The Projection Median of a Set of Points in $\mathbb{R}^d$

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Abstract

The projection median of a finite set of points in $\mathbb{R}^2$ was introduced by Durocher and Kirkpatrick [5]. They proved the projection median in $\mathbb{R}^2$ provides a better approximation of the 2-dimensional Euclidean median, while maintaining a fixed degree of stability, than other standard estimators, like the center of mass or the rectilinear median. In this paper we study the projection median of a set of points in $\mathbb{R}^d$ for $d \geq 3$. We prove new bounds on the approximation factor and stability of the projection median in $\mathbb{R}^d$, which show that the d-dimensional projection median also maintains a fixed degree of stability and provides a better approximation of the d-dimensional Euclidean median than the d-dimensional rectilinear median. For the special case of $d = 3$, our results imply that the 3-dimensional projection median is a (3/2)-approximation of the 3-dimensional Euclidean median, which settles a conjecture posed by Durocher [4].

1 Introduction

A *median function* on $\mathbb{R}^d$ is a function from the set of all finite non-empty sets contained in $\mathbb{R}^d$ to $\mathbb{R}^d$. The median of a set S of n real numbers is a point $\mathcal{M}(S)$ which partitions the points in S such that there are at most $n/2$ points of S greater than $\mathcal{M}(S)$ and at most $n/2$ points of S that are less than $\mathcal{M}(S)$. Let $S = \{p_1, p_2, \dots, p_n\}$ be a set of n distinct real numbers arranged in increasing order. If $n = 2m + 1$ is odd, the median of S is the point p_{m+1}, and when $n = 2m$ is even any point on the line segment joining p_m and p_{m+1} is a median of S. In such cases, the midpoint of the line segment joining p_m and p_{m+1} is often selected to represent $\mathcal{M}(S)$.

Several attempts have been made to generalize the notion of median to higher dimensions. Hayford [7] suggested the *vector-of-medians* of orthogonal coordinates. This involves selecting an orthogonal coordinate system and then computing the coordinate-wise univariate median along these axes. However, this definition of multivariate median depends on the choice of the orthogonal coordinate system. It is easy to see that the vector-of-medians of a set S in $\mathbb{R}^d$ is a point in $\mathbb{R}^d$, to be denoted by $\mathcal{M}_R(S)$, which minimizes $\sum_{s\in S} |\boldsymbol{s} - \boldsymbol{x}|$, when $\boldsymbol{x} = \mathcal{M}_R(S)$ and where $|.|$ denotes the ℓ_1 norm. For this reason, the vector-of-medians is also referred to as the *rectilinear median*. The rectilinear median is invariant under translation and uniform scaling, but not under rotation or reflection. If there are an even number of points in S, then $\mathcal{M}_R(S)$ may not be unique, and we select $\mathcal{M}_R(S)$ to be the midpoint of the d-dimensional rectangular region of points that define rectilinear medians of S. Since the one-dimensional median defined above can be computed in $O(n)$ time, the d-dimensional rectilinear $\mathcal{M}_R(S)$ can be computed in $O(dn)$ time by computing d independent one dimensional medians.

Analogous to Hayford's definition, the Euclidean median of a set S in $\mathbb{R}^d$ (to be denoted by $\mathcal{M}_E(S)$) is defined as the point in $\mathbb{R}^d$ which minimizes $\sum_{s\in S} ||\boldsymbol{s} - \boldsymbol{x}||$, when $\boldsymbol{x} = \mathcal{M}_E(S)$ and where $||.||$ denotes the ℓ_2 norm. The Euclidean median problem on three points in the plane was first posed by Fermat and solved geometrically by Torricelli early in the 17-th century [8]. The problem was later revived by Weber [11] in 1909 in the context of optimal facility location. The Euclidean median is invariant under uniform scaling, reflection, translation, and rotation. This makes it much more suitable candidate for a multivariate median compared to the rectilinear median. However, solving for the exact location of the Euclidean median in two or more dimensions is, in general, difficult. Bajaj [2] showed that even for 5 points, the coordinates of the Euclidean median may not be representable even if we allow radicals, and that it is impossible to construct an optimal solution by means of ruler and compass. The most famous of all existing algorithms is the iterative algorithm due to Weiszfeld [13].

We say that a median function $\mathcal{M}$ is a λ-*approximation* of the Euclidean median $\mathcal{M}_E$, if $\sum_{p\in S} ||\boldsymbol{p} - \mathcal{M}(S)|| \leq \lambda \sum_{p\in S} ||\boldsymbol{p} - \mathcal{M}_E(S)||$ for all nonempty finite sets S in $\mathbb{R}^d$. Recently, motivated from several problems in mobile facility location, Durocher and Kirkpatrick [5] introduced the notion of stability of a median function, which measures the behavior of the median function to slight perturbations of the data. Given $\varepsilon > 0$ and a finite set S of $\mathbb{R}^d$, a function $f : S \to \mathbb{R}^d$ is an ε-*perturbation* on S if for all $\boldsymbol{p} \in S$, $||\boldsymbol{p} - f(\boldsymbol{p})|| \leq \varepsilon$. Let $\mathcal{F}_\varepsilon(S)$ denote the set of all ε-perturbations on S. A median function $\mathcal{M}(S)$ is κ-*stable* if for all $\varepsilon > 0$ and for all $f \in \mathcal{F}_\varepsilon(S)$,

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$\kappa||\mathcal{M}(S) - \mathcal{M}(f(S))|| \le \varepsilon$, for all nonempty finite sets S in $\mathbb{R}^d$.

Using a small 4-point example, Durocher and Kirkpatrick [5] showed that the Euclidean median is not continuous even for small point sets, thus proving that the Euclidean median is not κ-stable for any $\kappa > 0$. They also showed that no median function can ensure any fixed degree of stability while also guaranteeing an arbitrarily-close approximation of the Euclidean median sum.

It is well known that the center of mass of a set S of n points in $\mathbb{R}^d$ is the point in $\mathbb{R}^d$ given by $\frac{1}{n}\sum_{p\in S}\boldsymbol{p}$. The center of mass is invariant under affine transformations and it is the unique point that minimizes the sum of the squares of the distances to the points of S [12]. It follows from results of Bereg et al. [3] that the center of mass of a set of n points in $\mathbb{R}^d$ is 1-stable and provides a $(2-2/n)$-approximation of the d-dimensional Euclidean median, and both the bounds are tight.

Bereg et al. [3] also proved that the rectilinear median in $\mathbb{R}^2$ provides a $\sqrt{2}$-approximation of the Euclidean median. Later, Durocher [5] showed that the d-dimensional rectilinear median provides a $\sqrt{d}$-approximation of the Euclidean median, and proved a $\frac{1+\sqrt{d-1}}{\sqrt{d}}$ lower bound on the approximation factor for any $d \ge 1$. Generalizing the results of Bereg et al. [3], Durocher [5] also proved a tight stability bound of $(1/\sqrt{d})$ on the d-dimensional rectilinear median for any $d \ge 1$.

The main result of Durocher and Kirkpatrick [5] is the introduction of the notion of the projection median. Given a fixed positive integer $d \ge 2$ and a finite set of S points in in $\mathbb{R}^d$, the d-dimensional projection median of S is defined as

$$\mathcal{M}_P(S) = d\frac{\int_{S^{d-1}} \mathsf{med}(S_{\boldsymbol{u}})d\boldsymbol{u}}{\int_{S^{d-1}} d\boldsymbol{u}} \qquad (1)$$

where $S^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d : ||\boldsymbol{x}|| = 1\}$ is the unit d-dimensional hypersphere and $\mathsf{med}(S_{\boldsymbol{u}})$ is the median of the projection of S onto the line through the origin parallel to vector $\boldsymbol{u}$.

They show that for any set S in $\mathbb{R}^2$, $\mathcal{M}_P(S)$ is $(\pi/4)$-stable and it provides a $4/\pi$-approximation to the Euclidean median $\mathcal{M}_E(S)$, thus proving that the projection median in $\mathbb{R}^2$ maintains a fixed degree of stability while providing a better approximation of the 2-dimensional Euclidean median than the center of mass or the rectilinear median. They also showed that the stability bound is tight and the lower bound on the approximation factor is $\sqrt{4/\pi^2+1}$.

In this paper, we study the projection median of a set of points in $\mathbb{R}^d$. Using results from the theory of integration over topological groups we show that the d-dimensional projection median provides a $J(d)$-approximation to the d-dimensional Euclidean median, where $J(d) = (d/\pi)B(d/2, 1/2)$ and $B(\alpha, \beta)$ denotes the Beta function. We also show that the d-dimensional projection median is $1/J(d)$-stable and the bound is tight. From this it follows that the projection median in $\mathbb{R}^d$ also provides a better approximation of the d-dimensional Euclidean median than the d-dimensional rectilinear median, and maintains a fixed degree of stability. For the special case $d = 3$, our results imply that the 3-dimensional projection median is a $(3/2)$-approximation of the 3-dimensional Euclidean median, which settles a conjecture posed by Durocher [4].

2 Topological Preliminaries

In this section we present the basics of the theory of integration over topological groups, which gives us the necessary mathematical machinery to deal with the projection median of a finite set of points in $\mathbb{R}^d$. A rotation ϑ is a isometry of $\mathbb{R}^d$, which keeps the origin and the orientation fixed. A rotation ϑ can be represented as a linear transformation $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{A}$ is a $d \times d$ orthogonal matrix with determinant 1. The group of all rotations in $\mathbb{R}^d$ with the operation of composition is denoted by $\mathsf{SO}(d)$, which stands for the special orthogonal group. Algebraically, the group $\mathsf{SO}(d)$ is the set of all orthogonal matrices of order d with determinant 1, under matrix multiplication. With natural topology, obtained by regarding the matrices in $\mathsf{SO}(d)$ as points in $\mathbb{R}^{d^2}$, it is a compact group.

It follows from a general theorem of topological groups, that there exists an unique Borel probability measure on $\mathsf{SO}(d)$, which is invariant under the action of the elements of $\mathsf{SO}(d)$. This is called the *Haar measure* of $\mathsf{SO}(d)$ [10]. We denote by ν the *normalized Haar measure* of $\mathsf{SO}(d)$.

As mentioned before, S^{d-1} denotes the unit hypersphere in $\mathbb{R}^d$. As a subset of $\mathbb{R}^d$, S^{d-1} is a compact and separable metric space. We denote the normalized uniform measure over S^{d-1} by μ, which is invariant under the action of $\mathsf{SO}(d)$.

We now state the following change of variable result, which will be crucial in the proofs of the following claims. The proof of this result follows from a general change of variable theorem in measure spaces (Theorem 1.6.12, Ash and Doléans-Dade [1]) and a theorem of group actions on locally compact Hausdorff spaces (Theorem 14.6.25, Royden [10]).

Result 1 *Consider the function $\psi : \mathsf{SO}(d) \to S^{d-1}$ given by $\psi(\boldsymbol{A}) = \boldsymbol{A}\boldsymbol{u}_0$, for $\boldsymbol{A} \in \mathsf{SO}(d)$ and a fixed $\boldsymbol{u}_0 \in S^{d-1}$. If $f : S^{d-1} \to \mathbb{R}$ is a continuous function, then $\int_{\mathsf{SO}(d)} f(\psi(\boldsymbol{A}))d\nu(\boldsymbol{A}) = \int_{S^{d-1}} f(\boldsymbol{x})d\mu(\boldsymbol{x})$.* $\square$

3 The Projection Median in $\mathbb{R}^d$

We now proceed to study the projection median of a set of points in $\mathbb{R}^d$. Let $S = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\}$ be a set of n points in $\mathbb{R}^d$ in a d-dimensional orthogonal coordinate system which we denote by C. Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_d$ be the canonical basis of $\mathbb{R}^d$, which also corresponds to the direction vectors along the d orthogonal coordinate axes of C. Given a vector or a matrix $\boldsymbol{M}$, we denote its transpose as $\boldsymbol{M}'$. For two vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^d$, denote by $\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle$ the standard Euclidean inner product between the vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

Let $C_{\boldsymbol{A}}$ denote the coordinate system obtained by rotating C by an orthogonal matrix $\boldsymbol{A} \in \mathsf{SO}(d)$. The coordinate axes of $C_{\boldsymbol{A}}$ are then given by $\boldsymbol{A}\boldsymbol{e}_1, \boldsymbol{A}\boldsymbol{e}_2, \ldots, \boldsymbol{A}\boldsymbol{e}_d$. Let $S_{\boldsymbol{A}} = \{\boldsymbol{A}'\boldsymbol{p}_1, \boldsymbol{A}'\boldsymbol{p}_2, \ldots, \boldsymbol{A}'\boldsymbol{p}_d\}$ be the points of S in $C_{\boldsymbol{A}}$. Let $\mathcal{M}^*_R(S_{\boldsymbol{A}})$ be the rectilinear median of the points of $S_{\boldsymbol{A}}$ in $C_{\boldsymbol{A}}$, and $\mathcal{M}_R(S_{\boldsymbol{A}})$ the coordinate of the point $\mathcal{M}^*_R(S_{\boldsymbol{A}})$ in C.

Now, we have the following simple observation:

Observation 1 *In the coordinate system C, $\mathcal{M}_R(S_{\boldsymbol{A}}) = \sum_{i=1}^d \mathsf{med}(S_{\boldsymbol{A}\boldsymbol{e}_i})$.*

Proof. By definition, $\mathsf{med}(S_{\boldsymbol{A}\boldsymbol{e}_i}) = \mathcal{M}(\{\langle \boldsymbol{p}_j, \boldsymbol{A}\boldsymbol{e}_i\rangle | \boldsymbol{p}_j \in S\})\boldsymbol{A}\boldsymbol{e}_i$. Now, observe that

$$\begin{aligned}\mathcal{M}_R(S_{\boldsymbol{A}}) &= \sum_{i=1}^d \mathcal{M}(\{\langle \boldsymbol{p}_j, \boldsymbol{A}\boldsymbol{e}_i\rangle | \boldsymbol{p}_j \in S\})\boldsymbol{A}\boldsymbol{e}_i \\ &= \sum_{i=1}^d \mathsf{med}(S_{\boldsymbol{A}\boldsymbol{e}_i}).\end{aligned}$$

□

Durocher [4] showed that in $\mathbb{R}^2$ the projection median and the rectilinear median satisfy the following identity: $\mathcal{M}_P(S) = \frac{2}{\pi}\int_0^{\pi/2} \mathcal{M}_\phi(S)d\phi$, where $\mathcal{M}_\phi(S)$ denotes the rectilinear median relative to a rotation by ϕ of the reference axis. The main obstacle in the generalization of their results to higher dimensions is the difficulty in obtaining an analogous result in higher dimensions.

In the following lemma, we obtain a generalization of this result to higher dimensions by integrating the rectilinear median over the group of rotations $\mathsf{SO}(d)$ with respect to the normalized Haar measure.

Lemma 1 *For a finite set S of points in $\mathbb{R}^d$, $\mathcal{M}_P(S) = \int_{\mathsf{SO}(d)} \mathcal{M}_R(S_{\boldsymbol{A}})d\nu(\boldsymbol{A})$.*

Proof. Observation 1 implies that $\int_{\mathsf{SO}(d)} \mathcal{M}_R(S_{\boldsymbol{A}})d\nu(\boldsymbol{A}) = \sum_{i=1}^d \int_{\mathsf{SO}(d)} \mathsf{med}(S_{\boldsymbol{A}\boldsymbol{e}_i})d\nu(\boldsymbol{A})$. Consider the map $\psi_i : \mathsf{SO}(d) \to S^{d-1}$ given by $\psi(\boldsymbol{A}) = \boldsymbol{A}\boldsymbol{e}_i$ and the function $f : S^{d-1} \to \mathbb{R}^d$ given by $f(\boldsymbol{x}) = \mathsf{med}(S_{\boldsymbol{x}})$, where $||\boldsymbol{x}|| = 1$. Therefore, we have $\int_{\mathsf{SO}(d)} \mathsf{med}(S_{\boldsymbol{A}\boldsymbol{e}_i})d\nu(\boldsymbol{A}) = \int_{\mathsf{SO}(d)} f(\psi_i(\boldsymbol{A}))d\nu(\boldsymbol{A}) = \int_{S^{d-1}} f(x)d\mu(\boldsymbol{x}) = \int_{S^{d-1}} \mathsf{med}(S_{\boldsymbol{x}})d\mu(\boldsymbol{x})$ by Result 1, where μ is the normalized uniform measure over S^{d-1}. Therefore, $\int_{\mathsf{SO}(d)} \mathcal{M}_R(S_{\boldsymbol{A}})d\nu(\boldsymbol{A}) = \sum_{i=1}^d \int_{\mathsf{SO}(d)} \mathsf{med}(S_{\boldsymbol{A}\boldsymbol{e}_i})d\nu(\boldsymbol{A}) = d\int_{S^{d-1}} \mathsf{med}(S_{\boldsymbol{x}})d\mu(\boldsymbol{x})$.

Next, observe that the denominator in the definition of the projection median is the volume of the d-dimensional unit sphere. Therefore, from Equation 1, we get that $\mathcal{M}_P(S) = d\int_{S^{d-1}} \mathsf{med}(S_{\boldsymbol{x}})d\mu(\boldsymbol{x}) = \int_{\mathsf{SO}(d)} \mathcal{M}_R(S_{\boldsymbol{A}})d\nu(\boldsymbol{A})$. □

3.1 Approximation

Equipped with the results of the previous section, we now proceed to determine the approximation factor of the projection median with respect to the Euclidean median in $\mathbb{R}^d$. In order to find the approximation factor, we need to bound the following ratio:

$$\lambda(d) = \frac{\sum_{i=1}^n ||\mathcal{M}_P(S) - \boldsymbol{p}_i||}{\sum_{i=1}^n ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \tag{2}$$

Using Lemma 1, we now write the above ratio as:

$$\begin{aligned}\lambda(d) &= \frac{\sum_{i=1}^n ||\int_{\mathsf{SO}(d)} \mathcal{M}_R(S_{\boldsymbol{A}})d\nu(\boldsymbol{A}) - \boldsymbol{p}_i||}{\sum_{i=1}^n ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\ &= \frac{\sum_{i=1}^n ||\int_{\mathsf{SO}(d)} (\mathcal{M}_R(S_{\boldsymbol{A}}) - \boldsymbol{p}_i)d\nu(\boldsymbol{A})||}{\sum_{i=1}^n ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\ &\leq \frac{\sum_{i=1}^n \int_{\mathsf{SO}(d)} ||\mathcal{M}_R(S_{\boldsymbol{A}}) - \boldsymbol{p}_i||d\nu(\boldsymbol{A})}{\sum_{i=1}^n ||\mathcal{M}_E(S) - \boldsymbol{p}_i||}\end{aligned}$$

where the last step follows from triangle inequality. Now, let $\boldsymbol{u}_i = \frac{\mathcal{M}_E(S) - \boldsymbol{p}_i}{||\mathcal{M}_E(S) - \boldsymbol{p}_i||}$ and observe that for all $\boldsymbol{A}$ and $\boldsymbol{x}$, $||\boldsymbol{x}|| \leq |\boldsymbol{x}|_{\boldsymbol{A}}$, where $|\boldsymbol{x}|_{\boldsymbol{A}}$ is the ℓ_1 norm of $\boldsymbol{x}$ in the coordinate system $C_{\boldsymbol{A}}$. This implies that

$$\begin{aligned}\lambda(d) &\leq \frac{\sum_{i=1}^n \int_{\mathsf{SO}(d)} |\mathcal{M}_R(S_{\boldsymbol{A}}) - \boldsymbol{p}_i|_{\boldsymbol{A}}d\nu(\boldsymbol{A})}{\sum_{i=1}^n ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\ &\leq \frac{\sum_{i=1}^n \int_{\mathsf{SO}(d)} |\mathcal{M}_E(S) - \boldsymbol{p}_i|_{\boldsymbol{A}}d\nu(\boldsymbol{A})}{\sum_{i=1}^n ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\ &= \frac{\sum_{i=1}^n ||\mathcal{M}_E(S) - \boldsymbol{p}_i|| \int_{\mathsf{SO}(d)} |\boldsymbol{u}_i|_{\boldsymbol{A}}d\nu(\boldsymbol{A})}{\sum_{i=1}^n ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\ &= \frac{\sum_{i=1}^n ||\mathcal{M}_E(S) - \boldsymbol{p}_i|| \int_{\mathsf{SO}(d)} \sum_{j=1}^d |\langle \boldsymbol{u}_i, \boldsymbol{A}\boldsymbol{e}_j\rangle| d\nu(\boldsymbol{A})}{\sum_{i=1}^n ||\mathcal{M}_E(S) - \boldsymbol{p}_i||} \\ &= \frac{\sum_{i=1}^n ||\mathcal{M}_E(S) - \boldsymbol{p}_i|| \sum_{j=1}^d \int_{\mathsf{SO}(d)} |(\boldsymbol{A}'\boldsymbol{u}_i)'\boldsymbol{e}_j| d\nu(\boldsymbol{A})}{\sum_{i=1}^n ||\mathcal{M}_E(S) - \boldsymbol{p}_i||}.\end{aligned} \tag{3}$$

We can now simplify Equation 3 using Result 1.

Observation 2 $\lambda(d) \leq d\int_{S^{d-1}} |\boldsymbol{x}'\boldsymbol{e}_1| d\mu(\boldsymbol{x})$.

Proof. Consider the function $\psi_i : \mathsf{SO}(d) \rightarrow S^{d-1}$ given by $\psi_i(\boldsymbol{A}) = \boldsymbol{A}'\boldsymbol{u}_i$, for $i \in \{1, 2, \ldots, n\}$. Let for $j \in \{1, 2, \ldots, d\}$, $f_j : S^{d-1} \rightarrow \mathbb{R}$ be defined as $f_j(\boldsymbol{x}) = |\boldsymbol{x}'\boldsymbol{e}_j|$, where $||\boldsymbol{x}|| = 1$. Then from Result 1, $\int_{\mathsf{SO}(d)} |(\boldsymbol{A}'\boldsymbol{u}_i)'\boldsymbol{e}_j| d\nu(\boldsymbol{A}) = \int_{\mathsf{SO}(d)} f_j(\psi_i(\boldsymbol{A})) d\nu(\boldsymbol{A}) = \int_{S^{d-1}} f_j(\boldsymbol{x}) d\mu(\boldsymbol{x}) = \int_{S^{d-1}} |\boldsymbol{x}'\boldsymbol{e}_j| d\mu(\boldsymbol{x})$. Since the integral are taken over all the units vectors in S^{d-1}, it is easy to see that for $j \neq k$ we have, $\int_{S^{d-1}} |\boldsymbol{x}'\boldsymbol{e}_j| d\mu(\boldsymbol{x}) = \int_{S^{d-1}} |\boldsymbol{x}'\boldsymbol{e}_k| d\mu(\boldsymbol{x})$. The proof now follows from Equation 3. □

We shall now prove the main result of this paper, where we determine a upper bound on $\lambda(d)$.

Theorem 2 *For any $d \geq 2$, the d-dimensional projection median provides a $J(d)$-approximation of the d-dimensional Euclidean median, where $J(d) = (d/\pi)B(d/2, 1/2)$.*

Proof. Let $\phi_1, \phi_2, \ldots, \phi_{d-1}$ denote the angular coordinates in the d-dimensional spherical coordinates where the last angle ϕ_{d-1} has a range of 2π while the other angles have a range of π [9]. Observe that $\boldsymbol{x}'\boldsymbol{e}_1$ is the first coordinate of the vector $\boldsymbol{x}$. Using d-dimensional spherical coordinates we get $\boldsymbol{x}'\boldsymbol{e}_1 = \cos\phi_1$ [9]. If $d_{S^{d-1}}V$ denotes the area element of S^{d-1}, which is also the non-normalized uniform measure over S^{d-1}, then using the d-dimensional spherical coordinate system we obtain:

$$\begin{aligned} \lambda(d) &\leq d\frac{\int_0^{\pi}\int_0^{\pi}\cdots\int_0^{2\pi} |\cos(\phi_1)| d_{S^{d-1}}V}{\int_0^{\pi/2}\int_0^{\pi/2}\cdots\int_0^{\pi} d_{S^{d-1}}V} \\ &= d\frac{\int_0^{\pi} |\cos(\phi_1)| \sin^{d-2}(\phi_1) d\phi_1}{\int_0^{\pi} \sin^{d-2}(\phi_1) d\phi_1}. \end{aligned} \quad (4)$$

The result now follows after minor simplifications and on using the fact that for any two reals $a, b > -1$, $\int_0^{\pi/2} \sin^a\theta \cos^b\theta d\theta = \frac{1}{2} \cdot B(\frac{a+1}{2}, \frac{b+1}{2})$ [6]. □

Since d is an integer, using standard formulae of Beta functions [6], it is possible to find out the explicit expression for $J(d)$ as follows

$$J(d) = \begin{cases} \frac{2d}{\pi} \cdot \frac{(d-2)(d-4)\ldots 4.2}{(d-1)(d-3)\ldots 5.3}, & \text{if } d \text{ is even;} \\ d.\frac{(d-2)(d-4)\ldots 3.1}{(d-1)(d-3)\ldots 4.2}, & \text{if } d \text{ is odd.} \end{cases}$$

For the special case $d = 3$, $J(d) = 3/2$, which implies that in $\mathbb{R}^3$ the projection median gives a (3/2)-approximation of the Euclidean median, thus answering a question posed by Durocher [4].

As mentioned earlier, Durocher and Kirkpatrick [5] proved the lower bound of $\sqrt{4/\pi^2 + 1}$ on the approximation factor of the projection median in $\mathbb{R}^2$. Note that the same quantity provides a lower bound on the approximation factor in $\mathbb{R}^d$ as well. However, the problem of obtaining a non-trivial lower bound on the approximation factor in $\mathbb{R}^d$ remains open.

3.2 Stability

In 2D, Durocher and Kirkpatrick [5] showed the tight bound on the stability of the projection median is the reciprocal of the upper bound on the approximating factor. We claim that same is true for higher dimensions as well. The proof follows by directly generalizing the techniques of Durocher [4] to higher dimensions, which finally involves integration of functions over the area element of the d-dimensional sphere.

Theorem 3 *For any $d \geq 2$, the d-dimensional projection median is $1/J(d)$-stable and this bound is tight.* □

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