THE THEORY OF FIRST-PRICE, SEALED-BID AUCTIONS

1. Within the class of first-price, sealed-bid auctions, there are a number of possible variations in environment, information, and rules:

(1) The number of potential bidders is either known, or unknown with a distribution that is common knowledge.
(2) There may be no reservation price, so that the item will definitely be sold, or there may be a reservation price which is announced or unannounced in advance of the auction. If there is a reservation price that is unannounced, then it may be made known after the auction is complete (e.g., as a losing bid), or the item may simply be withdrawn from the auction. If there are ties, there may be a tie-breaking mechanism, or subsequent rounds of bids between those tied.
(3) The winning bid may be announced, or may be private information shared by the winner and seller, so that non-winners have only the (unverified) information that their bid was lower. Non-winning bids may be announced, or may be private information shared by a losing bidder and the seller. Finally, a third party acting as an agent for the seller and buyers may designate a winner (or announce the withdrawal of the item from the auction) without revealing the amounts of bids to either seller or buyers, or revealing the reservation price. Of course, if an item is sold, then the winning bid is known at least to the buyer and seller. These information differences are not germane in a “one-shot” auction, but are relevant if there is a possibility of resale or re-auction.
(4) There may or may not be an opportunity for negotiation after the auction. One alternative is that bids are binding. Another is that they may contain contingencies whose value and prospects for clearing offer players the opportunity for post-auction negotiation. If such contingencies are allowed, then there must be a mechanism for terminating negotiations with the winning bidder if the contingencies are not satisfied, and negotiating with a lower bidder or re-starting the auction. An obvious consequence of allowing contingencies in bids is that setting contingencies becomes part of the bidder’s strategy, where a high bid with contingencies offers the opportunity to engage in bilateral negotiations with less effective competition from potential rival buyers. Finally, if one player (not necessarily a bidder) has a right of first refusal (RFR), then all bidders must take into account the possibility that this right will be exercised so that the RFR holder preempts the winning bidder and acquires the item at the winning bid, and that the item may subsequently be available following the auction through contracting with the party holding the RFR.
(5) If the item is not sold, there may or may not be opportunities for the item to be re-auctioned. If the item is sold, there may or may not be opportunities for resale.

(6) The value placed on the item by a potential buyer may be known to this buyer, may be unknown with a known distribution, or may be unknown with a distribution that is not completely known (e.g., in a class known up to location). This is also true for the seller. No player knows the value of any other player, but each player has beliefs about the distributions from which other player's values are drawn. These beliefs might be symmetric among all buyers, or not, and might be rational or not (rational the sense that each value is in fact drawn from the distribution that other players believed to hold). The distribution of beliefs about a given player may be common to all other players, or may be specific to each other player. If there is uncertainty about values, it may be independent across players, or may contain common uncertainties (e.g., different buyer's values for oil leases are influenced by common uncertainties about the future price of oil as well as individual uncertainty about the capacity of the lease). The information situation on buyer's values is critical if there is a possibility that the item will be withdrawn and subsequently re-auctioned to the same potential buyers.

2. The simplest, case is J > 1 buyers, with J common knowledge, no reservation price so the item will definitely be sold to the highest bidder, with ties broken by random assignment among those tied. Later when a reservation price is introduced, assume that in case of ties, all bidders have priority over the seller. Bids are binding, without contingencies. There are no resale possibilities. Buyers know their own values with certainty. They do not know the values of other bidders, but all know that these values are independent draws from a distribution G(v) that is common knowledge. Under these assumptions, buyer k will have a (mixed) strategy, depending on his value v, described by a cumulative distribution function (CDF) F(b|v) with a support B(v). If there is a pure strategy, then F(b|v) has unit mass at the singleton B(v). When B(v) is a singleton, call it the <u>bid function</u>. Define $F(b\text{-}|v) = \sup_{b'<b} F(b'|v)$ to be the probability of a bid strictly less than b, $H(b\text{-}) = \int F(b\text{-}|v)G(dv)$ to be the expected probability of a bid strictly less than b, $H(b) = \int F(b|v)G(dv)$ to be the expected probability of a bid less than or equal to b, and h(b) = H(b) - H(b-) to be the point mass at b. Then, H(b) is the probability that bidder k will make a bid no larger than b, and h(b) is the probability that he will make a bid of b. In the symmetric information case assumed here, this will describe the strategies of all rivals of bidder k as well.

Since H(b) is non-decreasing, there are at most a countable number of values of b at which it can have jumps at which h(b) > 0. Taking account of the tie-breaking mechanism, the probability of buyer k winning with bid b is

$$\begin{aligned} P(b) &= H(b\text{-})^{J-1} + H(b^{-})^{J-2}h(b)(J-1)/2 + \dots \\ &\quad + H(b\text{-})^{J-1-m}h(b)^m(J-1)!/(m+1)!(J-1-m)! + \dots + h(b)^{J-1}/J \\ &= [H(b)^J - H(b\text{-})^J]/J[H(b) - H(b\text{-})]. \end{aligned}$$

When h(b) = 0, $P(b) = H(b)^{J-1}$. This is also the limit of the last formula as H(b-) =

H(b) - h(b) approaches H(b). Note that P(b) is always non-decreasing. It is continuous at any b where h(b) = 0, and jumps at any b where h(b) > 0. At a jump, it is a proper weighted average of its left and right limit, $H(b-)^{J-1} < P(b) < H(b)^{J-1}$. The payoff to buyer k is his expected profit, (v-b)P(b). In Nash equilibrium, each $b \in B(v)$ maximizes this payoff.

The figures below plot P(b) and contours of the form p = A/(v-b) for various A and v. B(v) consists of the points of contact of P(b) and the northwest-most contour p = A/(v-b) which touches P(b). This has some general implications: A bid b is in B(v) only if P(b) is at least as steeply sloped as the tangent contour to the immediate left of the tangency and no more steeply sloped than the contour to the immediate right of the tangency. This rules out the possibility of points in B(v) with the property that P(b′) = P(b) for b′ to the immediate left of b, or the property that b is a jump point of P. As v increases, the contours rotate clockwise, and as a result the contact points will necessarily roll to the right (or remain fixed). Figure 1 shows a case in which B(v) is a singleton that is increasing in v.

Fig.1

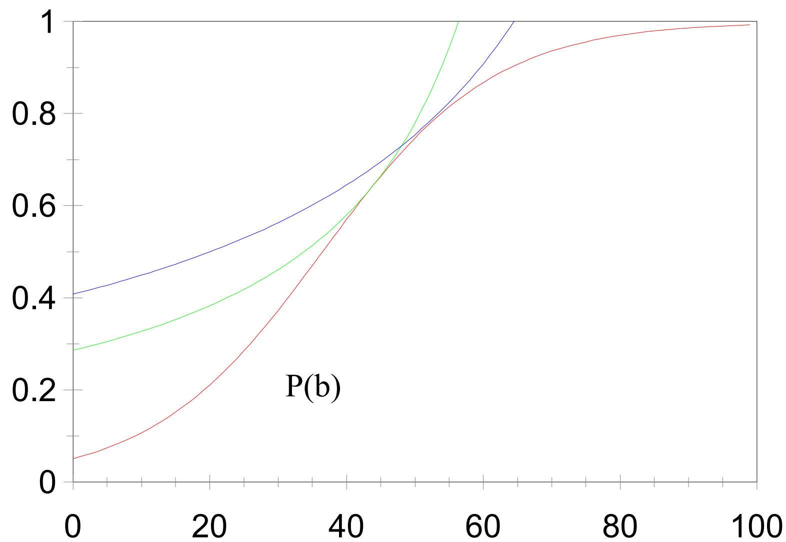

Figure 2 illustrates a case where P(b) has a kink, resulting in B(v) being fixed for an interval of v. But there is then a positive probability of a tie at b, and by the previous analysis, P(b) must then have a jump at b. Then, the situation in Figure 2 is impossible. Consequently, P(b) cannot have kinks, and instead must be differentiable at each b.

Fig. 2

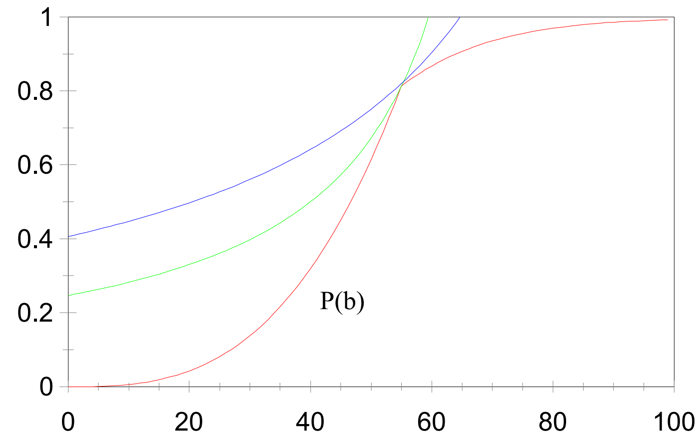

Figure 3 shows a situation in which B(v) contains an interval for some v.

Fig. 3

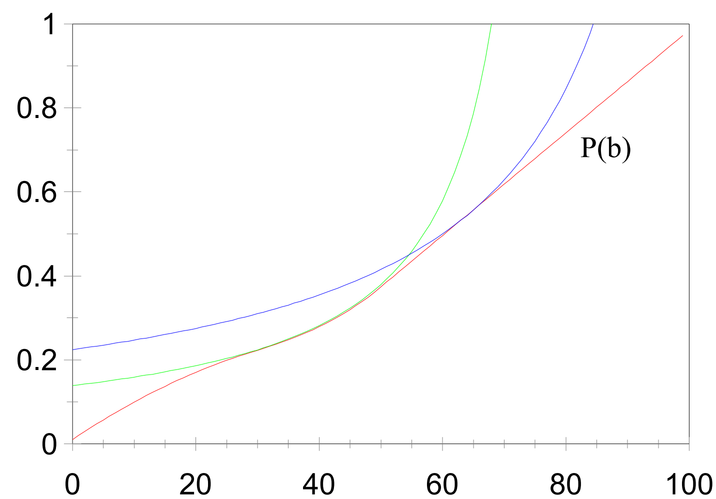

Fig. 4

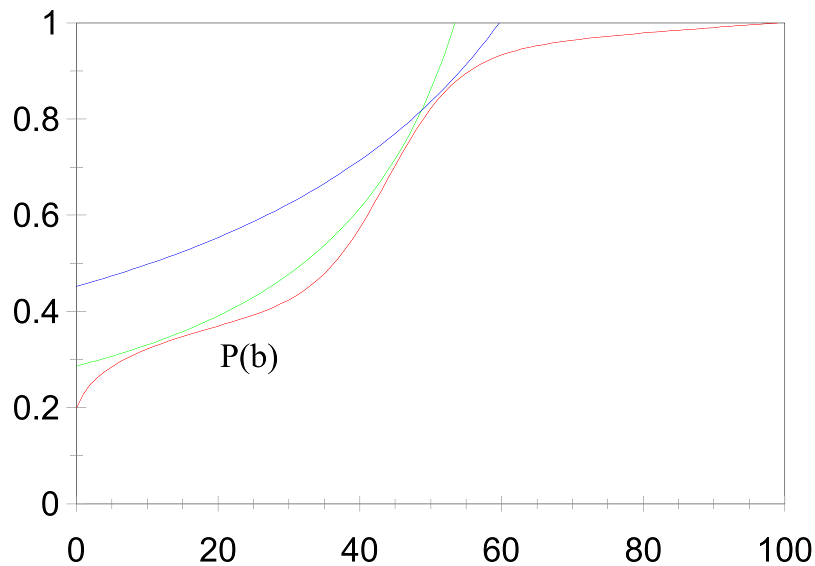

Figure 4 depicts a situation in which B(v) is not a singleton for some v, containing isolated points. When P(b) is sufficiently convex in some regions, this outcome will typically occur for some v, and the range B* of B(v) will not be an interval.

The following argument formalizes the properties of B(v) that are obvious from the geometry. Suppose b,b' are maximizing for v and $v' = v+\Delta$, respectively, with $\Delta > 0$. Then, $(v-b)P(b) \geq (v-b')P(b')$ and $(v+\Delta-b')P(b') \geq (v+\Delta-b)P(b)$. Adding these inequalities, $\Delta\cdot[P(b')-P(b)] \geq 0$. This implies $b' \geq b$. Therefore, B(v) is non-decreasing in v. For every value of v for which B(v) is not a singleton, points in B(v) bracket an open interval. Since the number of disjoint open intervals is countable, there are at most a countable number of v for which B(v) is not a singleton. Since G has a bounded density, it follows that B(v) is a singleton with probability one.

Next suppose a bid b at which $h(b) > 0$, and let $\alpha = H(b^-)/H(b)$ and $\beta = h(b)/H(b)$. Then,

$$P(b) = H(b)^{J-1} \sum_{m=0}^{J-1} \alpha^{J-1-m}\beta^m (J-1)!/(m+1)!(J-1-m)!$$

$$< H(b)^{J-1} \sum_{m=0}^{J-1} \alpha^{J-1-m}\beta^m (J-1)!/m!(J-1-m)! = H(b)^{J-1}.$$

Since the number of b values with $h(b) > 0$ is countable, there exist $b' \searrow b$ with $P(b') = H(b')^{J-1} \geq H(b)^{J-1}$. Therefore, $\lim (v-b')P(b') > (v-b)P(b)$. In this case, a maximum does not exist. Nevertheless, it contradicts the supposition of a bid b at which there is a positive probability of a tie, and hence implies that when a maximum is achieved, P(b) is differentiable. Summarizing, B(v) is increasing in v, and is a singleton, with probability one. It can jump, so that its range B* is not necessarily an interval. It has an inverse v = V(b) that is strictly increasing on B*. Note that the results so far could also have been established without the symmetry assumption of a common probability P(b) of winning. However, with symmetry,

$$H(b) = G(V(b)),\ b \in B^*, \text{ or } H(B(v)) = G(v).$$

The preceding results establish that B(v) almost surely satisfies a first-order condition for maximization of $(v-b)H(b)^{J-1}$,

$$0 \equiv -P(B(v)) + (v-B(v))P'(B(v)).$$

From the condition $P(B(v)) \equiv H(B(v))^{J-1} \equiv G(v)^{J-1}$ in the absence of ties, this implies $P'(B(v))B'(v) = (J-1)G(v)^{J-2}G'(v)$, and hence

$$G(v)^{J-1} = (v-B(v))(J-1)G(v)^{J-2}G'(v)/B'(v).$$

This is a differential equation in B, $B'(v)/(v-B(v)) = (J-1)G'(v)/G(v)$, that has the solution

$$B(v) = \int_{t=0}^{v} t(J-1)G(t)^{J-2}G'(t)dt/G(v)^{J-1}.$$

But this is just the Conditional Second Value (CSV), the expected maximum value among bidder k's rivals, conditioned on this maximum value being no greater than bidder k's value v. This is the primary result.

How can the seller expect to do in this auction? The density of the maximum value among all bidders is $JG(v)^{-1}G'(v)$, and hence the expected revenue from the sale is

$$\int_{v=0}^{\infty} B(v)JG(v)^{J-1}G'(v)dv = \int_{t=0}^{\infty}\int_{v=t}^{\infty} t(J-1)G(t)^{J-2}G'(t)dtJG'(v)dv$$

$$= \int_{t=0}^{\infty} tJ(J-1)G(t)^{J-2}G'(t)[1-G(t)]dt.$$

To illustrate these solutions, consider the concrete case $G(v) = v^{\theta}$ for $0 < v < 1$, where θ is a positive parameter. In this case, $B(v) = v\theta(J-1)/[1+\theta(J-1)]$ is proportional to v, with a proportion that is closer to one when J and/or θ are larger. The expected revenue to the seller is $[J\theta/(1+J\theta)][(J-1)\theta/(1+(J-1)\theta)]$. This gets close to one when J and/or θ are large.

Question: Are the theoretical results above completely general, or can they fail if the smoothness of B and connectedness of the range B* of B fail? Note that a condition like H(b) concave is sufficient to make B* an interval: The objective function $(v-b)H(b)^{J-1}$ is maximized when $(v-b)^{1/(J-1)}H(b)$ is maximized. But the term $(v-b)^{1/(J-1)}$ is positive, concave, and decreasing, so that if H(b) is positive and increasing, the product is concave. On the other hand, consider examples like $G(v) = v^2$ for $0 < v < 1$. Do the $B(v) = 2(J-1)v/(1+2(J-1))$ and $H(b) = b^2[(1+2(J-1))/2(J-1)]^2$ that solve the differential equation also solve the original Nash problem?

3. There is no difficulty in making J unknown, with a distribution that is common knowledge. This changes the P(b) function to a mixture over the possible values of J. In particular, when the distribution of J is geometric, there is a nice simplification. Suppose the event $J \geq 1$, and consider the strategy of buyer 1. Let π_J be the unconditional probability of J bidders. Given the event that there is at least one bidder, the first bidder sees the conditional probability $\pi^*_J = \pi_J/(1-\pi_0)$ of J-1 rival bidders. Each additional potential buyer will have a value v distributed with the CDF G(v). The probability that the highest value among other potential buyers is no greater than v is given by

$$Q(v) = \sum_{J=1}^{\infty} G(v)^{J-1}\pi^*_J \equiv P(B(v)).$$

Recall that for fixed J, B(v) is strictly increasing in v. This followed from the monotonicity of P(b) and the geometry of the payoff function, and remains true when P(v) is obtained by the mixture over J given above. Then, bidder 1 will win the auction only if his value v_1 exceeds the maximum value of all other potential buyers. The probability of this event is $Q(v_1)$ when ties are ruled out. Substitute this into the first-order condition $0 \equiv -P(B(v)) + (v-B(v))P'(B(v))$ to get

$$Q(v) = (v-B(v))Q'(v)/B'(v).$$

This has the solution

$$B(v) = \int_{t=0}^{v} tQ'(t)dt/Q(v).$$

Then, the optimal strategy for bidder 1 is to bid the expected highest value of all remaining potential buyers, conditioned on the event that this value is less than the value of bidder 1, simply taking account the probability of various numbers of bidders.

The formulas above simplify when the number of potential buyers has a geometric distribution, $\pi_j = (1-\lambda)\lambda^j$. Then, $\pi^*_J = (1-\lambda)\lambda^{j-1}$ and $Q(v) = (1-\lambda)/(1-\lambda G(v))$. Note that when λ is small, $Q(0)$ is large, and $B(v)$ will be near zero. As an example, consider $G(v) = v$ for $0 < v < 1$, which in the case of fixed J leads to $B(v) = v(J-1)/J$. In this case, one obtains $B(v) = v + \log(1-\lambda v)/\lambda$.

4. Suppose that the auction rules are changed to have an announced reservation price r. For the analysis of this case, again assume a fixed number of potential buyers $J > 1$. From the standpoint of a potential buyer, it is worth entering the auction if this buyer has a value greater than r. With common knowledge on the distribution of values $G(v)$, this implies there is a probability $1-G(r)$ that a potential buyer will enter the auction, and if a buyer enters, he will bid between r and his value. Suppose there is a Nash equilibrium at a symmetric bid function $B(v)$ that is differentiable and invertible, and let $V(b)$ denote its inverse. Then, as before, the probability of winning at a bid $b > r$ is $P(b) = G(V(b))^{J-1}$. This takes account fully of the fact that some rivals may not bid at all. Then, as before, we obtain the differential equation for $B(v)$, $B'(v)/(v-B(v)) = (J-1)G'(v)/G(v)$. The one difference is that we now have the boundary condition that $B(r) = r$, with a potential buyer not bidding if his $v < r$. The solution with this boundary condition for $v > r$ is

$$B(v) = \int_{t=0}^{v} \max(r,t)(J-1)G(t)^{J-2}G'(t)dt/G(v)^{J-1}.$$

The seller's tradeoff in fixing r is that increasing r lowers the probability that the item will be sold, but raises the expected bid if it is sold. The probability of no bids above the reservation price is $(1-G(r))^J$. The expected revenue to the seller is then

$$\begin{aligned}
&\int_{v=r}^{\infty} B(v)JG(v)^{J-1}G'(v)dv \\
&= \int_{t=0}^{r}\int_{v=t}^{\infty} J(J-1)rG(t)^{J-2}G'(t)G'(v)dvdt + \int_{t=r}^{\infty}\int_{v=t}^{\infty} J(J-1)tG(t)^{J-2}G'(t)G'(v)dvdt \\
&= \int_{t=0}^{r} J(J-1)rG(t)^{J-2}G'(t)[1-G(t)]dt + \int_{t=r}^{\infty} J(J-1)tG(t)^{J-2}G'(t)[1-G(t)]dt \\
&= rJG(r)^{J-1} - r(J-1)G(r)^J + \int_{t=r}^{\infty} J(J-1)tG(t)^{J-2}G'(t)[1-G(t)]dt.
\end{aligned}$$

An additional issue is whether, in the extensive game where the seller receives the bids and then declares a winner, he is obligated to stick to his declared reservation price. Unless he can by some mechanism as a binding legal contract pre-commit to this reservation price, he has an incentive in the end game to accept any maximum bid above his true value for retaining the object. Then, absent pre-commitment, potential buyers expect the stated reservation price to be meaningless, and the de facto reservation price to be the seller's true value, which generally would be unknown, but perhaps with a commonly known distribution (see the next section). However, when the seller is able to pre-commit to the reservation price r, then he can choose r to maximize expected revenue.

5. Suppose the situation is as in 4, but the reservation price is <u>not</u> announced in advance of the auction. From the standpoint of potential buyers, the seller in this case acts just like another buyer, in effect putting in a bid and buying the item back for himself if his bid exceeds the others that are submitted. Then, what matters are the potential buyers' beliefs about the distribution of reservation prices of the seller, and whether this is common knowledge. Assume that the seller's distribution of reservation prices is known to be M*(b). It will be convenient to define a distribution M(v) = M*(B(v)), the apparent distribution of values for the seller which if he followings the same bidding strategy as the buyers B(v) produces his actual M* distribution. The probability that a potential buyer will win at a bid b when all other buyers are using a bid function B(v) is, absent ties,

$$P(b) = G(V(b))^{J-1}\cdot M(V(b)).$$

A bidder will choose b to maximize (v-b)P(b), leading to the first-order condition P(b) = (v-b)P′(b). If B(v) is a Nash equilibrium bid function, then

$$G(v)^{J-1}M(v) = (v-B(v))[(J-1)G(v)^{J-2}G'(v)M(v) + G(v)^{J-1}M'(v)]/B'(v).$$

This differential equation has the solution

$$B(v) = \int_{t=0}^{v} t[(J-1)G(t)^{J-2}M(t)G'(t)+G(t)^{J-1}M'(t)]dt/[G(v)^{J-1}M(v).$$

Since M(v) = M*(B(v)), this relationship given G and M* defines B(v) implicitly, and it will not be possible to obtain an explicit form for B(v) in most cases.

6. **<u>Asymmetric auctions</u>**. Consider a first-price sealed bid auction of a single item with two bidders, B and C. Suppose that there is an announced reservation price of 300. Suppose resale is prohibited. Bidder B attaches value 400 to the item. Bidder C attaches value 300 with probability one-half and value 800 with probability one-half, so his expected value is 550. Assume a tie-breaking rule that the item goes among those tied first to bidder B, second to bidder C, and last to the seller.

7. In a first-price sealed bid auction with buyer's values drawn from the same probability distribution that is common knowledge, we know that Nash equilibrium strategies are for each buyer to bid his Conditional Second Values (CSV); i.e., given his own value, find the conditional expectation of the highest value of the remaining bidders, conditioned on its being less than his value. In the asymmetric value auction described above, the CSV of B is the expectation of bidder C's value given it is less than the value 400 of bidder B. This CSV_B is 300. The $CSV_C(v_C)$ of C is 400 at value v_C = 800, and is 300 at value v_C = 300, the reservation price. If the bidders follow CSV strategies, then with probability ½, v_C = 800 and C wins with bid 400 and payoff 400, and with probability ½, v_C = 300 and B wins with bid 300 and payoff 100, so that the expected payoff to B is 50, and the expected payoff to C given v_C = 300 is zero and given v_C = 800 is 400. The expected payment to the seller is 350. However, these strategies are not a Nash equilibrium (NE): Since B bids 300 with probability one, if C bids 300 if v_C = 300 and 301 if v_C = 800, C's expected payoff given v_C = 300 is still zero, but C's expected payoff given v_C = 800 improves to 499. This shows that in an asymmetric auction, CSV bidding is not necessarily a NE.

8. We claim that the NE strategies for the bidders are mixed, with cumulative distribution functions

$F_B(b)$ = 450/(800-b) for 300 < b $\leq$ 350, with probability $F_B(300)$ = 0.9 of a bid at 300.
$F_C(b|300) = 1(b \geq 300)$ when v_C = 300
$F_C(b|800)$ = (b-300)/(400-b) for 300 $\leq$ b $\leq$ 350 when v_C = 800

The forms of these distributions come from the proposition that a mixed strategy is NE when the strategies of others is given only if the payoff to a player is constant on the support of the player's strategy, and no higher outside the support. Thus, the payoff to player C if v_C = 300 is $(300-b_C)F_B(b_C^-)1(b_C \geq 300)$, where the first term is the difference between value and bid, the product of the second and third terms is the probability that the reservation price is reached and the seller will sell, and B will not win. This expression is maximized at zero when b_C = 300, and is negative for higher bids. If v_C = 800, the payoff is $(800-b_C)F_B(b_C^-)1(b_C \geq 300)$. Before considering the set of points on which this is maximized, form the similar payoff for B, $(400-b_B)(½ + ½F_C(b_B|800))1(b_C \geq 300)$, where the second term is the probability that the item will be sold and C will not win. The question is to find a pair F_B and $F_C(\cdot|800)$ which make these expressions constant over some intervals, and no larger elsewhere. But clearly the first expression is constant over an interval only if F_B has the form K_1/(800-b) on this interval, where K_1 is a constant, and the second expression is constant over an interval only if ½ + ½$F_C(b|800)$ is inversely proportional to 400-b on this interval, or $F_C(b|800)$ has the form K_2/(400-b) - 1, where K_2 is a constant. Since C loses ties, it will never assign positive probability to 300 when v_C = 800, so $F_C(300|800)$ = 0. This gives K_2 = 100, and hence $F_C(b|800)$ = (b-300)/(400-b). Since $F_C(b|800) \leq 1$, the upper limit of the support is 350. Verify that on [300,350], B has expected payoff 50 from any bid, and that the expected payoff from any bid above 350 is less than 50. Finally, if C's bids never exceed 350, then B can never gain from bidding more than 350, and hence $K_1/(800\text{-}b)$ must be one at b = 350. This establishes $K_1 = 450$.

We now verify directly our claim that the mixed strategies we have given are a NE. With these strategies, B always wins if $v_C = 300$, receiving expected payoff

$$\int_{300}^{350}(400\text{-}b)F_B(db) = .9\cdot 100 + 450\int_{300}^{350}[(400\text{-}b)/(800\text{-}b)^2]db$$
$$= 50 + 450\ \ln(10/9) = 97.41223$$

If $v_C = 800$, then the expected payoff to B is

$$\int_{300}^{350}(400\text{-}b)F_C(b|800)F_B(db) = 450\int_{300}^{350}[(b\text{-}300)/(800\text{-}b)^2]db = 50 - 450\ \ln(10/9) =$$
2.5878.

The expected payoff to B, the average of these payoffs, is 50. If $v_C = 800$, then the expected payoff to C is

$$\int_{300}^{350}(800\text{-}b)F_B(b)F_C(db) = 450\int_{300}^{350}F_C(db) = 450.$$

We now show these mixed strategies are a NE. Consider C's strategy when v_C = 300. In this case, C knows that B playing F_B will always win, so that C's payoff to any bid between zero and 300 is zero, and to any bid above this is negative. Hence, the bid of 300 weakly maximizes his payoff. Next, consider C's strategy when $v_C = 800$. Bids at or below 300 never win, and bids above 350 always win, so the support of C's strategy is contained in (300,350]. If C plays a mixed strategy G, then its payoff is

$$\int_{300}^{350}(800\text{-}b)F_B(b)G(db) = 450\int_{300}^{350}G(db) = 450,$$

for any G(b) with this support, so that $F_C(b|800)$ is weakly maximizing.

Finally, consider B's strategy. Bids below 300 never win and bids above 350 always win, so the support of B's strategy is contained in [300,350]. Suppose B plays a mixed strategy G with this support. Either $v_C = 300$, so that C's bid is 300 and B always wins, or $v_C = 800$ and C's bid has distribution $(b\text{-}300)/(400\text{-}b)$ for $300 \le b \le 350$. The expected payoff to B is then

$$½\int_{300}^{350}(400\text{-}b)G(db) + ½\int_{300}^{350}(b\text{-}300)G(db) = 50\int_{300}^{350}G(db) = 50.$$

Then $F_B(b)$ which achieves this expected payoff is weakly maximizing. This proves that F_B, F_C are a mixed strategy Nash equilibrium. Note that a bid of 350 for B wins with probability one, with payoff 50, and no higher bid can yield a payoff this high; this is the condition that determines the upper limit 350 of the support. At the Nash equilibrium, one has the following properties of the equilibrium bids:

$$Eb_B = \int_{300}^{350} bF_B(db) = 350 - \int_{300}^{350} F_B(b)db = 350 - 450 \ln(10/9) = 302.5878.$$

$$E(b_C|v_C=800) = \int_{300}^{350} bF_C'(b)db = 350 - \int_{300}^{350} F_C(b|v_C=800)db$$

$$= 300 + 100 \ln(2) = 330.6853$$

$$Eb_C = ½(300 + E(b_C|v_C=800)) = 315.3426$$

$$E \max(b_B,b_C) = ½[Eb_B + \int_{300}^{350} bd(F_B(b)F_C(b))]$$

$$= ½Eb_B + ½[350 - \int_{300}^{350} F_B(b)F_C(b)db]$$

$$= ½[350 - 450 \ln(10/9)+ 350 - 450 \int_{300}^{350} [(b-300)/(800-b)(400-b)]db] = ½[350 - 450 \ln(10/9) + 350 - 450*\ln(2) - (1125/2)*\ln(25/9)] = 316.937$$

Consider, as an alternative mechanism, a second-value sealed bid auction. Suppose C bids his value. Then, any bid from B from 301 to 799 is equally good, giving a payoff of 100 with probability ½. B's value of 400 is weakly optimal in this range. Suppose B bids his value. Then, C's optimal response when $v_C = 800$ is to bid his value, or any bid from 401 to 800, giving a payoff of 400. Hence, his value of 800 in this case is weakly optimal. The expected revenue to the seller is 350. A completed ascending bid auction with public bidding will start with bids of 300 from B and C. If C does not bid again, the item goes to B at bid 300, given the tie-breaking rule. If C does bid again, say with bid 301, then this establishes that he has $v_C = 800$, and will continue the bidding until B stops before or at 400. Then, B has no incentive to either continue or stop bidding, and the auction could end at the bid 301, or continue until B bids 400 and C wins with 401. The final bids 349 for B and 350 for C are weakly optimal. Thus, revenue equivalence fails in general, and the first-price sealed bid auction is for the seller inferior to the second-price auction.

Summarizing, the optimal bidding strategies in the first-price sealed-bid auction are asymmetric. The average bid for B is 302.5878 and the average bid for C is 300 if $v_C = 300$ and 330.6853 if $v_C = 800$, an overall average of 315.3426. Thus, B bids higher than its CSV of 300, while C bids substantially less than its CSV of 400 in the case $v_C = 800$. The expected revenue to the seller is 316.937 versus the expected CSV of 350 that it could attain in a second-price sealed-bid auction. Thus, this auction fails to meet the hypotheses of the revenue equivalence theorem. The first-price sealed bid auction is not efficient, since bidder B wins the auction with a positive probability when C has value 800.

9. Now drop the assumption that no resale is possible. Suppose that resale of the item from B to C is possible, but because of the auction rules (e.g., a standstill agreement by B), resale of the item from C to B is not possible. Also exclude the possibility of resale to third parties. This is justified because if third parties had high value, they would have incentives similar to C to participate in the primary auction. Then, the ex ante value of the item to B before the auction is the larger of his own value for holding the item, and the expected price at which it could be resold in the event that he wins the auction. In general, the possibility of profit from resale will give B some incentive to bid more aggressively, while the possibility of acquiring the item through resale will give C some incentive to bid less aggressively. The strength of these incentives and their impact on bidding will depend substantially on the bargaining power of the two players in the resale market. Assume for concreteness that in the event of a resale from B to C, the item is traded at the average of their values. (This is the Nash bargaining solution[1] when each player' own value establishes its threat point.) First consider player C's options. If $v_C = 300$, C cannot resell and cannot profit, and consequently will submit the minimum bid of 300. If $v_C = 800$, then the expected profit of C at bid b is

$$(800 - b)F_B(b) + 200(1 - F_B(b)) = (600 - b)F_B(b) + 200,$$

where the first term is the expected profit obtained through winning the auction, and the second term is the expected profit obtained from losing the auction and then obtaining the item through resale. The expected profit of B at bid b is the sum of the probability of $v_C = 300$ times the expected payoff from winning and holding the item, and the probability of $v_C = 800$ times the probability of winning and reselling at the average of v_B and $v_C = 800$, or

$$½(400 - b) + ½(600 - b)F_C(b|800) = ½(600 - b)[F_C(b|800) + 1] - 100.$$

In a Nash equilibrium, these payoffs must be constant on a common support, and

[1] The term <u>Nash bargaining solution</u> is a different concept than Nash equilibrium, and is one proposed solution to a cooperative game between two agents, each of which has a monetary payoff with a "threat" level that each can achieve if no bargain is made. Under some plausible axioms on behavior, a bargain will be struck at a division that maximizes the product of the excess payoffs that a bargain gives relative to the players' threat points.

larger than the payoffs in other bid ranges. We show that this is satisfied by $F_B(b) = 150/(600-b)$ for $300 \leq b \leq 450$ and $F_C(b|300) = 1(b \geq 300)$ and $F_C(b|800) = (b-300)/(600-b)$ for $300 < b < 450$. First, at this F_B, the payoff of C when $v_C = 800$ is 350 for $300 < b < 450$, and bids above 450 always win and yield lower payoff. Second, at this $F_C(b|800)$, the payoff of B is 50 for $300 < b < 450$, and bids above 450 win with certainty and have an expected payoff less than 50. Therefore, the pair above is proven to be a Nash equilibrium. Note that in this NE, B will with some probability bid higher than his own value, and hence with some probability will incur a loss. Some features of this equilibrium are

$$Eb_B = 450 - \int_{300}^{450} F_B(b)db = 450 - 150 \ln(2) = 346.0279.$$

$$E(b_C|800) = 450 - \int_{300}^{450} F_C(b)db = 392.0558.$$

$$Eb_C = 369.0428$$

$$Emax(b_B,b_C) = ½(\int_{300}^{450} bF_B(db) + \int_{300}^{450} bd(F_B(b)F_C(b|800))$$

$$= ½(450 - \int_{300}^{450} F_B(b)db + 450 - \int_{300}^{450} (F_B(b)F_C(b|800)db)$$

$$= 346.0279/2 + 225 - ½ \int_{300}^{450} F_B(b)F_C(b|800)db = 375.$$

Then, the payoff to the seller is substantially higher when resale by B is permitted than in the market where resale was prohibited. Bidder B gains nothing from the availability of resale, receiving an expected payoff of 50 as in the previous case, so that the seller gains all of the rents from resale through its impact on bidding in the primary auction. Further, the expected payoff to the seller exceeds the expected payoff from a second-price sealed bid auction where resale is prohibited, showing that in the absence of symmetric buyer values, this auction format is not revenue-maximizing for the seller.

10. Now suppose resale from B to C, but not from C to B, is possible, due to a standstill agreement signed by B, and suppose there are no other potential buyers. So far, this is the same as the previous case. However, now suppose a third player D has a right-of-first-refusal (RFR), meaning that this player can supplant the winning bidder in the primary auction, paying the winning bid to the original seller, and then either hold the item or sell it to C. Suppose that D has value zero for

holding the item, and that it will exercise its RFR if and only if B wins and C agrees to buy at a price at which D has a positive profit. From the state in which B wins with bid b, consider the extensive game in which C offers D a conditional purchase agreement at price (b+800)/2, the Nash bargaining solution when $v_C = 800$ and D's threat point is its cost to exercise the RFR, and D then decides to exercise when (b+800)/2 > b, or b < 800. Then, C will make a purchase offer if $v_C = 800$ and B wins at any bid b < 800, the RFR will be exercised, and C will attain a payoff (800-b)/2. Now, the expected payoff to C from a bid b is

$$(800-b)F_B(b) + ((800-b)/2)(1-F_B(b)) = (800-b)[1+F_B(b)]/2.$$

On the other hand, bidding 300 is a dominant strategy for B, since it then wins with maximum payoff 100 if $v_C = 300$, and B in any case receives payoff zero if $v_C = 800$. Given this, the optimal strategy for C when $v_C = 800$ is to bid 301, win, and attain payoff 499. Thus, there is a NE in which the RFR is not exercised, the expected payment to the seller is 300.5, B has expected payoff 50, C has expected payoff zero if $v_C = 300$ and expected payoff 499 if $v_C = 800$, and D has payoff zero. In this case, B neither gains nor loses compared to the cases of no resale, or resale available to B, and the RFR holder does not gain any positive rent. Nevertheless, the presence of the RFR holder essentially eliminates the ability of the seller to utilize its market power to garner rents above its reservation price.

The analysis in cases 3 and 4, where resale is possible, and where in case 4 there is an RFR holder, depends critically on the solution to the bilateral bargaining game between C and the winning bidder or the RFR holder. The solution also depends critically on what information is available to the RFR holder regarding C's value and the outcome of bargaining at the time the RFR must be exercised. For example, an assumption at one extreme is that the RFR holder can exercise the RFR and then make a "take it or leave it" ultimatum to any bidder, winner or loser. In this case, the RFR holder will indeed exercise the RFR, make the ultimatum price of 799 to C, and the ultimatum price of 399 to B if C does not accept. Then, neither B or C have any incentive to bid above the 300 level necessary to put the item in play, and the RFR holder gets essentially all the rents available in the market. An assumption at the other extreme is that the RFR holder must decide to exercise or not without knowing who the winner is or what resale contracts might be possible, say because the auction rules state that it must submit its own RFR reservation price in advance. Assume further that it is C as potential buyer that makes an ultimatum "take it or leave it" offer of p > 0 to the RFR holder when the RFR is exericsed and $v_C = 800$. Then, by backward recursion, a dominant strategy for the RFR holder is to accept an offer of p if the RFR is exercised, receiving an expected payoff of p/2 - b, and hence to not exercise the RFR unless p > 2b. Then, I claim that primary auction bids of 300 from B and 300+!($v_C = 800$) from C, followed if B wins by a repurchase offer of b+1 from C in case $v_C = 800$, is a subgame perfect NE in which B will win with maximum payoff if $v_C = 300$, C wins with payoff 499 otherwise, the RFR is not exercised, and the initial seller expected revenue is 300.5. In this case, the rents go essentially to C.

11. The cases considered are contrasted in the table below:

	Eb_B	Eb_C	$E\max(b_B,b_C)$
NO RESALE			
Conditional Second Values (not a NE)	**300**	**350**	**350**
First-Price Sealed Bid	**302.5878**	**315.3426**	**316.937**
Second-Price Sealed bid	**300**	**350**	**350**
RESALE FROM B TO C PERMITTED			
First-Price Sealed Bid	**346.0279**	**392.0558**	**369.0428**
RFR HELD BY D, SALE TO C PERMITTED			
First-Price Sealed Bid	**300**	**300**	**300**