

Blocking Delaunay Triangulations*

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Abstract

Given a set B of n blue points in general position, we say that a set of red points R blocks B if in the Delaunay triangulation of $B \cup R$ there is no edge connecting two blue points. We give the following bounds for the size of the smallest set R blocking B : (i) $3n/2$ red points are always sufficient to block a set of n blue points, (ii) if B is in convex position, $5n/4$ red points are always sufficient to block it, and (iii) at least $n - 1$ red points are always necessary, and there exist sets of blue points that require at least n red points to be blocked.

1 Introduction

Proximity graphs were originally defined to capture different notions of proximity in a set of points [7]. A particular proximity graph in a set of points S is defined by assigning to every pair of points in S a region (or family of regions). Then, the edge pq is part of the graph if and only if at least one of the regions corresponding to the pair is empty of points of S . Examples of such graphs are the Gabriel graph, the relative neighborhood graph, and the Delaunay triangulation.

Recently, the notion of proximity graphs has been extended with the concept of *witness proximity graphs* [2, 3, 4]. In this generalized setting, we have a second set of points W , the *witnesses*, which account for the existence of an edge between two points of S .

Motivated by the study of the witness Delaunay graph, the following stabbing problem is considered

in [3]. Let S be a set of n points in general position (no three points are collinear, and no four are cocircular) and let \mathcal{D} be the set of Delaunay circles of S , i.e., circles passing through at least two points of S with no point of S in the interior. What is the smallest number of points needed to pierce all the circles in \mathcal{D} ? In [3] it is stated that $2n - 2$ points are always sufficient and that sometimes n points are necessary. Similar problems have been studied from an algorithmic point of view in [1].

The following formulation, which we are using in the rest of the paper, is equivalent: Let B be a set of n blue points. We say that a set of red points R blocks B if in the Delaunay triangulation of $B \cup R$ there is no edge connecting two blue points. What is the size of the smallest set blocking a set of n blue points?

We show the following results:

- In general, $3n/2$ red points are always sufficient to block B .
- If B is in convex position, then $5n/4$ red points are always sufficient to block B .
- For every set of n blue points, at least $n - 1$ red points are always needed to block it, and there are sets that require at least n red points.

Recently, de Berg et al. [5] proved NP-hardness of a related problem: Given a set of red and blue sites, what is the minimum number of blue points that have to be removed so that the Voronoi cells of the red points form a connected region?

Throughout this paper, we denote the Delaunay triangulation of S with $DT(S)$, the Voronoi diagram of S with $V(S)$, and the Voronoi region of point $p \in S$ in $V(S)$ with $V_p(S)$.

2 An upper bound for general point sets

We start with a constructive approach for blocking general point sets that utilizes the duality between Delaunay triangulations and Voronoi diagrams.

Theorem 1 *Let B be a set of n blue points in general position. There always exists a set R of at most $3n/2$ red points that blocks B .*

*Partially supported by the FWF Joint Research Program ‘Industrial Geometry’ S9205-N12, Projects MEC MTM2008-04699-C03-02, and MICINN Programa Nacional de Movilidad de Recursos Humanos, Plan Nacional de I-D+i 2008-2011.

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Proof. Let I be the biggest independent set in $DT(B)$, and $C = B \setminus I$ its complement. Because the triangulation is 4-colorable, we know that $C \leq 3n/4$. We are going to show that B can be blocked by adding two red points in a close neighborhood of each point in C .

First, for each $p \in C$ we choose a point $\eta(p) \in C$ among the neighbors of p in $DT(B)$. This is always possible, because if pqr is a triangle in $DT(B)$, then it cannot happen that q and r are both in I .

Then, for each $p \in C$ we choose a point p_1 (not in B) in the interior of the Voronoi cell $V_p(B)$, and with the following conditions: (i) The ray p_1p intersects the boundary of $V_p(B)$ in the Voronoi edge of $V(B)$ which is separating $V_p(B)$ from $V_{\eta(p)}(B)$. Let p_b be this point of intersection. (ii) In the case in which $q = \eta(p)$ and $p = \eta(q)$, p_1 and q_1 have to be chosen in such a way that $p_b \neq q_b$ (see Figure 1).

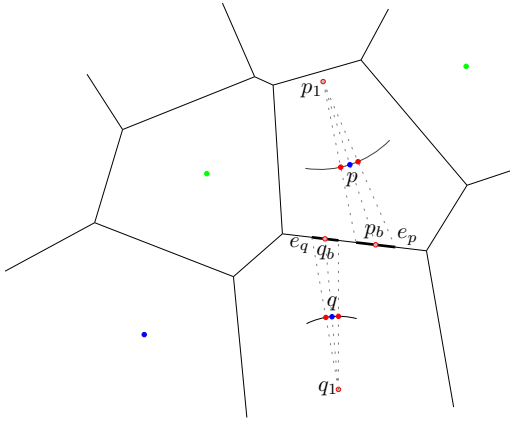


Figure 1: Isolating a blue point by placing two red points in its Voronoi cell.

Now we assign a segment e_p to each point p such that e_p is centered at p_b and contained in the edge of $V(B)$ separating the Voronoi regions of p and $\eta(p)$. If $q = \eta(p)$ and $p = \eta(q)$, we choose the intervals e_p and e_q small enough to be disjoint.

Next, we add two red points in $V_p(B)$ in the following way. Consider the circle that is centered at p_1 and passes through p , and place the red points at the intersections of this circle with the line segments defined by p_1 and the endpoints of e_p . Note that p_1 does not belong to our set of red points.

Once we have done this for every point $p \in C$, we claim that in the Voronoi diagram of the resulting set no pair of blue points have adjacent regions. The only area where p could be closer to some blue point than (one of) the two shielding red points (constructed for it) is inside the wedge defined by the bisectors of p and these two red points. The apex of the wedge is $p_1 \in V_p(B)$, and only point $q = \eta(p)$ has the possibility to be a Voronoi neighbor for p . But by construction, the intervals e_p and e_q are disjoint, so this does not happen. \square

3 An upper bound for convex sets - coloring combinatorial triangulations

For the special case of point sets in convex position we improve our general upper bound by translating the problem into a combinatorial setting.

We call a triangle of a triangulation an *ear* if it contains a vertex (the *tip*) which is not incident to inner edges. We call a triangle an *inner triangle* if it consists solely of inner edges.

Considering the properties of the Delaunay triangulation, we propose the following two simple operations to block Delaunay edges. *Blocking a single edge* is done by placing a red point arbitrarily close to the center of the edge. For inner edges this can be done on any of its two sides, and for edges of the convex hull the red point has to be placed slightly outside the convex hull. *Blocking a vertex p* is done by placing two red points outside the convex hull, one at each incident convex hull edge, and arbitrarily close to p (such that the two red points are Delaunay neighbors). This way all Delaunay edges incident to p are blocked.

Reconsidering the presented blocking operations we transform the whole setting into a combinatorial framework. We call blocking a single edge *coloring an edge* with cost 1, and blocking a vertex *coloring a vertex* with cost 2, where the latter also colors all incident edges. Thus, our task can be rephrased as coloring all edges of a given triangulation with minimal cost. Let $C(n)$ denote the maximum minimal cost over all sets of n points in convex position. Clearly, an upper bound on the occurring cost is an upper bound on the number of red points needed in the original setting, while the inverse statement is not true in general. In fact, we can apply our combinatorial setting to any triangulation, not only to the Delaunay triangulation.

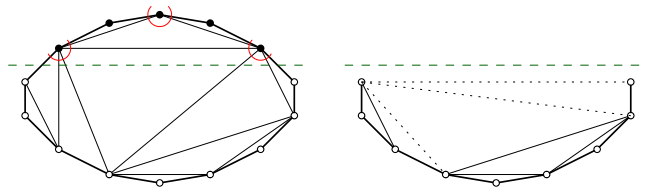


Figure 2: An (n, a, k) -cut and the retriangulated subset.

An (n, a, k) -cut of a triangulation T of a set of n points is a separation of the n points into two disjoint groups A and B with $|A| = a$ and $|B| = n - a$, plus a coloring of A with cost k such that any edge of T incident to a point in A is colored, see Figure 2.

Lemma 2 *If for a triangulation T of a convex n -gon, there exists an (n, a, k) -cut, then the cost of coloring T is at most $C(n - a) + k$.*

Proof. Let A and B be the two sets as defined for the (n, a, k) -cut. We use the coloring of A given by the cut and remove all colored vertices and edges. We complete the remaining graph of B to a full triangulation of the convex set B by (arbitrarily) retriangulating the holes induced by removing A (cf. Figure 2, right), and color this triangulation of B with cost at most $C(n-a)$. Combining the two colorings of A and B (where we can ignore colored edges of B which are not part of T), we obtain a coloring of T with cost at most $C(n-a)+k$. \square

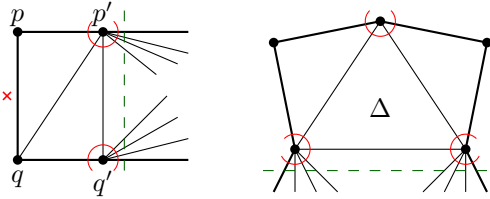


Figure 3: The two cases for a convex set: removing an ear (left), and removing an inner triangle with two incident ears (right).

Theorem 3 $C(n) \leq \frac{5n}{4}$.

Proof. We prove the statement by induction on the number n of vertices. For the induction base it is straightforward that for $n \leq 3$ we have $C(n) \leq n$. So assume the statement is true for any set of size $n' < n$, and consider a triangulation T of n points. We distinguish two cases.

Case 1. Assume that there exists an ear of T with tip p such that a neighbor q of p (neighborhood is with respect to the convex hull) has precisely one incident inner edge, see Figure 3(left). We color the two other neighbors p' and q' of p and q , respectively, as well as the edge pq . With $A = \{p, q, p', q'\}$ this induces an $(n, 4, 5)$ -cut of T . By Lemma 2 we have $C(n) \leq 5 + C(n-4) \leq 5 + \frac{5(n-4)}{4} = \frac{5n}{4}$, where the last inequality follows from the induction.

Case 2. Otherwise, all neighbors of the tip of an ear are incident to at least two inner edges. This is equivalent to the fact that all ears share an inner edge with an inner triangle. As in any triangulation (of a convex set) the number of ears is larger than the number of inner triangles by two (this follows by considering the dual tree), there exists at least one inner triangle Δ which is incident to two ears. We color the three vertices of Δ , see Figure 3(b). The tips of the two ears incident to Δ together with the three vertices of Δ form our set A . As A has cardinality 5, this induces an $(n, 5, 6)$ -cut of T , and similar as before we have $C(n) \leq 6 + C(n-5) < \frac{5n}{4}$. \square

Corollary 4 For any set B of n blue points in convex position, (the Delaunay triangulation of) B can be blocked by at most $\frac{5n}{4}$ red points.

4 Lower Bounds

In this section we provide a general lower bound on the number of points needed to block any given set, again using independent vertices. In addition, we apply a method from economics to show that there exist sets requiring n points.

Lemma 5 For every point set S with n points, the number of independent vertices in the Delaunay triangulation $DT(S)$ is at most $\lfloor \frac{n+1}{2} \rfloor$.

Proof. It is known that every Delaunay triangulation contains a perfect matching of its vertices [6]. Consider such a perfect matching M , and an independent set I . Then for every edge in M , at most one of its endpoints can be in I . If n is odd, then the non-matched point can be in I as well. \square

Note that this is a special property of the Delaunay triangulation, as there exist sets of n points, which can be triangulated in a way that the triangulation has an independent set of size as much as $\frac{2n-2}{3}$. For example, take a set of k red points and triangulate it arbitrarily. Place one blue point in the interior of each red triangle. Further, place one blue point outside but close to each convex hull edge. Complete the full set of $n = 3k - 2$ points to a triangulation with k red and $2k - 2$ independent blue points.

Theorem 6 At least $n - 1$ red points are always necessary to block a set B of n blue points.

Proof. Assume that the red point set R , $|R| = m$, blocks B . Then the joint Delaunay triangulation $DT(B \cup R)$ does not contain any edge between two blue vertices, which implies that B is an independent set in $DT(B \cup R)$. Thus, by Lemma 5, we have $n \leq \lfloor \frac{n+m+1}{2} \rfloor$, and consequently $m \geq n - 1$. \square

Proposition 7 For every $n \geq 3$, there exist point sets B with $|B| = n$ such that at least n red points are necessary to block B .

Proof. Consider a set of n coins that are placed in a way that every coin touches exactly two other coins (i.e., they form a set of cycles), which gives n touching points in total. Let these be the n blue points of B . Then for each coin c , the connection between its two touching points is an edge in $DT(B)$ (because c does not contain any other blue point). These edges form a set of (not necessarily convex) cycles, see Figure 4. For blocking B we need to place at least one red point on each coin. As the coins are pairwise disjoint, we need at least n red points. \square



Figure 4: Euros proving a lower bound of n : the cycles induced by the coins are drawn solid. The dashed edges complete the Delaunay triangulation.

5 Discussion

To block a set B of n blue points we have shown that $3n/2$ red points are sufficient for general sets, and $5n/4$ red points are sufficient for sets in convex position. Note that both proofs for the upper bounds are constructive, directly providing an algorithm. Moreover, we know that for any set with n blue points we need at least $n - 1$ red points, and that there exist sets which require n red points.

So far we have not been able to construct a set that needs more than n red points to be blocked, and to the best of our knowledge, no example is known that can in fact be blocked with only $n - 1$ points. Thus we state the following conjecture.

Conjecture 1 *For any set B of n blue points in convex position, n red points are necessary and sufficient to block B .*

In fact, from what is currently known, the conjecture might be true even for general point sets.

Independently, the algorithmic issue of finding a minimal set of blocking red points arises as a natural ques-

tion for future work. This can also be seen in the light of the NP-hardness result of [5].

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