Six-way Equipartitioning by Three Lines in the Plane

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Abstract

Let K be a convex body in the plane with area 1. It is well-known that there exists three concurrent lines that divide K into six regions, each with area $1/6$. Probably less well-known is the fact that three lines can *never* partition K into *seven* regions, each with area $1/7$. It is also known that if three lines partition K into seven regions, and if six of them have equal area, then it must be the central, triangular region that has area different from the other six. It is somewhat curious that the main existence question for such a partitioning had remained open. Here we settle this question by showing that such partitions always exist and, in a certain sense, are unique.

1 Introduction

In 1949 Buck and Buck [7] showed that for any convex body K in the plane, there always exists three concurrent lines that equipartition K; that is, each of the six sectors they define cuts off the same area in K. Courant and Robbins [8] later gave a simple proof by continuity, and there has followed a succession of interesting results [1], [2], [3], [4], [5], [6], [12], [13], [15], [16], [17], [19], [20] about the possibilities/impossibilities of partitioning measures in various ways, facts that are consequences of analysis, topology, and algebra (see especially Matoušek's beautiful book on the Borsuk-Ulam theorem [14]).

Here we reopen an apparently unexplored aspect of partitioning a convex body with three lines. First, we will assume $A(K) = 1$, where we write $A(S)$ for the area of the set $S \subset R^2$. In general, three lines will divide the plane into seven regions. The central, bounded one is a triangle T that degenerates into a point if the lines are concurrent. In [7] Buck and Buck also showed that no convex set K can be partitioned by three lines into seven regions, each with area $1/7$. They then asked whether there are partitions where six of the seven regions each has area $(1-z)/6$, and the seventh has area z; the previous statement shows this is impossible for *every* K when $z = 1/7$, and their original result shows it is always possible for *every* K when $z = 0$. Finally they showed that if such a six-way partitioning of K did exist for some $z > 0$, then it must be the central triangle T that has area z.

They also stated the conjecture that if K has a six-way equipartition with three lines, then $A(T) = z$ is at most $z_0 = 1/49$. This is the value that does occur when K is, itself, a triangle; it is easily seen that when K is a triangle, it may be six-way equipartitioned by lines parallel to its own sides. The fact that NO convex body K can have a six-way equipartition in which the central triangle has $A(T) > z_0$ was later proved by Sholander [18] (see also [9], [10], [11]). He thus showed the triangle to be the extreme convex body K admitting a six-way equipartitioning - it has the largest possible area in the unequal region.

However the general existence question for six-way equipartitions of planar convex bodies has remained open since the original paper of Buck and Buck! This is a surprising fact. We reopen the question here and elucidate the existence of such partitions. To make things concrete we give

Definition 1: *Given a convex body K with $A(K) = 1$, lines ℓ_1, ℓ_2, ℓ_3 form a six-way equipartition of K if*

1. *the points $P_{ij} = \ell_i \cap \ell_j, i < j$ are in K,*
2. *the triangle $T = \Delta P_{12}P_{13}P_{23}$ has area z, and*
3. *each of the six regions of $K \setminus T$ has area $(1-z)/6$.*

The main new fact is

Theorem 1 *Given a convex body $K \subset R^2$ with $A(K) = 1$ and a unit vector $v \in R^2$, there exists a unique trio of lines that form a six-way equipartition of K, with one of them having normal vector v.*

Before this result, the basic question of the existence had been open, except when $z = 0$.

According to Theorem 1, for each convex body K and $\theta \in [0, 2\pi)$ there is a unique six-way equipartition of K where one of the lines has normal vector $v = (\cos(\theta), \sin(\theta))$ and we write $f_K(\theta)$ as the area t of the central triangle in the partition. We can use this function to characterize certain convex sets K. For example f is identically zero if K is radially symmetric.

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Also, there is an interesting algorithmic problem when seeking six-way partitions of n given points in general position in R^2; this is a variant of the original problem where we use counting measure instead of Lebesgue measure for the "size" of a set. Finally, we can pose an analogous partitioning problem where the object is to partition a convex body K using three lines, but now the goal is to have the area of the smallest region as close as possible to the area of the largest region. We conjecture that the triangle is the extreme convex body for this property as well. These final two points will be postponed for the journal version. Here we mainly discuss Theorem 1 and the ideas used in its proof.

2 Sketch of the Proof

The theorem depends on continuity and a geometric property of convex sets that may be of independent interest (Lemma 4). Without loss of generality we may take the given normal vector to be $v = (1, 0)$, so one of the three lines will be a vertical line which we denote by l_0. It will have equation $x = t$ and we write $l_0(t)$ to describe its position. We coordinatize R^2 so that $l_0(0)$ is the vertical line that bisects K (see Figure 1); i.e., K has area $1/2$ on both sides of l_0.

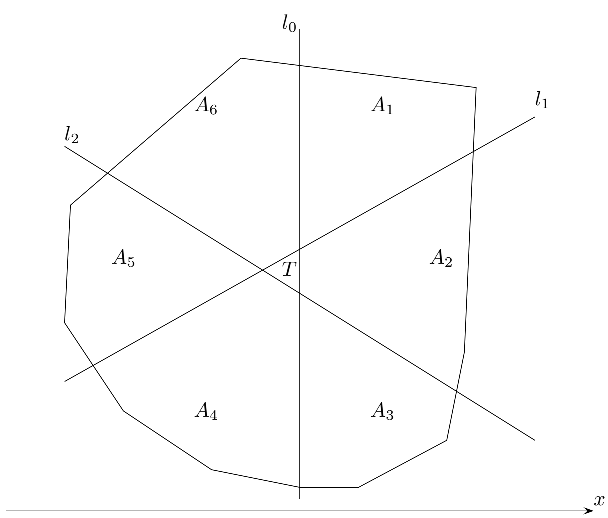

Figure 1: Three lines and seven regions. l_0, l_1 and l_2 are bisectors of K when $t = 0$.

We choose two other bisecting lines for K, l_1 and l_2, both uniquely determined by the ham-sandwich theorem to have the following properties: l_1 has above it $1/3$ of the part of K that is to the right of l_0, and $2/3$ of the part of K that is to the left of l_0; l_2 has above it $1/3$ of the part of K that is to the left of l_0 and $2/3$ of the part of K that is to the right of l_0. This is the initial configuration based on $l_0(t)$ when $t = 0$. If these lines are concurrent, T is empty and we have a six-way equipartition. The uniqueness argument will show there is no other six-way equipartition where one line is vertical.

Otherwise $T \neq \phi$ and without loss of generality we may assume it lies to the left of l_0 as shown in Figure 1. In this case ($t = 0$) the seven regions shown in Figure 1 have areas $A_1(t), \ldots, A_6(t)$ and $z > 0$ (for the area of T). By construction, $A_1(0) = A_2(0) = A_3(0) = 1/6$, $A_4(0) = 1/6 - z = A_6(0)$, and $A_5(0) = 1/6 + z > 1/6$. Therefore there is no six-way equipartitioning using $l_0(t), t = 0$.

Next, we observe that if three lines (m_0, m_1 and m_2) have their pairwise intersections in K, they create a six-way equipartition if and only if

(P1) Each line m_i divides K into two parts: K_i^- with area $\alpha \leq 1/2$, and the other part, K_i^+, with area $1 - \alpha$;

(P2) For each i, K_i^- is partitioned by the other two lines into three regions of equal area.

Clearly, when m_0, m_1 and m_2 satisfy both (P1) and (P2), the central triangle $T = \bigcap_{i=1}^{3} K_i^+$.

As there is no six-way equipartition when $t = 0$ and $z = Area(T) > 0$, we translate vertical line l_0 to $l_0(t)$, $t \neq 0$, and write $\alpha < 1/2$ for the area of K_0^-, the smaller part of K cut off by by $l_0(t)$. By the ham-sandwich theorem there is a unique pair of lines l_1 and l_2 that satisfy the following invariants:

(I1) $\text{Area}(l_1^+ \cap K_0^-) = \alpha/3$; $\text{Area}(l_1^+ \cap K_0^+) = 2\alpha/3$.

(I2) $\text{Area}(l_2^+ \cap K_0^-) = 2\alpha/3$; $\text{Area}(l_2^+ \cap K_0^+) = 1 - 5\alpha/3$.

Here, l_i^+ denotes the halfspace above $l_i, i = 1, 2$. The region below l_2 is the smaller part of K.

Its easy to verify the following statements (see e.g., Fig. 1, but think of $t > 0$ so K_0^- is on the right of l_0):

Fact 1: *If lines l_1, l_2 satisfy both invariants (I1) and (I2), the trio automatically satisfies (P1).*

Fact 2: *It is also the case that if l_1 and l_2 meet within K_0^+, then the trio l_0, l_1, and l_2 will also satisfy (P2), as long as A_5 has the same area as A_1:* I1 and I2 imply that A_1, A_2, A_3 each have area $\alpha/3$; if A_5 also has area $\alpha/3$, the invariants imply that A_4 and A_6 do as well. In this case, the lines form a six-way equipartition.

Fact 3: *When l_0, l_1, l_2 equipartition, T must have area $z = 1 - 2\alpha$, α the area in K of the smaller halfspace of each l_i.*

These facts hold as well if $t < 0$.

We begin as in Figure 1 with $l_0(t), t = 0$, the vertical line bisecting K. We will move l_0 continuously and for each t, choose $l_1(t)$ and $l_2(t)$ to satisfy invariants I_1 and I_2. We consider all t for which $l_0(t)$ meets K and

$l_1(t) \cap l_2(t) \in K$. A basic fact in the argument - probably already known - is

Lemma 2 *The functions $A_i(t)$, $i = 1, \ldots, 6$ and $z(t) = Area(T(t))$ are continuously differentiable.*

The proof is elementary and left as an exercise.

Returning now to the situation shown in Figure 1 at $t = 0$, lines l_0, l_1, l_2 each bisect K, and l_1 and l_2 meet to the left of l_0. We will first argue (Lemma 3) that there is no equipartitioning using $l_0(t)$ when $t < 0$. Next, let R denote the x-coordinate of the right vertical tangent to K. We will show that as t moves continuously from 0 toward R, there is a smallest $t = t' > 0$ at which $A_1(t') = \cdots = A_6(t')$. The last part of the proof of Theorem 1 argues uniqueness showing that t' is the only value where all six areas are the same.

We first observe that there can be no six-way equipartition when $t < 0$.

Lemma 3 *Suppose $l_0(t), l_1(t), l_2(t)$ are bisecting lines as in Figure 1, $t = 0$, and with the central triangle on the left of l_0. Then there is no six-way equipartition of K using $l_0(t)$ if $t < 0$.*

Proof: Fix $t < 0$, and move l_0 to $x = t$, $t > L$, $x = L$ the left vertical tangent of K. In this case, K_0^- is on the left of l_0 and has area $\alpha < 1/2$. If l_1 and l_2 dont meet in K there is nothing to prove. Otherwise, if triangle $T(t)$ is again on the left of l_0, the invariants imply that $A_6(t) = \alpha/3 - z(t)$ and $A_5(t) = \alpha/3 + z(t)$, where $z(t)$ is the area of $T(t)$. These areas can be equal only if $z(t) = 0$, contradicting the assumption that T is on the left of $l_0(t)$.

On the other hand suppose that triangle $T(t)$ is on the right of $l_0(t)$, $t < 0$. Lemma 2 implies that there exists $t' \in (t, 0)$ where the three lines are concurrent. The invariants imply this can occur only when all three lines bisect K, clearly impossible if $t' \neq 0$. ■

So we start with $l_0(t)$, $t = 0$, where it is the vertical bisector of K, and continuously translate until $t = R$, where it is the right vertical tangent to K. For each t we maintain the invariants for l_1 and l_2, as long as they meet in K. Using the same reasoning as in the proof of Lemma 3, if T is on the left of $l_0(0)$, it remains on the left as l_0 moves to the right.

Initially, with T on the left of $l_0(0)$, the invariants imply that $A_5(0) = 1/6 + z$ and that $A_1(0) = A_2(0) = A_3(0) = 1/6$. Therefore $f(t) \equiv A_5(t) - A_1(t)$ is positive at $t = 0$. We argue that there is a $t^* \in (0, R)$ for which l_1 and l_2 meet on the boundary of K. Therefore $A_5(t^*) = 0$, so $f(t^*) < 0$. By the continuity of f there must be a $\bar{t} \in (0, t^*)$ for which $f(\bar{t}) = 0$, so by the earlier **Fact 2**, l_0, l_1, l_2 form a six-way equipartition. Finally, we can show that A_5, A_1 and f are all decreasing, a fact that guarantees the uniqueness of the equipartition. A main tool behind these statements is the following lemma, possibly of independent interest, and based on the situation described in Figure 2.

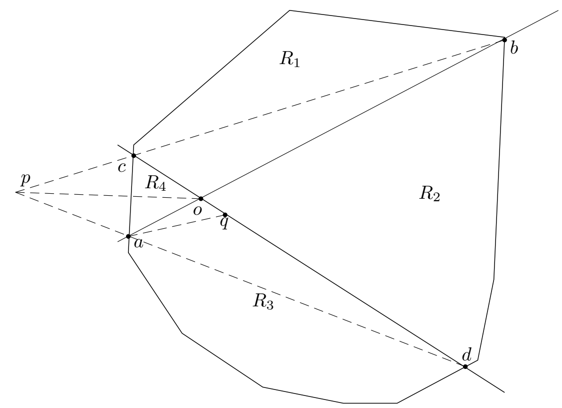

Figure 2: Lengths of segments of two cuts.

Lemma 4 *Let K be a convex body of area 1, and let a and b be two points on the boundary, in clockwise order, so that the region in K above the line $\overline{ab}$ has area $\alpha < 1/2$. Let c be between a and b (in clockwise order) and d be points so that the region in K below the line $\overline{cd}$ also has area α. These lines meet at a point $o \in K$, and divide K into four regions: R_1 above both lines; R_3 below both lines; R_2 above $\overline{cd}$ and below $\overline{ab}$; and R_4, as in Figure 2. We have the following inequalities:*

$$\frac{|ao|}{|ab|} + \frac{|co|}{|cd|} \geq \frac{Area(R_4)}{Area(R_1) + Area(R_4)},$$

$$\frac{|bo|}{|ab|} + \frac{|do|}{|cd|} \geq \frac{Area(R_2)}{Area(R_1) + Area(R_2)}.$$

Also, the larger of the sums on the left hand side is at least 1, and if one of them is 1, then both are.

Proof: Draw the lines $\overline{ad}$ and $\overline{bc}$. If they are parallel, then both sums are 1.

Otherwise, w.l.o.g, $\overline{ad}$ meets $\overline{bc}$ at some point p, as in Figure 2. Join o and p and draw the line $\overline{aq}$ which is parallel to $\overline{bc}$ and meets $\overline{od}$ at q. It is easy to check that

$$\frac{|bo|}{|ab|} + \frac{|do|}{|cd|} = \frac{|co|}{|cq|} + \frac{|do|}{|cd|} > \frac{|co|}{|cd|} + \frac{|do|}{|cd|} = 1$$

so we have

$$\frac{|co|}{|cd|} = \frac{Area(\triangle pco)}{Area(\triangle pcd)} \text{ and } \frac{|ao|}{|ab|} = \frac{Area(\triangle pao)}{Area(\triangle pab)}.$$

If Area$(\triangle pab) \geq Area(\triangle pcd)$,

$$\frac{|ao|}{|ab|} + \frac{|co|}{|cd|} \geq \frac{Area(pcoa)}{Area(pcoa) + Area(\triangle ocb)}.$$

By the convexity of K, R_4 is contained in quadrilateral $pcoa$, and $\triangle ocb$ is contained in R_1. Therefore

$$\frac{|ao|}{|ab|} + \frac{|co|}{|cd|} \geq \frac{Area(R_4)}{Area(R_4) + Area(\triangle ocb)}$$

$$\geq \frac{Area(R_4)}{Area(R_1) + Area(R_4)}.$$

It is also clear that for the smaller sum, the inequality becomes equality only when R_4 is exactly the quadrilateral $pcoa$ (that is, $\triangle pac \subseteq R_4$), and R_1, R_3 are triangles $\triangle ocb$ and $\triangle oad$ which have the same area. ∎

3 Some Related Issues

According to Theorem 1, for every convex body K of area 1, and for every $\theta \in (0, 2\pi]$ there is a unique trio of lines that form a six-way equipartition of K where $(\cos\theta, \sin\theta)$ is normal to one of the lines. Writing $A(T)$ for the area of the central triangle in this partition, the function

$$f_K(\theta) \equiv A(T)$$

is well defined. Since $\theta + \pi$ will give the same equipartition as θ we only need to consider f on $(0, \pi]$. It is easy to see that f is always continuous and must be zero for at least three distinct values of $\theta \in (0, \pi]$ (once for each line in a concurrent six-way equipartition). By Sholander's results, $\max f_K(\theta) \leq 1/49$ for all $K \in R^2$ with area 1. In fact $f_K(\theta) = 1/49$ only when K is a triangle and one side has normal $(\cos\theta, \sin\theta)$.

It is not hard to prove the following statements:

Lemma 5 *If K is a centrally symmetric convex body, three lines in a 6-way equipartion must always be concurrent; i.e., $f_K(\theta) \equiv 0$.*

Lemma 6 *If K is symmetric about a line l, there exists a concurrent 6-way equipartition where l is one of lines.*

Finally f behaves nicely for regular polygons. Specifically:

- For even n, $f_K(\theta) \equiv 0$ since the regular n-gon has a center of symmetry.
- For odd n, the regular n-gon has n lines of symmetry, one through each vertex. If n is divisible by 3, the concurrent 6-way equipartition with one line incident with a vertex must have all lines of a six-way equipartition incident with vertices. So we have n distinct values where $f_K(\theta) = 0$.
- If n is odd and not divisible by 3, we have n concurrent 6-way equipartitions with one of the lines incident with a vertex. Therefore all n concurrent 6-way equipartitions are different. So $f_K(\theta) = 0$ for at least $3n$ distinct θ's in $[0, \pi]$.

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