in[0,1]^d} \inf_{m,y_1,\ldots,y_m\notin[0,1]^d} L_{n+m}(x_1, \ldots, x_n, y_1, \ldots, y_m) = L^* \in (0, \infty).$$

It takes a moment to verify that the spanning ratio of a Delaunay triangulation is a standard parameter, even though we may not know L^*. We note here that if we know that $L^* \geq L^{**}$, then (1-2) hold with L^* replaced by L^{**}. By Theorem 1 we can set $L^{**} = 1.581$.

In the full version of this paper we prove a general theorem on standard parameter sequences that essentially says that a copy of some pessimal construction can be expected in a random point set under weak restrictions; the probability density f mentioned in the theorem can have unbounded support, and could possibly fail to be continuous at almost all x.

Theorem 3 *Let $L_n \colon (\mathbb{R}^d)^n \to \mathbb{R}$ be a standard parameter sequence for $n \geq 1$, with a lower bound L^* on its limit, and let $X_1, \ldots, X_n$ be i.i.d. random vectors drawn from a common density f in $\mathbb{R}^d$, then for every $\epsilon > 0$,*

$$\lim_{n\to\infty} \mathcal{P}\left\{L_n(X_1, \ldots, X_n) \geq L^* - \epsilon\right\} = 1.$$

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