Stirling numbers and some partial sums of powers and products

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Abstract: We give new expressions for Stirling numbers, and some partial sums of powers and products.

AMS 2000 subject classifications: Primary 41A58; Secondary 05A10.

Keywords and phrases: Stirling numbers; Sums of powers and products.

1 Introduction and summary

Stirling numbers have applications in many areas of mathematics, probability, statistics, operations research, engineering, chemistry, physics, computer science, biology, ecology and education. Some examples of applications include: moments of the Poisson distribution, moments of fixed points of random permutations, rhyming schemes and the cereal box problem.

In this note, we derive some new expressions related to Stirling numbers. The results are organized as follows.

In Section 2, we give new expressions for

$$A_r(n) = \sum_{k=1}^{n-1} k^r$$

for $n = 2, 3, \ldots$. Section 3 gives expressions for

$$D_k(r) = \sum_{0<i_1<\cdots<i_k<r} i_1 \cdots i_k \text{ for } 0 < k < r \tag{1.1}$$

and for the Stirling numbers of the *first* kind, $s(r,k)$, (sometimes denoted as S_r^k), defined by

$$(x)_r = \sum_{k=0}^{r} s(r,k)\ x^k \tag{1.2}$$

for $r = 0, 1, \ldots$, where

$$\begin{aligned}(x)_r &= x(x-1)\cdots(x-r+1)\\ &= \Gamma(x+1)/\Gamma(x+1-r)\\ &= \sum_{k=0}^{r-1}(-1)^k D_k(r) x^{r-k},\end{aligned} \tag{1.3}$$

and we set $D_0(r) = 1$. This gives

$$s(r, r-k) = (-1)^k D_k(r).$$

Stirling numbers of the *second* kind $S(n,k)$, sometimes denoted as $\mathcal{S}_n^k$, are defined by

$$x^n = \sum_{m=0}^{n} S(n,m)(x)_m \tag{1.4}$$

for $n = 0, 1, \ldots$.

The third equality in (1.3) is not difficult to see. Note that $x(x-1)\cdots(x-r+1)$ can be expanded as $x^r + \omega_1 x^{r-1} + \cdots + \omega_{r-1}x$. The coefficient ω_k is precisely the sum of all k products (product of k distinct terms) out of $\{-1, -2, \ldots, -(r-1)\}$. So, it follows from the definition, (1.1), that $\omega_k = (-1)^k D_k(r)$.

Section 4 gives an expression for

$$C_k(r) = \sum_{0 \le i_1 \le \cdots \le i_k \le r} i_1 \cdots i_k$$

for $k \le r$. We prove the remarkable result (and a generalization of it) that

$$C_k(r) = D_k(-r),$$

where $D_k(x)$ is the extension of $D_k(r)$ to x in $\mathbb{R}$. Some related functions are also considered.

2 Sums of powers

Theorem 2.1 gives expressions for

$$A_r(n) = \sum_{k=1}^{n-1} k^r$$

for $r = 0, 1, \ldots$ and $n = 2, 3, \ldots$.

Theorem 2.1 *We have*

$$A_r(n) = \left\{B_{r+1}(n) - B_{r+1}\right\} / (r+1) = \sum_{k=0}^{r} \frac{1}{k+1} S(r,k)(n)_{k+1} \tag{2.1}$$

for $r = 1, 2, \ldots$, where $B_r(n)$ and $B_r = B_r(0)$ are the rth Bernoulli polynomial and rth Bernoulli number.

Proof: The first equality in (2.1) is equation (23.1.4) in Abramowitz and Stegun (1964, page 804). The second equality follows from the first formula in the 'Relation to falling factorial' section of http://en.wikipedia.org / wiki / Bernoulli_polynomials. □

3 Stirling numbers and sums of products

The Stirling numbers of the first and second kind, defined by (1.2) and (1.4) above, have generating functions

$$\sum_{n=m}^{\infty} s(n,m) x^m / m! = \left\{\ln(1+x)\right\}^m / m! \text{ for } |x| < 1,$$

$$\sum_{n=m}^{\infty} S(n,m) x^n / n! = \left[\exp(x) - 1\right]^m / m!,$$

$$\sum_{n=m}^{\infty} S(n,m) x^{n-m} = \prod_{i=1}^{m} (1 - ix)^{-1} \text{ for } |x| < m^{-1}.$$

See Abramowitz and Stegun (1964) for closed forms, tables, recurrence formulas and related formulas. Comtet (1974, page 135) notes that

$$|s(n,m)| = B_{n,m}\left(0!, 1!, 2!, \ldots\right),$$
$$S(n,m) = B_{n,m}\left(1, 1, 1, \ldots\right),$$

where $B_{n,m}(x_1, x_2, \ldots)$ is the partial exponential Bell polynomial defined by

$$\left(\sum_{n=1}^{\infty} z^n x_n/n!\right)^m /m! = \sum_{n=m}^{\infty} z^n B_{n,m}\left(x_1, x_2, \ldots\right)/n!,$$

and tabled on page 307 there. Comtet (1974, page 144) shows that the infinite matrix $\{s(n,k), n \geq 0, k \geq 0\}$ is the inverse of the infinite matrix $\{S(n,k), n \geq 0, k \geq 0\}$. Comtet (1974) also gives their role in Taylor series for $g(\exp(t)-1)$ and $f(\ln(1+u))$ for general functions g, f. He gives asymptotic expansions for them for n large on page 293 and tables them on page 310 for $n \leq 15$ giving references for larger n.

The sum of products $D_k(r)$ may be computed using the recurrence relation given by Theorem 3.1.

Theorem 3.1 *We have the recurrence relation*

$$D_k(r) = \sum_{i=1}^{r-1} i D_{k-1}(i)$$

with the initial value

$$D_1(r) = \binom{r}{2}.$$

Proof: By definition, $D_k(r)$ is the sum of all k products of the form $i_1 \cdots i_k$ for $1 \leq i_1 < \cdots < i_k \leq r-1$. Now consider only those k products with $i_k = i$ for $1 \leq i \leq r-1$. The sum of all such k products is $iD_{k-1}(i)$. So, the result follows. □

Riordan (1958, page 82, problem 7) gives a formula for $S(n, n-k)$ in terms of what he calls the associated Stirling numbers of the second kind.

For the connection between Stirling numbers and Stirling polynomials see, for example, Erdélyi et al. (1955, page 257). For a table of the Stirling numbers and their expression as a multiple of a generalized Bernoulli number, see David and Barton (1962, pages 287 and 294).

We end this section with another recurrence relation for the partial sums D_k given by Theorem 3.2.

Theorem 3.2 *We have the recurrence relation*

$$\left(2^r - 1\right) D_r(j) + \sum_{n=1}^{r-1} (-1)^{j-1} a_{j,j-n} D_{r-n}(j-n) \equiv 0, \qquad (3.1)$$

where $a_{j,r} = (-1)^n (2j-r-1)_{2n} 2^{-n}/n!$ *at* $n = j - r$.

Proof: The result comes from the identity (see Comtet (1974))

$$2^j (k)_j = \sum_{r=1}^{j} a_{j,r} (2k)_r$$

for $j = 1, 2, \ldots$. Taking the coefficient of $(2k)^{j-r}$ for $r = 1, 2, \ldots, j-1$ gives (3.1). □

4 Related results

Here, we give expressions for

$$C_k(r) = \sum_{0 \le i_1 \le \cdots \le i_k \le r} i_1 \cdots i_k. \tag{4.1}$$

This was computed iteratively from

$$C_1(r) = \binom{r+1}{2},$$

$$C_k(r) = \sum_{i=1}^{r} r C_{k-1}(r)$$

using a MACSYMA program. This gave the remarkable formula

$$C_k(r) = D_k(-r). \tag{4.2}$$

The relation (4.2) is a particular case of (4.3) in Theorem 4.1.

Theorem 4.1 *We have*

$$c_{a,b,\ldots}(r) = (-1)^{a+1+b+1+\cdots} d_{a,b,\ldots}(-r), \tag{4.3}$$

where

$$c_{a,b,\ldots}(r) = \sum_{0 \le i \le j \le \cdots \le r} i^a j^b \cdots ,$$

$$d_{a,b,\ldots}(r) = \sum_{0 < i < j < \cdots < r} i^a j^b \cdots .$$

Proof: The proof of (4.3) follows from Theorem 4.2. □

A generalization of (4.3) is given by Theorem 4.2.

Theorem 4.2 *Suppose $p, q, \ldots$ are polynomials (or more generally functions with power series expansions) such that $p(-r) = (-1)^a p(r)$, $q(-r) = (-1)^b q(r)$, $\ldots$ for some integers $a, b, \ldots$ so that*

$$d_{p,q,\ldots}(r) = \sum_{0<i<j<\cdots<r} p_a(i) q_b(j) \cdots$$

and

$$c_{p,q,\ldots}(r) = \sum_{0\le i\le j\le\cdots\le r} p_a(i) q_b(j) \cdots$$

also are polynomials in r (or more generally have power series expansions in r). Then

$$c_{p,q,\ldots}(r) = (-1)^{a+1+b+1+\cdots} d_{p,q,\ldots}(-r).$$

Proof: Let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. For i in $\mathbb{Z}^+$ let $\alpha_{i,j}$ be any real number. For k in $\mathbb{Z}^+$ and r in $\mathbb{Z}$, define $C_k(r)$ and $D_k(r)$ as follows:

$$D_k(r) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k > 0 \text{ and } r = 0, \\ \alpha_{k,r-1} D_{k-1}(r-1) + D_k(r-1), & \text{otherwise} \end{cases}$$

and

$$C_k(r) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k > 0 \text{ and } r = 0, \\ \alpha_{k,r} C_{k-1}(r-1) + C_k(r-1), & \text{otherwise.} \end{cases}$$

Then one can prove by induction that for $r > 0$,

$$D_k(r) = \sum_{0<i_1<\cdots<i_k<r} \alpha_{1,i_1} \cdots \alpha_{k,i_k} \tag{4.4}$$

and

$$C_k(r) = \sum_{0\le i_1\le\cdots\le i_k\le r} \alpha_{1,i_1} \cdots \alpha_{k,i_k}. \tag{4.5}$$

Consider for instance (4.4). Induction is trivial for the cases $k = 0$ or $r = 0$. So, if $k > 0$ and $r > 0$ assume (4.4) holds. Then

$$
\begin{aligned}
D_{k+1}(r) &= \sum_{0<i_1<\cdots<i_k<i_{k+1}<r} \alpha_{1,i_1}\cdots\alpha_{k,i_k}\alpha_{k+1,i_{k+1}} \\
&= \text{sum of all products with } i_{k+1} = r-1 \\
&\quad +\text{sum of all products with } i_{k+1} < r-1 \\
&= \sum_{0<i_1<\cdots<i_k<r-1} \alpha_{1,i_1}\cdots\alpha_{k,i_k}\alpha_{k+1,r-1} \\
&\quad + \sum_{0<i_1<\cdots<i_k<i_{k+1}<r-1} \alpha_{1,i_1}\cdots\alpha_{k,i_k}\alpha_{k+1,i_{k+1}} \\
&= \alpha_{k+1,r-1} \sum_{0<i_1<\cdots<i_k<r-1} \alpha_{1,i_1}\cdots\alpha_{k,i_k} \\
&\quad + \sum_{0<i_1<\cdots<i_k<i_{k+1}<r-1} \alpha_{1,i_1}\cdots\alpha_{k,i_k}\alpha_{k+1,i_{k+1}} \\
&= \alpha_{k+1,r-1}D_k(r-1) + D_{k+1}(r-1).
\end{aligned}
$$

Also

$$
\begin{aligned}
D_k(r+1) &= \sum_{0<i_1<\cdots<i_k<r+1} \alpha_{1,i_1}\cdots\alpha_{k,i_k} \\
&= \text{sum of all products with } i_k = r \\
&\quad +\text{sum of all products with } i_k < r \\
&= \sum_{0<i_1<\cdots<i_{k-1}<r} \alpha_{1,i_1}\cdots\alpha_{k,r} \\
&\quad + \sum_{0<i_1<\cdots<i_k<r} \alpha_{1,i_1}\cdots\alpha_{k,i_k} \\
&= \alpha_{k,r} \sum_{0<i_1<\cdots<i_{k-1}<r} \alpha_{1,i_1}\cdots\alpha_{k-1,i_{k-1}} \\
&\quad + \sum_{0<i_1<\cdots<i_k<r} \alpha_{1,i_1}\cdots\alpha_{k,i_k} \\
&= \alpha_{k,r}D_{k-1}(r) + D_k(r).
\end{aligned}
$$

So, (4.4) follows by induction. The proof of (4.5) is similar.

Now suppose there are integers $m_1, m_2, \ldots$ such that

$$\alpha_{i,-j} = (-1)^{m_i}\alpha_{i,j}$$

for $i \geq 1$ and j in $\mathbb{Z}$. Define

$$C'_k(r) = (-1)^{\sum_{j=1}^{k}(m_k+1)} D_k(-r).$$

Then by applying the above recurrence formula for D we find that C' satisfies the above recurrence formula for C and so $C'_k(r) = C_k(r)$. The proof is complete. □

An alternative expression for $C_k(r)$ of (4.1), and so by (4.2) for $D_k(r)$ is to use the expression given by Theorem 4.3.

Theorem 4.3 *For any fixed numbers $x_1, \ldots, x_n$, we have*

$$C_{k,n} = C_k(x_1, \ldots, x_n) = \sum_{1 \le j_1 \le \cdots \le j_k \le n} x_{j_1} \cdots x_{j_k} = \sum_{r=1}^{k} \widehat{B}_{k,r}(\mathbf{a})/r!$$

for $k \ge 1$, where $a_j = S_j/j$, $\mathbf{a} = (a_1, a_2, \ldots)$, and $\widehat{B}_{k,r}(\mathbf{a})$ is the partial ordinary Bell polynomial tabled on page 309 of Comtet (1974), where

$$S_j = \sum_{k=1}^{n} x_k^j$$

are the power sums.

Proof: We have

$$-\sum_{k=1}^{n} \ln(1 - x_k z) = \sum_{j=1}^{\infty} a_j z^j = T$$

say. So,

$$\prod_{k=1}^{n} (1 - x_k z)^{-1} = \sum_{r=0}^{\infty} T^r/r!. \tag{4.6}$$

By definition,

$$T^r = \sum_{j=r}^{\infty} \widehat{B}_{k,r}(\mathbf{a}) z^k.$$

Taking the coefficient of z^k in (4.6), since the left hand side of (4.6) is equal to

$$1 + \sum_{k=1}^{\infty} z^k C_{k,n},$$

we obtain the required expression for $C_{k,n}$. □

Acknowledgments

The authors would like to thank the Editor and the referee for careful reading and for their comments which greatly improved the paper.

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