Morpion Solitaire 5D: a new upper bound of 121 on the maximum score

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Abstract

Morpion Solitaire is a pencil-and-paper game for a single player. A move in this game consists of putting a cross at a lattice point and then drawing a line segment that passes through exactly five consecutive crosses. The objective is to make as many moves as possible, starting from a standard initial configuration of crosses. For one of the variants of this game, called 5D, we prove an upper bound of 121 on the number of moves. This is done by introducing *line-based analysis*, and improves the known upper bound of 138 obtained by potential-based analysis.

Keywords: *pencil-and-paper game, lattice points, line-based analysis.*

1 Introduction

Morpion Solitaire, also known as *Join Five*, is a game played alone with a pencil and paper, and it is popular in several countries [4]. A move in this game consists of drawing a cross and a line segment on an infinite square lattice. The line segment has to pass through exactly five consecutive crosses including the one that has just been placed. The objective is to make as many moves as possible starting from a given initial configuration. We call the number of moves the *score*. There are two variants of this game according to how two line segments can touch each other.

Demaine et al. [6] studied generalizations of the game and their computational complexity, and show that a generalized Morpion Solitaire is NP-hard and that its maximum score is hard to approximate. Another target of interest is the maximum scores or their lower and upper bounds. Recently, computing maximum scores was used as a test problem to evaluate the effectiveness of the Monte-Carlo tree search method, which has been attracting rising attention as a promising approach in game programming [5, 9].

In this paper, we focus on the 5D variant of the game, and show improved upper bounds on the maximum score. We first show that the known upper bound of 138 can be improved to 136 by pushing on the existing potential-based approach. Next we introduce a line-based approach and further improve the bound to 121. We also try to organize and present related results, since there are relatively few research papers on this topic.

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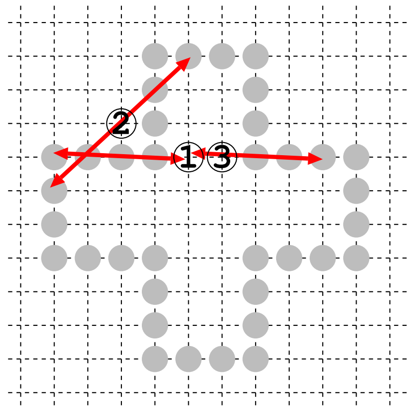

Figure 1: The standard initial board layout for Morpion Solitaire 5D and 5T, and an example of the first three moves. Each cross placed in these moves is denoted by a number surrounded by a circle. Move 3 is allowed in 5T (touching) but not in 5D (disjoint).

2 Rules and Records

2.1 Rules

Morpion Solitaire is played on an infinite square lattice. Initially 36 crosses are drawn on lattice points so that they form a large cross shape with edge length 4 as shown in Figure 1. In this figure, a cross is denoted by a circle. (In this paper, the length of a line segment means the number of crosses covered by it.)

A *move* consists of the following two steps applied in this order. The objective of this game is to maximize the number of moves.

1. Draw a new *cross* on a lattice point which is empty (no cross exists) on the current board.
2. Draw a segment of length 5 (called a *line*) that passes through exactly five consecutive crosses including the one drawn in step 1 of this move. Here, the line can be drawn in either one of the four directions, vertical, horizontal, or diagonal. Two lines in the same direction may not overlap.

There are two variants of this game depending on whether two lines in the same direction can touch (5T) or have to be disjoint (5D) (Figure 1). We mainly discuss about 5D in this paper.

When a line L passes a cross C, we say that L *covers* the cross or the lattice point on which it is drawn. We sometimes call a board after move N a board at move N. Also we sometimes denote a cross and a line drawn in move N by C_N and L_N, respectively.

2.2 Records

The above definition of the game can be extended to αD and αT, where the lines have length α and the edges of the large cross in the initial configuration have length $\alpha - 1$, however, the maximum scores are known for all variants except $\alpha = 5$. For 3T and 3D, the maximum scores are not bounded, as there are sequences of moves that can be repeated infinitely [6]. For 6T and 6D, we can easily see that the maximum score is 12. For 4T and 4D, there used to be gaps between the maximum achieved scores and the upper bounds in the past, but in 2007, 62 and 35 moves were achieved for 4T and 4D, respectively [7], and these scores were proved to be optimal in 2008 [4].

Table 1 [4] shows the current maximum scores of 5T and 5D. We briefly explain how the records of these two variants have been developed.

5T. Bruneau achieved 170 in 1976 by hand [2]. In 2010, by computer, Akiyama, Komiya and Kotani [1] used Monte-Carlo tree search to achieve 145 and 146, which were still less than human's record at that time. From 2010 to 2011, also by computer, Rosin achieved 172, beating human's record [3]. Rosin [9] improved the record to 177 in 2011, and the current record is 178 [10]. An upper bound of 705 on the maximum score is known [6].

5D. According to Demaine et al. [6], 68 moves was achieved by hand in 1999. Cazenave [5] established 80 in 2008, and then Rosin [9] improved it to 82 in 2010, both by computers. As for upper bounds, Demaine et al. [6] showed 141 in 2006 [6] and Karjalainen showed 138 in 2011 [8].

Recent records of maximum scores of both 5T and 5D were obtained by computers. The framework used for this was Monte-Carlo tree search or its extensions, which are known to produce excellent results in designing computer programs, for example, for playing Shogi or Go against humans.

Table 1: Records on Morpion Solitaire 5T and 5D: their maximum achieved scores and proven upper bounds.

game type	best achieved score	upper bound
5T	178	705
5D	82	138

Hereafter, in this paper, we focus only on 5D variant and aim to improve the upper bound on its maximum score, which is known to be 138.

3 Potential-based Analysis of Upper Bounds

The known upper bound of 138 on the maximum score of Morpion Solitaire 5D is obtained by arguments using 'potentials'. In this section, we explain potentials and the related results, and then show that the upper bound can be improved to 136 by a more detailed analysis based on this approach.

3.1 Preceding Research

The notion of potential in the analysis of Morpion Solitaire seems to have been originally introduced in folklore discussions and was used by Demaine et al. [6]. The *potential* of a cross on a board is the number of additional lines that can cover it. Since a cross can be covered by at most four lines (in the vertical, horizontal and two diagonal directions), the potential of a cross C is formally given by

$$4 - (\text{number of lines that cover } C).$$

We define the *total potential* of a board to be the sum of the potentials of all crosses on that board.

Now we can observe the following three facts about Morpion Solitaire 5D.

Observations

(i) The total potential of the initial board is 144.
(ii) The total potential decreases at least by 1 in every move.
(iii) At any time, playing the next move requires at least a total potential 4.

We have (i) because initially there are 36 crosses, each of which has potential 4. We have (ii) because step 1 of a move in 5D adds 4 to the total potential, and step 2 decreases the potential by 5.

Demaine et al. [6] showed the following upper bound based on the above three observations.

Theorem 1 ([6]) *The number of moves in Morpion Solitaire 5D cannot exceed* 141*.*

To see this, let M be the maximum score (the number of moves). The total potential after $M - 1$ moves must be at least 4, that is, $144 - (M - 1) \geq 4$.

Karjalainen [8] improved this argument and obtained the following result by showing that the total potential at any time is at least 6.

Theorem 2 ([8]) *The number of moves in Morpion Solitaire 5D cannot exceed* 138*.*

To see this, let M be the maximum score and consider the last three moves. The crosses drawn in the last three moves M, $M-1$ and $M-2$ are eventually covered by one line, by at most two lines, and by at most three lines, respectively. In other words, those crosses have potentials 3, ≥ 2, and ≥ 1, respectively, at the end of the game. This implies $144-M \geq 6$, and thus $M \leq 138$.

3.2 Improvements

We next show some small improvements of maximum scores in the framework of potential-based analysis. Our improvements are obtained by focusing on the last four moves. We denote the potential of a cross C on a board by $p(C)$.

Lemma 3 *The sum of the potentials of the three crosses that are drawn in the last three moves is greater than or equal to* 7.

Proof. Consider the board at move N. According to the arguments for Theorem 2, $p(C_N) = 3$, $p(C_{N-1}) \geq 2$ and $p(C_{N-2}) \geq 1$ hold for crosses C_N, C_{N-1} and C_{N-2} at moves N, $N-1$ and $N-2$, respectively. Here, $p(C_N) = 3$, $p(C_{N-1}) = 2$ and $p(C_{N-2}) = 1$ cannot be satisfied simultaneously. Suppose they can. Then line L_{N-1} has to cover cross C_{N-2} as well as C_{N-1}, and line L_N has to cover both crosses C_{N-2} and C_{N-1} as well as cross C_N, and this forces such two lines L_{N-1} and L_N to overlap. This contradicts the rules of Morpion Solitaire, and thus $p(C_N) + p(C_{N-1}) + p(C_{N-2}) > 6$ holds. □

Lemma 3 alone improves an upper bound to 137, and we can save one more move.

Theorem 4 *The number of moves in Morpion Solitaire 5D cannot exceed* 136.

Proof. Let M be the maximum score, and consider a board at move $M-1$. First, we can see that in order that move M is feasible, there exists a cross C other than C_{M-1}, C_{M-2} and C_{M-3} with $p(C) \geq 1$. Then we determine the total potential of board $M-1$ by a case analysis; whether line L_M drawn in move M covers all three crosses C_{M-1}, C_{M-2} and C_{M-3}, or not.

Case 1: line L_M covers all crosses C_{M-1}, C_{M-2} and C_{M-3}. In this case, three crosses C_{M-1}, C_{M-2} and C_{M-3} lie on a common lattice line. Since no two lines can overlap, line L_{M-2} that covers C_{M-3} and line L_{M-1} that covers both C_{M-2} and C_{M-3} are not compatible. Hence, $p(C_{M-1}) = p(C_{M-2}) = p(C_{M-3}) = 3$ holds. This, together with the fact that there exists a cross C with $p(C) \geq 1$ other than C_{M-1}, C_{M-2} and C_{M-3} guarantees $p(C_{M-1})+p(C_{M-2})+p(C_{M-3})+p(C) \geq 10$.

Case 2: line L_M does not cover at least one of crosses C_{M-1}, C_{M-2} or C_{M-3}. In this case, there must exist two different crosses C and C' with $p(C) \geq 1$ and $p(C') \geq 1$. Therefore, together with Lemma 3, $p(C_{M-1}) + p(C_{M-2}) + p(C_{M-3}) + p(C) + p(C') \geq 9$ holds.

To put both cases together, the total potential of an arbitrary board of move $M-1$ is greater than or equal to 9. That is, $144 - (M-1) \geq 9$ holds, which implies $M \leq 136$. □

4 Line-based Analysis of Upper Bounds

In this section, we introduce a new approach for deriving better upper bounds, which we call the line-based analysis. It is based on the relationship between the number of lines on a board and the number of lattice points they cover.

The following observation is easy but crucial.

Fact After N moves, there are $N+36$ crosses and N lines.

Let $c(N)$ denote the minimum number of lattice points that are covered by N lines of length 5 in an arbitrary layout on a board (lattice plane). Then in order for a board of move N to be feasible (realizable), it has to satisfy that $c(N) \leq N+36$. Conversely, for N that satisfies $c(N) > N+36$, such a move N is infeasible. Here, since this game proceeds move by move, if a board of move N is infeasible then all boards of moves greater than N are infeasible. Hence, these observations imply the following property.

Property (Board Infeasibility Condition) If there exists N that satisfies $c(N) > N+36$, then an upper bound on the maximum score is $N-1$.

In the subsequent discussions, we derive new upper bounds on the maximum score by fully utilizing this property. In this case, however, since it is not easy to obtain $c(N)$ directly, we compute a lower bound $c'(N)$ on $c(N)$, and we try to find N that satisfies the Board Infeasibility Condition for that $c'(N)$.

4.1 An Upper Bound of 132

Here, we count the number of lattice points covered by lines by focusing on lines in one direction among four that we draw arbitrarily. Then we have the following lower bound on $c(N)$.

Claim 5 *For any move* N, $c(N) \geq \lceil \frac{N}{4} \rceil \times 5$ *holds.*

Proof. Since we draw N lines in all, there is a direction in which at least $\lceil \frac{N}{4} \rceil$ lines are drawn. They cover at least $\lceil \frac{N}{4} \rceil \times 5$ lattice points. □

By Claim 5, we have the following upper bound.

Theorem 6 *The number of moves in Morpion Solitaire 5D cannot exceed* 132.

Proof. In case that $N = 133$, $c(N) \geq \lceil \frac{133}{4} \rceil \times 5 = 170$ holds according to the claim. On the other hand, $N+36 = 169$ and this N satisfies the Board Infeasibility Condition. □

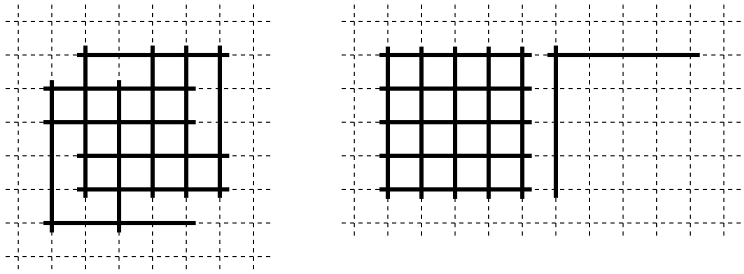

Figure 2: Six lines in each of two directions cover at least 34 lattice points.

4.2 An Upper Bound of 121

In the previous arguments, to count the number of lattice points covered by lines, we focused on the lines only in one direction. By considering two directions, we will obtain a tighter lower bound on $c(N)$. We first prove a technical lemma.

Lemma 7 *Suppose that* $5k+\beta$ $(k \geq 0; 5 > \beta \geq 0)$ *lines of length* 5 *are drawn in each of two different directions (among the possible four). Then they cover at least* $(5k+\beta)\times 5+5\beta-\beta^2$ *lattice points.*

Proof. We assume without loss of generality that the two different directions are vertical and horizontal. We color vertical lattice lines on the board periodically with different five colors, and consider the situation where $5k+\beta$ lines are drawn arbitrarily along the vertical and horizontal directions on that board. Notice here that the number of lattice points covered by line is the same in both the vertical and horizontal directions, that is $(5k+\beta)\times 5$. Then we can observe that

(i) in all the lattice points covered by lines drawn in horizontal directions, there are exactly $5k+\beta$ points colored in each one of five colors, and

(ii) if we classify the lattice points covered by vertical lines by their colors, there are at least $5k+5$ points in some β colors out of five.

Therefore, at least $\beta(5-\beta)$ out of $5k+5$ lattice points are not covered by horizontal lines. Consequently, these lines cover $(5k+\beta)\times 5+\beta(5-\beta)$ lattice points. □

Figure 2 shows two different layouts of lines where this lemma holds for $k=1$. Moreover, Lemma 7 can be generalized as follows for different lengths of lines.

Lemma 8 *Suppose that* $k\alpha+\beta$ $(k \geq 0; \alpha > \beta \geq 0)$ *lines of length* α *are drawn in each direction of two different directions on board. Then they cover at least* $\alpha(k\alpha+\beta)+\beta\alpha-\beta^2$ *lattice points.*

In the following claim, we use Lemma 7 with $\beta=1$. That is, if we draw $5k+1$ lines in each of two different directions, they cover at least $(5k+1)\times 5+4$ lattice points.

Claim 9 *For a move* N*, if* $N \not\equiv 1 \pmod 4$ *and* $\lceil \frac{N}{4} \rceil \equiv 1 \pmod 5$*, then* $c(N) \geq \lceil \frac{N}{4} \rceil \times 5+4$.

Proof. If the maximum number of lines drawn in a certain direction is greater than or equal to $\lceil \frac{N}{4} \rceil+1$, the number of lattice points covered by some line is at least $\lceil \frac{N}{4} \rceil \times 5+5$ and the statement trivially holds. So suppose otherwise, that is, the maximum number of lines drawn in one direction is equal to $\lceil \frac{N}{4} \rceil$. Since $N \not\equiv 1 \pmod 4$, at least $\lceil \frac{N}{4} \rceil$ lines are drawn in more than one direction. Since this number $\lceil \frac{N}{4} \rceil$ equals $5k+1$ for some k by assumption, we can apply Lemma 7 to conclude that the lines drawn in these two directions cover at least $\lceil \frac{N}{4} \rceil \times 5+4$ lattice points. This implies the desired inequality. □

Using this fact, we obtain a new upper bound.

Theorem 10 *The number of moves in Morpion Solitaire 5D cannot exceed* 121.

Proof. When $N=122$, since $122 \equiv 2 \pmod 4$ and $\lceil \frac{122}{4} \rceil = 1 \pmod 5$, the hypothesis of Claim 9 is satisfied, and thus $c(122) \geq 31 \times 5+4 = 159$. Since this exceeds $N+36=158$, we have the Board Infeasibility Condition. □

4.3 Remarks

We mention that a similar argument to Claim 9 holds when $N \not\equiv 1 \pmod 4$ and $\lceil \frac{N}{4} \rceil \equiv 2$ or $3 \pmod 5$). In this case, if the maximum number of lines drawn in a certain direction is $\lceil \frac{N}{4} \rceil$, the number of lattice points covered by some line is at least $\lceil \frac{N}{4} \rceil \times 5+6$. On the other hand, if the maximum number of lines drawn in a certain direction is equal to or greater than $\lceil \frac{N}{4} \rceil+1$, that is at least $\lceil \frac{N}{4} \rceil \times 5+5$. So putting these two cases together, we have $c(N) \geq \lceil \frac{N}{4} \rceil \times 5+5$. However, such N that satisfies this hypothesis and the Board Infeasibility Condition is at least 126, and thus we know that an upper bound on the maximum score can be improved to 125 at best.

We also note a limitation of this approach of trying to use $c(N)$: we cannot obtain an upper bound smaller than 102 by proving the Board Infeasibility Condition. This is because we have $c(N) \leq N+36$ for all $N \leq 102$. Figure 3 proves this inequality for $N=102$, and we can also easily confirm that it holds for all smaller N.

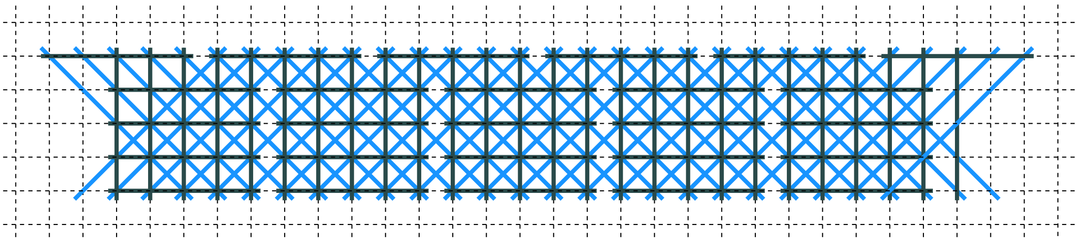

Figure 3: 102 lines cover 138 lattice points.

5 Conclusion

Although the ultimate goal of this game is to achieve the true maximum score, there are some other interesting questions.

It is possible that some idea based on our line-based analysis can further improve the upper bound on the maximum score of 5D. For example, we may consider more than two directions in those arguments. Also we may somehow take the initial layout of 36 crosses into account, which we did not in this paper.

We can also try to apply our line-based analysis to 5T. There are variants of 5T called 5T+ and 5T++, defined by relaxing the original rules about the relationship between the numbers of crosses and the lines; see Boyer's web page [4]. On this web page, he shows how to play 317 moves in the 5T++ variant, and expects that this number may be the best possible (and hence may give an upper bound for 5T). Our line-based approach may help prove upper bounds close to this.

References

[1] H. Akiyama, K. Komiya and Y. Kotani. Nested Monte-Carlo search with AMAF heuristic. *Proc. International Conference on Technologies and Applications of Artificial Intelligence (TAAI)*, pp. 172–176 (2010).

[2] P. Berloquin. Mini-morpion et nouveaux problèmes de dominos. *Science and Vie*, pp. 130–131 (1976).

[3] C. Boyer. Morpion Solitaire, le record est enfin battu!, *Science and Vie*, pp. 144–147 (2010).

[4] C. Boyer. Morpion Solitaire. `http://www.morpionsolitaire.com/`

[5] T. Cazenave. Nested Monte-Carlo search. *Proc. 26th International Joint Conference on Artificial Intelligence*, pp. 456–461 (2009).

[6] E. D. Demaine, M. L. Demaine, A. Langerman and S. Langerman. Morpion Solitaire. *Theory of Computing Systems*, 39(3), pp. 439–453 (2006).

[7] H. Hyyrö and T. Poranen. New heuristics for Morpion Solitaire. `http://www.sis.uta.fi/~tp54752/pub/morpion-article.pdf` (2007).

[8] P. Karjalainen. Bounding the upper limit of moves in the game of Morpion Solitaire 5D. `http://www.morpionsolitaire.com/Karjalainen.pdf` (2011).

[9] C. D. Rosin. Nested rollout policy adaptation for Monte Carlo tree search. *Proc. 27th International Joint Conference on Artificial Intelligence*, pp. 649–654 (2011).

[10] C. D. Rosin. A new Morpion Solitaire record via Monte-Carlo search. `http://www.chrisrosin.com/morpion/`