

Constant-Rounds, Linear Multi-party Computation for Exponentiation and Modulo Reduction with Perfect Security ^{*}

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Abstract

Bit-decomposition is an important primitive in multi-party computation (MPC). Given a sharing of secret x , it allows the parties to compute the sharings of the bits of x in constant rounds. With the help of bit-decomposition, we will be able to construct constant-rounds protocols for various MPC problems, such as *equality test*, *comparison*, *public modulo reduction* and *private exponentiation*, which are four main applications of bit-decomposition. However, when considering perfect security, bit-decomposition does *not* have a linear communication complexity. Thus any protocols involving bit-decomposition inherit this inefficiency, i.e. the communication complexity is non-linear. Constructing protocols for MPC problems without relying on bit-decomposition is a meaningful work because this may provide us with perfect secure protocols with linear communication complexity. It is already proved that *equality test*, *comparison* and *public modulo reduction* can be solved without involving bit-decomposition and the communication complexity can be reduced to linear. However, it remains an open problem whether *private exponentiation* could be done without relying on bit-decomposition. In this paper, maybe somewhat surprisingly, we show that it can. That is to say, we construct a *constant-rounds, linear, perfect secure* protocol for private exponentiation *without* relying on bit-decomposition though it seems essential to this problem. Compared with the previous solution for this problem (with perfect security), our protocol has a lower round complexity and a much lower communication complexity.

In a recent work, Ning and Xu proposed a generalization of bit-decomposition which can, given a sharing of secret x and an integer $m \geq 2$, compute the sharings (or bitwise sharings) of the base- m digits of x . They also proposed a linear protocol for public modulo reduction as a simplification of their generalization. In this paper, we show that their generalization can be further generalized. More importantly, as a simplification of our further generalization, we propose a public modulo reduction protocol which is more efficient than theirs. Specifically, the round complexity of our (modulo reduction) protocol is the same with theirs, but the communication complexity can be considerably lower.

Keywords. Multi-party Computation, Perfect Secure, Constant-Rounds, Linear, Exponentiation, Modulo Reduction, Bit-Decomposition.

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Contents

1	Introduction	3
1.1	Our Results	3
1.2	Related Work	4
2	Preliminaries	5
2.1	Notations and Conventions	5
2.2	Known Primitives	6
3	Multi-party Computation for Private Exponentiation with BD	7
3.1	The Public Exponentiation Protocol	7
3.2	The Bit Exponentiation Protocol	8
3.3	The Private Exponentiation Protocol with BD	9
4	Linear Multi-party Computation for Private Exponentiation	10
4.1	The Private Exponentiation Protocol without BD	10
4.2	A Further Improvement	11
5	Further Generalization of BD and Improved Solution for Public Modulo Reduction	13
6	Discussions	16
7	Conclusion and Future Work	16
A	Extending to \mathbb{Z}_N	18
B	An Overview of Known Primitives	19
C	An Overview of The New Protocols	19
D	Further Discussions on the Complexities of Private Exponentiation Protocols	20
E	Hierarchy of The Protocols for Exponentiation	20
F	A Concrete Example	21
G	Intuitions behind Protocols	22
G.1	Intuition behind Bit-Decomposition(BD)	22
G.2	Intuition behind Base- m Digit-Decomposition & Digit-Bit-Decomposition	23
G.3	Intuition behind Hybrid-base Digit-Decomposition & Digit-Bit-Decomposition	24
G.4	Intuition behind the Pub-MR(\cdot) Protocol in [NX10]	25
G.5	Intuition behind Our Pub-MR ⁺ (\cdot) Protocol	26

1 Introduction

Multi-party computation (MPC) is a powerful and interesting tool in cryptology. It allows a set of n mutually un-trusted parties to compute a predefined function f with their private information as inputs. After running the MPC protocol, the parties obtain only the predefined outputs but nothing else, and the privacy of their inputs is guaranteed. Although generic solutions for MPC (which can compute any function f) already exist [BGW88, GMW87], these solutions tend to be inefficient and thus not applicable for practical use. So, to fix this problem, we focus on constructing efficient protocols for specific functions.

Recently, in the work [DFK⁺06], Damgård *et al.* proposed a novel technique called *bit-decomposition* which can, in constant rounds, convert a polynomial sharing of secret x into the sharings of the bits of x . Bit-decomposition (which will often be referred to as **BD** hereafter for short) is a very useful tool for MPC. For example, after getting the sharings of the bits of some shared secrets using BD, we can securely perform Boolean operations on these secrets (such as computing the Hamming Weight, XOR, etc). Thus we can say that BD can be viewed as a “bridge” (in the world of MPC) connecting the arithmetic circuits and the Boolean circuits. What’s more, with the help of BD, we can construct constant-rounds protocols for some very important basic problems in MPC, such as *equality test*, *comparison*, *public modulo reduction* and *private exponentiation*, which will be referred to as four main applications of BD. After getting the bitwise sharings of the shared inputs to these problems (using BD), we will be able to use the *divide and conquer* technique to solve these problems.

However, a problem is, BD is relatively expensive in terms of round and communication complexities, and thus all the protocols relying on BD inherit this inefficiency. For example, the communication complexity of BD (with perfect security) is non-linear, thus all the protocols involving BD have a non-linear communication complexity. A feasible solution for this problem is to construct protocols for MPC problems without relying on BD. It is already proved that, three of the four main applications of BD, i.e. *equality test*, *comparison* and *public modulo reduction*, can be realized without relying on BD [NO07, NX10] and the main advantage is that the communication complexity can be reduced to linear (under the premise of ensuring perfect security). Thus a natural problem is whether a similar conclusion can be arrived at for another important application of BD: *private exponentiation*. This is generally believed to be impossible before (e.g. [NO07], Page 2; [RT09], Page 2), however, in this paper, we show that it can. What’s more, we show an improvement of the public modulo reduction protocol (without BD) proposed in [NX10]. The details of our results are presented below. Here we’d like to argue that although these four applications of BD can be realized without involving BD, this does *not* mean BD is meaningless for these problems because all these protocols (without relying on BD) depend heavily on the ideas, techniques and sub-protocols of BD and thus can be viewed as an extension of the research on BD.

1.1 Our Results

First we introduce some necessary notations. In this paper, we concern mainly about MPC based on linear secret sharing schemes (LSSS). Assume that the underlying LSSS is built on field \mathbb{Z}_p where p is a prime with bit-length l (i.e. $l = \lceil \log p \rceil$). For an element $x = (x_{l-1}, \dots, x_1, x_0) \in \mathbb{Z}_p$, we use $[x]_p$ to denote “*the sharing of x* ”, and $[x]_B$ to denote “*the bitwise sharing of x* ” (which will also be referred to as “*the sharings of the bits of x* ” or “*the shared base-2 form of x* ” in this paper), i.e. $[x]_B = ([x_{l-1}]_p, \dots, [x_1]_p, [x_0]_p)$.

Our work is mainly about two basic problems in MPC: the *private exponentiation problem* and

the *public modulo reduction problem*. We construct efficient protocols, which are *constant-rounds*, *linear and perfect secure*, for these two problems. The details are presented below.

The *private exponentiation problem* can be formalized as:

$$[x^a \bmod p]_p \leftarrow \text{Private-Exponentiation}([x]_p, [a]_p)$$

where $x, a \in \mathbb{Z}_p$.

Hereafter we will refer to $[x^a \bmod p]_p$ as $[x^a]_p$ for simplicity. For solving this problem, it seems that we must involve BD to get the bitwise sharing of the exponent, i.e. $[a]_B$. This is exactly the case in the private exponentiation protocol in [DFK⁺06]. However, in this paper we show that this is not necessary. That is to say, the private exponentiation problem can also be solved without relying on BD and the communication complexity can also be reduced to linear (*in the input length l*). Compared with the private exponentiation protocol in [DFK⁺06] (denoted as Pri-Expo-BD(\cdot)) in this paper), our protocol (denoted as Pri-Expo⁺(\cdot)) reaches lower round complexity and much lower communication complexity.

The *public modulo reduction problem* (which will be referred to as *Pub-MRP* for short) can be formalized as:

$$[x \bmod m]_p \leftarrow \text{Public-Modulo-Reduction}([x]_p, m)$$

where $x \in \mathbb{Z}_p$ and $m \in \{2, 3, \dots, p-1\}$.

Our work on this problem can be viewed as an extension of [NX10], in which Ning and Xu proposed a generalization of BD and, as a simplification of their generalization, they proposed a linear protocol for Pub-MRP without involving BD (denoted as Pub-MR(\cdot) in this paper). In this paper, we propose a further generalization of their generalization and, similarly and more importantly, as a simplification of our further generalization, we propose a protocol for Pub-MRP with improved efficiency (denoted as Pub-MR⁺(\cdot)). Specifically, the round complexity of our Pub-MR⁺(\cdot) protocol is the same with Pub-MR(\cdot) and, for relatively small m , the communication complexity is reduced by a factor of approximately 4.

Immodestly speaking, we believe (and of course not completely sure) that we have optimized the efficiency of the multi-party computation for both of these two basic problems; that is to say, any possible improvements may be minor. What's more, we'd like to stress that all the protocols constructed in this paper are constant-rounds and perfect secure. See Appendix C (Table 2) for an overview of our protocols.

1.2 Related Work

Both of the two problems considered in this paper, *exponentiation* and *modulo reduction*, are applications of bit-decomposition (BD). The problem of BD was first considered by Algesheimer *et al.* in [ACS02], in which a partial solution was proposed. The first full solution for BD in the secret sharing setting was proposed in [DFK⁺06] by Damgård *et al.* The main concern of this work is constant-rounds solution for BD and this is achieved by realizing various constant-rounds sub-protocols which are important building blocks for subsequent research including ours. What's more, as an application of BD, they also proposed a private exponentiation protocol which is the foundation of our work. Independently and concurrently, Schoenmakers and Tuyls [ST06] solved the problem of BD for MPC based on (Paillier) threshold homomorphic cryptosystems [CDN01, DN03] and they concern mainly about efficient variations of BD for practical use. In the work [NO07], Nishide and Ohta proposed solutions for interval test, comparison and equality test of shared

secrets without relying on the expensive BD protocol although it seems necessary. Their ideas and techniques play an important role in our work. Recently, Toft showed a novel technique that can reduce the communication complexity of BD to *almost linear* [Tof09]. This is a very meaningful work and some key ideas of our work come from it. In a followup work, Reistad and Toft proposed a linear BD protocol [RT09], however, the security of this protocol is non-perfect.

As for the public modulo reduction problem (Pub-MRP), Guajardo *et al.* proposed a protocol for it in the threshold homomorphic setting without relying on BD [GMS10]. Their protocol is very efficient (thus can be very useful for practical use) and is enlightening to this paper, however, they did not consider the general case (of Pub-MRP) where the inputs can be arbitrary size. In [NX10], Ning and Xu proposed a generalization of BD, and, as a simplification of their generalization, they proposed a linear protocol (without BD) for Pub-MRP which can deal with arbitrary inputs. Our work on Pub-MRP depends heavily on their work.

2 Preliminaries

In this section we introduce some important notations and known primitives.

2.1 Notations and Conventions

As mentioned above, the MPC considered in this paper is based on LSSS, such as Shamir’s [Sha79]. We denote the underlying field (of the LSSS) as \mathbb{Z}_p where p is a prime with bit-length $l = \lceil \log p \rceil$. For a secret $x \in \mathbb{Z}_p$, we use $[x]_p$ to denote the sharing of x and $[x]_B$ to denote the bitwise sharing of x . What’s more, assume that there are n participants in the MPC protocol.

As in previous works, such as [DFK⁺06, NO07], we assume that the underlying LSSS allows to compute $[x + y \bmod p]_p$ from $[x]_p$ and $[y]_p$ without communication, and that it allows to compute $[xy \bmod p]_p$ from (public) $x \in \mathbb{Z}_p$ and $[y]_p$ without communication. We also assume that the LSSS allows to compute $[xy \bmod p]_p$ from $[x]_p$ and $[y]_p$ through communication among the parties and we call this procedure *secure multiplication* (or *multiplication* for simplicity). One invocation of this *multiplication* will be denoted as

$$[xy \bmod p]_p \leftarrow \text{Sec-Mult}([x]_p, [y]_p)$$

in which $[xy \bmod p]_p$ will be referred to as $[xy]_p$ for simplicity. Obviously, for MPC protocols, this multiplication protocol is a dominant factor of complexity as it involves communication. So, as in previous works, the round complexity of the (MPC) protocols is measured by the number of rounds of parallel invocations of multiplication ($\text{Sec-Mult}(\cdot)$), and the communication complexity is measured by the number of invocations of multiplication. For example, if in all a protocol involves a multiplications in parallel and then another b multiplications in parallel, then we can say that the round complexity of this protocol is 2 and the communication complexity is $a + b$ multiplications. What’s more, if a procedure does not involve any secure multiplication, then it can be viewed as free and will not count for complexity. For example, if we get $[x]_B$, then $[x]_p$ can be freely obtained by a linear combination since $x = \sum_{i=0}^{l-1} x_i \cdot 2^i$.

As in [NO07], when we write $[C]_p$, where C is a Boolean test, it means that $C \in \{0, 1\}$ and $C = 1$ iff C is true. For example, we use $[x \stackrel{?}{=} y]_p$ to denote the output of the equality test protocol, i.e. $(x \stackrel{?}{=} y) = 1$ iff $x = y$ holds.

Given $[c]_p$, we need a protocol to reveal c , which is denoted by $c \leftarrow \text{Reveal}([c]_p)$. Note that although this protocol involves communication, it does not count for (both round and communication) complexity because the communication it involves can be carried out through a public

channel.

As in [Tof07], we will often use the *conditional selection command* below:

$$[C]_p \leftarrow [b]_p ? [A]_p : [B]_p$$

in which $A, B, C \in \mathbb{Z}_p$ and $b \in \{0, 1\}$, and which means the following:

If $b = 1$, then C is set to A ; otherwise, C is set to B .

It is easy to see that this command can be realized by setting

$$[C]_p \leftarrow [b]_p([A]_p - [B]_p) + [B]_p$$

which costs only 1 round and 1 multiplication. We will frequently use this *conditional selection command* in this paper because it can make our protocols easier to be understood.

2.2 Known Primitives

We will now simply introduce some existing primitives which will be of importance later on. We refer the readers to [DFK⁺06, NO07, Tof09] for detailed descriptions of these primitives.

- **Random Bit Protocol.** The **Random-Bit**(\cdot) protocol has no input and it will output a shared uniformly random bit $[b]_p$ which is unknown to all parties. In the secret sharing setting, it takes only 2 rounds and 2 multiplications [DFK⁺06].
- **Bitwise Less-Than Protocol.** Given two bitwise shared inputs, $[x]_B$ and $[y]_B$, the **Bit-LessThan**(\cdot) protocol can compute a shared bit $[x \stackrel{?}{<} y]_p$ which identifies whether $x < y$ holds. The complexity of this protocol can be referred to as 6 rounds and $14l$ multiplications when $l \geq 36$ holds which is often the case in practice [Tof09, NX10].
- **Secure Inversion Protocol.** Given a shared *non-zero* secret $[x]_p$ as input, the secure inversion protocol **Sec-Inver**(\cdot) will output $[x^{-1} \bmod p]_p$. This protocol will cost only 2 rounds and 2 multiplications [BB89, DFK⁺06, NO07].
- **Unbounded Fan-In Multiplication.** In this paper, we will often need to perform the unbounded fan-in secure multiplication [BB89, CFL83b], i.e. given l sharings $[A_0]_p, [A_1]_p, \dots, [A_{l-1}]_p$ where $A_i \in \mathbb{Z}_p$ for $i \in \{0, 1, \dots, l-1\}$, computing a sharing $[A]_p$ where $A = \prod_{i=0}^{l-1} A_i \bmod p$. By the detailed analysis in [NO07], we get to know that this protocol, denoted as **Sec-Prod***(\cdot) in this paper, can be realized in only 3 rounds and $5l$ multiplications.
- **Equal-Zero Test Protocol.** In [NO07], a linear protocol **Equ-Zero**(\cdot) was proposed for testing whether a given secret $[x]_p$ is 0 or not, i.e. we have $[x \stackrel{?}{=} 0]_p \leftarrow \text{Equ-Zero}([x]_p)$. Obviously, this protocol can also be used to test “whether two shared secrets $[x]_p$ and $[y]_p$ are equal” because “ $x = y$ ” \Leftrightarrow “ $(x - y) = 0$ ”. The complexity of this protocol is 8 rounds and $81l$ multiplications.
- **Generation of Bitwise Shared Random Value.** This protocol, denoted by **Solved-Bits**(\cdot), has no input and can output a bitwise shared random integer $[r]_B$ satisfying $r < p$. The complexity of this protocol can be referred to as 7 rounds and $56l$ multiplications when $l \geq 36$ [Tof09].
- **Bit-Decomposition (BD).** In the secret sharing setting, the function of BD can be described as converting $[x]_p$ to $[x]_B$, i.e. we have $[x]_B \leftarrow \text{BD}([x]_p)$ [DFK⁺06, Tof09]. (See Appendix G.1 for the intuition behind BD.) To the best of our knowledge, currently the most efficient version of BD (*with perfect security*) was proposed in [Tof09], whose complexity can be referred to as 23 rounds and $76l + 31l \log l$ multiplications when $l \geq 36$; in the text when analyzing the complexities of (exponentiation) protocols involving BD, we will refer to the complexity of BD

as above. We note that [Tof09] also proposed a BD protocol with *almost-linear* communication complexity (i.e. $O(l \log^* l)$ multiplications or even lower). This is of course a very meaningful work. However, inevitably the round complexity of this version of BD is relatively high and thus for obtaining (private exponentiation) protocols with close and comparable round complexities, (as well as for notational convenience) we do not refer to this BD protocol in detail in the text. (Although we focus mainly on the communication complexity of protocols, the round complexity should also be considered.) We also note that in [RT09], a linear BD is proposed, however, the security of this BD protocol is (at most) statistical; so in the text we will not refer to this BD protocol in detail neither, because we focus on protocols with perfect security.

See Appendix B (Table 1) for an overview of these known primitives.

3 Multi-party Computation for Private Exponentiation with BD

In [DFK⁺06], a constant-rounds private exponentiation protocol was constructed with the help of BD. This protocol is the foundation of our work and in our exponentiation protocol, we need to use the sub-protocols of it. So, in this section, we describe in detail this private exponentiation protocol *with* BD. We will first introduce two important sub-protocols of it, i.e. the *public exponentiation protocol* and the *bit exponentiation protocol*. All the protocols in this section are re-descriptions of the ones in [DFK⁺06] but with detailed analysis.

3.1 The Public Exponentiation Protocol

With a shared *non-zero* value $[x]_p$ (i.e. $x \in \mathbb{Z}_p^*$) and a public value $a \in \mathbb{Z}$ as inputs, the *public exponentiation protocol*, $\text{Pub-Expo}(\cdot)$, can compute $[x^a]_p$. The details are presented in Figure 1. Generally speaking, this protocol is a slightly improved version of the one in [DFK⁺06].

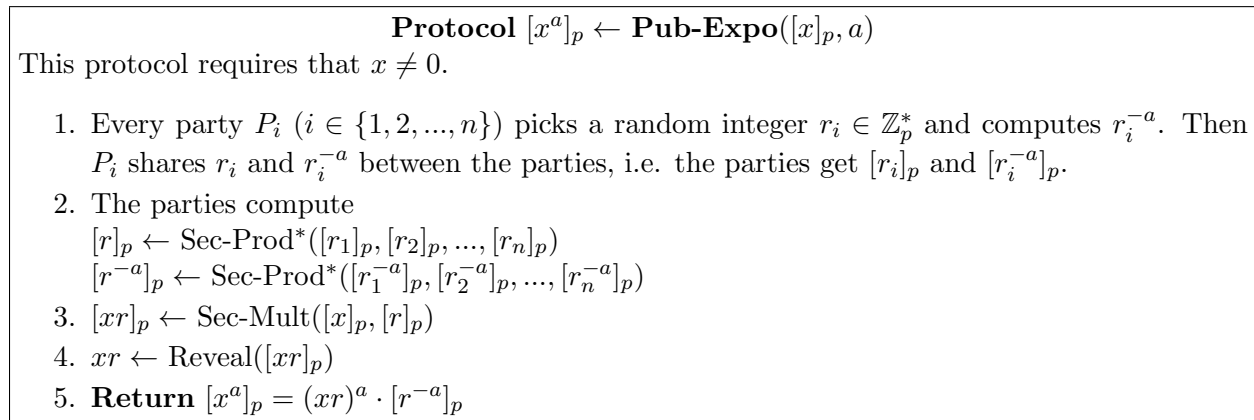


Figure 1: The Public Exponentiation Protocol

As for the correctness, notice that in Step 4 we need to reveal the value of xr where r is non-zero, and it is easy to see that $xr = 0 \Leftrightarrow x = 0$. This is just why this protocol requires that $x \neq 0$: if $x = 0$, then the parties will get to know this in this step. Also note that in this protocol the public exponent $-a$ is the additive inverse of a in the sense of mod $(p - 1)$ *rather than* mod p .

Privacy is straightforward.

The complexity will be discussed in two cases: the semi-honest case and the malicious case. The difference between these two cases lies in Step 1 where every party P_i ($i \in \{1, 2, \dots, n\}$) is required to distribute two sharings, $[r_i]_p$ and $[r_i^{-a}]_p$, between the parties. Below we will first analyze the complexity of this step. Before going on, recall the well-known fact that when considering the communication complexity of MPC protocols (in the LSSS setting), 1 invocation of the secure multiplication is equivalent to distributing n sharings between the n parties and thus the communication complexity of distributing 1 sharing (between the n parties) can be viewed as $\frac{1}{n}$ multiplications.

In the semi-honest case, all the n parties follow the protocol, so every party distributes 2 sharings between all the n parties, thus the complexity of Step 1 is 1 round and $\frac{2}{n} \cdot n = 2$ multiplications.

In the malicious case, as mentioned in [DFK⁺06], the complexity is much higher because we need to involve the *cut-and-choose* technique to make the protocol robust. Specifically, besides $[r_i]_p$ and $[r_i^{-a}]_p$, every party P_i ($i \in \{1, 2, \dots, n\}$) is required to distribute another two sharings $[s_i]_p$ and $[s_i^{-a}]_p$. Then the parties involve the Random-Bit(\cdot) protocol to jointly form a shared random bit $[b_i]_p$ and open it. Then they open $([s_i]_p, [s_i^{-a}]_p)$ or compute and open $([s_i r_i]_p, [s_i^{-a} r_i^{-a}]_p)$ according to the value of b_i and then verify that the first value is non-zero and that the second value is the $(-a)$ 'th power of the first. We call the above process *one instance of cut-and-choose*. For every party P_i ($i \in \{1, 2, \dots, n\}$), to get a lower error probability, we can repeat the above process k (which satisfies $k \geq 1$ and which will be referred to as “*the security parameter for cut-and-choose*”) times in parallel, leading to an error probability 2^{-k} . Then we can say that in all we need kn instances of cut-and-choose in parallel. As for the complexity of *one instance*, we notice the following facts: distributing $[s_i]_p$ and $[s_i^{-a}]_p$ between the parties involves $\frac{2}{n}$ multiplications; the generation of $[b_i]_p$ involves 2 rounds and 2 multiplications and can be scheduled in parallel with the process of distributing $[s_i]_p$ and $[s_i^{-a}]_p$; the computation of $([s_i r_i]_p, [s_i^{-a} r_i^{-a}]_p)$ involves 1 round and 2 multiplications and, obviously, on average we need only to compute $([s_i r_i]_p, [s_i^{-a} r_i^{-a}]_p)$ once every 2 instants of cut-and-choose because b_i is a uniformly random bit. So, on average, the complexity of one instance is (at most) $2 + 1 = 3$ rounds and $\frac{2}{n} + 2 + 2 \cdot \frac{1}{2} = \frac{2}{n} + 3$ multiplications. Recall that in all we need kn parallel instances of cut-and-choose. What's more, notice that the process of cut-and-choose can be scheduled in parallel with the process of distributing $[r_i]_p$ and $[r_i^{-a}]_p$. So, in the malicious case, the complexity of Step 1 is 3 rounds and $2 + kn \cdot (\frac{2}{n} + 3) = 2 + 2k + 3kn$ multiplications.

Then it is easy to see that, in the semi-honest case, the overall complexity of this Pub-Expo(\cdot) protocol is $R_{pub} \triangleq R_{pub}^{s-h} = 1 + 3 + 1 = 5$ rounds and $C_{pub} \triangleq C_{pub}^{s-h} = 2 + 5n \cdot 2 + 1 = 10n + 3$ multiplications; in the malicious case, the overall complexity is $R_{pub} \triangleq R_{pub}^{mal} = 3 + 3 + 1 = 7$ rounds and $C_{pub} \triangleq C_{pub}^{mal} = (2 + 2k + 3kn) + 5n \cdot 2 + 1 = 3kn + 10n + 2k + 3$ multiplications. Recall that n denotes the number of the parties and k is the security parameter for cut-and-choose. Hereafter, we will generally refer to the complexity of this protocol as R_{pub} rounds and C_{pub} multiplications. The values of R_{pub} and C_{pub} are determined by the adversaries considered; moreover, we can say that both R_{pub} and C_{pub} can be viewed as *constants* because they are independent from (the input length) l .

3.2 The Bit Exponentiation Protocol

With a shared *non-zero* value $[x]_p$ and a bitwise shared value $[a]_B = ([a_{l-1}]_p, \dots, [a_1]_p, [a_0]_p)$ as inputs, the *bit exponentiation protocol*, Bit-Expo(\cdot), can compute $[x^a]_p$. The details are seen in Figure 2.

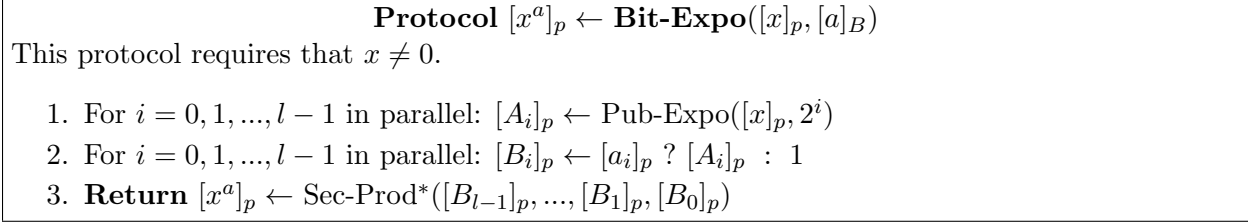


Figure 2: The Bit Exponentiation Protocol

Correctness and privacy is straightforward. The complexity of this protocol is $R_{pub} + 1 + 3 = R_{pub} + 4$ rounds and $C_{pub} \cdot l + l + 5l = (C_{pub} + 6)l$ multiplications.

3.3 The Private Exponentiation Protocol with BD

Here we come to the *private exponentiation protocol* relying on BD proposed in [DFK⁺06], which will be denoted by $\text{Pri-Expo-BD}(\cdot)$. Given two shared inputs $[x]_p$ and $[a]_p$, $\text{Pri-Expo-BD}(\cdot)$ will output $[x^a]_p$. This time, both x and a can be arbitrary values in \mathbb{Z}_p . See Figure 3 for the details.

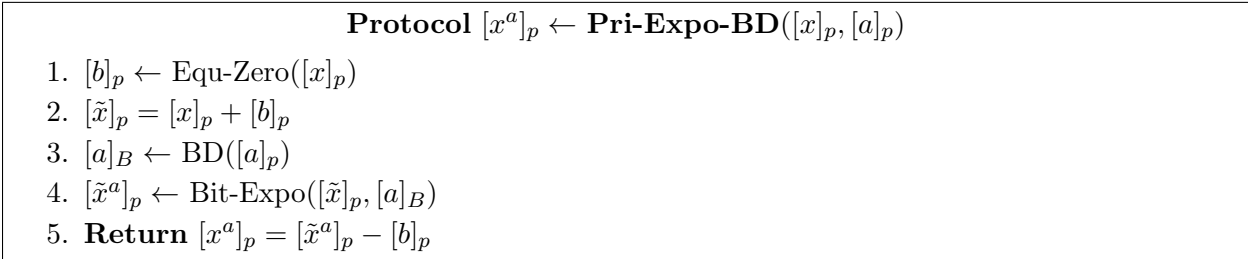


Figure 3: The Private Exponentiation Protocol *with* BD

As for the correctness, notice that $b = (x \stackrel{?}{=} 0)$ and that $[\tilde{x}]_p = [x]_p + [x \stackrel{?}{=} 0]_p$ is always non-zero and thus can be given to $\text{Bit-Expo}(\cdot)$ as the first input. What's more, it can be easily verified that $[x^a]_p = [\tilde{x}^a]_p - [x \stackrel{?}{=} 0]_p$ always holds no matter x is 0 or not. Using \tilde{x} to substitute x to perform the protocol is in fact the “exception trick” proposed in [DFK⁺06] for handling the special case where $x = 0$. This “exception trick” can be re-described using the *conditional selection command* as follows: first we set $[\tilde{x}]_p \leftarrow [x \stackrel{?}{=} 0]_p ? 1 : [x]_p$ thus \tilde{x} is always non-zero; then, after getting $[\tilde{x}^a]_p$, we can set the expected value $[x^a]_p$ to be $[x^a]_p \leftarrow [x \stackrel{?}{=} 0]_p ? 0 : [\tilde{x}^a]_p$.

Privacy follows readily from only using private sub-protocols.

As for the complexity, it is easy to see that only the three sub-protocols, i.e. $\text{Equ-Zero}(\cdot)$, $\text{BD}(\cdot)$ and $\text{Bit-Expo}(\cdot)$, count for complexity; what's more, $\text{Equ-Zero}(\cdot)$ and $\text{BD}(\cdot)$ can be scheduled in parallel. So the overall complexity of this protocol is $23 + (R_{pub} + 4) = R_{pub} + 27$ rounds and $81l + (76l + 31l \log l) + ((C_{pub} + 6)l) = 163l + C_{pub} \cdot l + 31l \log l$ multiplications. In the above complexity analysis, (as mentioned in Section 2.2) we view the BD involved as the one proposed in [Tof09] with $O(l \log l)$ communication complexity (which, taking both communication and round complexities into consideration, is now the most efficient BD with perfect security). See Appendix D (Table 3) for the complexity of this $\text{Pri-Expo-BD}(\cdot)$ protocol when different BDs are involved.

4 Linear Multi-party Computation for Private Exponentiation

In this section, we propose a private exponentiation protocol with constant round complexity and linear communication complexity. Specifically, we will first show how to remove the invocation of BD to get a protocol with linear communication complexity. Then we will further improve this linear protocol to reduce the communication complexity considerably.

4.1 The Private Exponentiation Protocol without BD

See Figure 4 for our private exponentiation protocol *without* BD which will be denoted as Pri-Expo(\cdot).

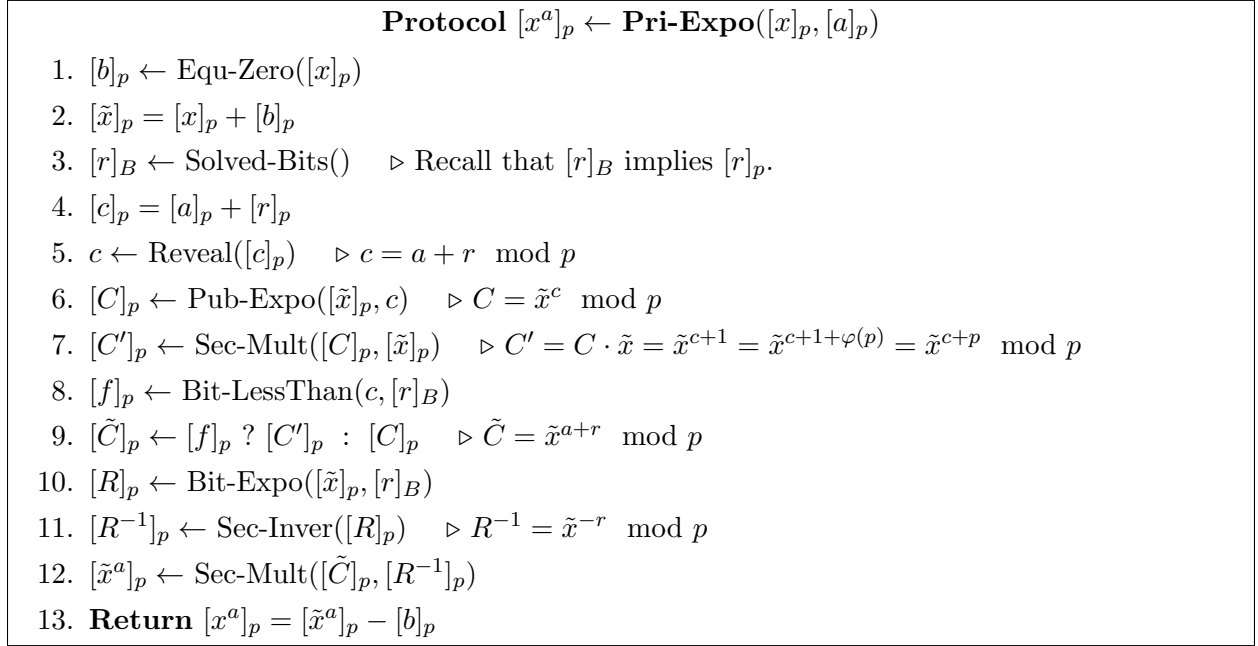


Figure 4: The Private Exponentiation Protocol *without* BD

As for the correctness, similar to the Pri-Expo-BD(\cdot) protocol (in Figure 3), we use the non-zero $[\tilde{x}]_p$ to substitute $[x]_p$ to perform the main process. The main idea of this protocol is as follows.

First we compute $[\tilde{C}]_p = [\tilde{x}^{a+r}]_p$. Notice that we have $c = a + r \pmod p$ and there are two cases: no wrap-around mod p occurs or there is a wrap-around. In the former case, $a + r = c$ holds over the integers (or we can say “ $a + r = c$ holds *unconditionally*”) and then we have $c \geq r$ because $a \geq 0$; similarly, in the latter case, $a + r = c + p$ holds over the integers (i.e. $a + r = c + p$ holds *unconditionally*) and then we have $c < r$ because $c = r + (a - p)$ and $a < p$. So, for computing $[\tilde{x}^{a+r}]_p$, we can compute both of the two possible values of it, $[\tilde{x}^c]_p$ and $[\tilde{x}^{c+p}]_p$, and then *select* the correct one; this selection can be carried out by testing whether $c < r$ holds. What’s more, when computing $[\tilde{x}^c]_p$ we need to involve the Pub-Expo(\cdot) protocol; however, this is not necessary when computing $[\tilde{x}^{c+p}]_p$ because we have: $\tilde{x}^{c+p} = \tilde{x}^{c+p-(p-1)} = \tilde{x}^{c+1} = \tilde{x}^c \cdot \tilde{x} \pmod p$.

Then, in the following steps, after getting $[R]_p = [\tilde{x}^r]_p$ using the Bit-Expo(\cdot) protocol, we can obtain $[\tilde{x}^a]_p$ based on the simple fact $\tilde{x}^a = \tilde{x}^{a+r} \cdot (\tilde{x}^r)^{-1} \pmod p$. Then finally $[x^a]_p$ can be easily obtained.

Privacy is straightforward because the only information revealed is the uniformly random c (Step 5).

As for the complexity, we notice the following facts: Step 1 and Step 3 can be scheduled in parallel; Step 6 through Step 9 can be schedule in parallel with Step 10 through Step 11; Step 6 through Step 7 can be scheduled in parallel with Step 8. So it is easy to see that, both in the semi-honest case and the malicious case, Step 6 through Step 9 involve $R_{pub} + 1 + 1 = R_{pub} + 2$ rounds and Step 10 through Step 11 involve $(R_{pub} + 4) + 2 = R_{pub} + 6$ rounds. So the round complexity of Step 6 through Step 11 is $R_{pub} + 6$. Thus, both in the semi-honest case and the malicious case, the complexity of this protocol (Pri-Expo(\cdot)) can be referred to as $8 + (R_{pub} + 6) + 1 = R_{pub} + 15$ rounds and

$81l + 56l + C_{pub} + 1 + 14l + 1 + (C_{pub} + 6)l + 2 + 1 = 157l + C_{pub} \cdot l + C_{pub} + 5$ multiplications. Recall that both R_{pub} and C_{pub} can be viewed as *constants*, so this is a constant-rounds protocol with linear communication complexity. Compared with the (perfect secure) Pri-Expo-BD(\cdot) protocol proposed in [DFK⁺06] (whose complexity is $R_{pub} + 27$ rounds and $163l + C_{pub} \cdot l + 31l \log l$ multiplications), our protocol has a lower round complexity and a significantly lower communication complexity. (See Appendix D (Table 3) for a detailed comparison.)

4.2 A Further Improvement

In this section, we make a further improvement of our Pri-Expo(\cdot) protocol above by improving one of the sub-protocols of it, Bit-Expo(\cdot), which is often the dominate factor of the communication complexity. The improved version of Pri-Expo(\cdot) and Bit-Expo(\cdot) will be denoted as Pri-Expo⁺(\cdot) and Bit-Expo⁺(\cdot) respectively. Generally speaking, by replacing the invocation of Bit-Expo(\cdot) with Bit-Expo⁺(\cdot) in our Pri-Expo(\cdot) protocol, we get our further improved private exponentiation protocol: Pri-Expo⁺(\cdot). The details are presented below.

In our Pri-Expo(\cdot) protocol (in Figure 4), Bit-Expo(\cdot) is a very important sub-protocol. Recall that the communication complexity of this sub-protocol is $(C_{pub} + 6)l$ multiplications; what's more, in the semi-honest case $C_{pub} = C_{pub}^{s-h} = 10n + 3$, and in the malicious case $C_{pub} = C_{pub}^{mal} = 3kn + 10n + 2k + 3$ (see Section 3.1). In many cases, Bit-Expo(\cdot) is relatively expensive. For example, in the malicious case, if we set $n = 20$ and $k = 10$, then the communication complexity of Bit-Expo(\cdot) will be $(C_{pub}^{mal} + 6)l = 829l$ multiplications; at the same time, the communication complexity of the (*whole*) Pri-Expo(\cdot) protocol is $(C_{pub}^{mal} + 157)l + C_{pub}^{mal} + 5 = 980l + 828$ multiplications. So we can see that, in this case, Bit-Expo(\cdot) is obviously a dominate factor of the communication complexity of Pri-Expo(\cdot). So, reducing the communication complexity of Bit-Expo(\cdot) is very meaningful.

The communication complexity of Bit-Expo(\cdot) comes mainly from the l invocations of Pub-Expo(\cdot) which is non-trivial (See Figure 2 and Figure 1). Here we show a technique that can reduce the number of invocations (of Pub-Expo(\cdot)) to $2\sqrt{l}$ (with slight increase in round complexity) and thus reduce the communication complexity significantly. The main idea is presented below.

Consider the case that we want to compute x^a . We divide the given exponent $a = (a_{l-1}, \dots, a_1, a_0)$ into s blocks, each of which contains t bits. Obviously, we have $s \cdot t = l$ and $1 \leq s, t \leq l$. We denote the i 'th block of a as $a_i^{s \times t}$ for $i \in \{0, 1, \dots, s-1\}$, and denote the j 'th bit of the i 'th block as $a_{i,j}^{s \times t}$ for $j \in \{0, 1, \dots, t-1\}$. That is to say, we have

$$\begin{aligned} a &= (a_{l-1}, \dots, a_1, a_0) = (a_{s-1}^{s \times t}, \dots, a_1^{s \times t}, a_0^{s \times t}) \\ &= \left(\left(a_{s-1,t-1}^{s \times t}, \dots, a_{s-1,1}^{s \times t}, a_{s-1,0}^{s \times t} \right), \dots, \left(a_{1,t-1}^{s \times t}, \dots, a_{1,1}^{s \times t}, a_{1,0}^{s \times t} \right), \left(a_{0,t-1}^{s \times t}, \dots, a_{0,1}^{s \times t}, a_{0,0}^{s \times t} \right) \right) \end{aligned}$$

Obviously, $a_i^{s \times t}$ can be viewed as the i 'th digit of the base- 2^t form of a . What's more, we have $a_{i,j}^{s \times t} = a_{i \cdot t + j}$. Now we have the following equations:

$$\begin{aligned} x^a &= x^{\sum_{i=0}^{s-1} a_i^{s \times t} \cdot (2^t)^i} = \prod_{i=0}^{s-1} x^{a_i^{s \times t} \cdot (2^t)^i} = \prod_{i=0}^{s-1} \left(x^{a_i^{s \times t}} \right)^{(2^t)^i} = \prod_{i=0}^{s-1} \left(x^{\sum_{j=0}^{t-1} a_{i,j}^{s \times t} \cdot 2^j} \right)^{(2^t)^i} \\ &= \prod_{i=0}^{s-1} \left(\prod_{j=0}^{t-1} x^{a_{i,j}^{s \times t} \cdot 2^j} \right)^{(2^t)^i} = \prod_{i=0}^{s-1} \left(\prod_{j=0}^{t-1} \left(x^{2^j} \right)^{a_{i,j}^{s \times t}} \right)^{(2^t)^i} = \prod_{i=0}^{s-1} \left(\prod_{j=0}^{t-1} \left(a_{i,j}^{s \times t} ? x^{2^j} : 1 \right) \right)^{(2^t)^i} \end{aligned}$$

Based on the above facts, we propose our improved Bit-Expo(\cdot) protocol, Bit-Expo⁺(\cdot), which is presented in Figure 5. Note that in Figure 5, for the convenience of the forthcoming discussions, the two variables, s and t , are not assigned. We will discuss how to assign them when analyzing the complexity of this protocol.

Protocol $[x^a]_p \leftarrow \mathbf{Bit-Expo}^+([x]_p, [a]_B)$

This protocol requires that $x \neq 0$.

1. For $j = 0, 1, \dots, t - 1$ in parallel: $[A_j]_p \leftarrow \text{Pub-Expo}([x]_p, 2^j)$
2. For $i = 0, 1, \dots, s - 1$ in parallel do
 - For $j = 0, 1, \dots, t - 1$ in parallel: $[B_{i,j}]_p \leftarrow [a_{i,j}^{s \times t}]_p ? [A_j]_p : 1 \quad \triangleright a_{i,j}^{s \times t} = a_{i \cdot t + j}$
 - $[B_i]_p \leftarrow \text{Sec-Prod}^*([B_{i,0}]_p, [B_{i,1}]_p, \dots, [B_{i,t-1}]_p)$
 - $[C_i]_p \leftarrow \text{Pub-Expo}([B_i]_p, (2^t)^i)$
 - End for
3. **Return** $[x^a]_p \leftarrow \text{Sec-Prod}^*([C_0]_p, [C_1]_p, \dots, [C_{s-1}]_p)$

Figure 5: The *Improved* Bit Exponentiation Protocol

Correctness and privacy is straightforward. As for the complexity, notice that there are invocations of Pub-Expo(\cdot) in both Step 1 and Step 2. One important point is, these two places of invocations can be scheduled partially in parallel. Specifically, when the invocations (of Pub-Expo(\cdot)) in Step 1 are proceeding with the first two steps of Pub-Expo(\cdot) (See Figure 1), the invocations in Step 2 can also proceed with them. That is to say, although these two places of invocations can *not* be scheduled (completely) in parallel, they will cost only 1 more round than one single invocation (note that Step 3 through Step 5 in Pub-Expo(\cdot) (Figure 1) involve only 1 multiplication). So, the complexity of this protocol is $R_{pub} + 1 + 3 + 1 + 3 = R_{pub} + 8$ rounds and $C_{pub} \cdot t + t \cdot s + 5t \cdot s + C_{pub} \cdot s + 5s \leq C_{pub} \cdot (s + t) + 11l$ multiplications. (Recall that $s \cdot t = l$ and $1 \leq s \leq l$.)

It remains to assign concrete values to s and t . Note that we have “ $s + t \geq 2\sqrt{s \cdot t} = 2\sqrt{l}$ ” and “ $s + t = 2\sqrt{l}$ iff $s = t = \sqrt{l}$ ”. So we should set $s = t = \sqrt{l}$, because in this case the communication complexity of this Bit-Expo⁺(\cdot) protocol will be the lowest, i.e. $C_{pub} \cdot 2\sqrt{l} + 11l$ multiplications. Then, if we replace the invocation of Bit-Expo(\cdot) in our Pri-Expo(\cdot) protocol (in Figure 4) with the Bit-Expo⁺(\cdot) protocol here, we will get an improved private exponentiation protocol (denoted as Pri-Expo⁺(\cdot)) whose complexity is $R_{pub} + 19$ rounds and $162l + C_{pub} \cdot 2\sqrt{l} + C_{pub} + 5$ multiplications. Compared with the Pri-Expo-BD(\cdot) protocol in [DFK⁺06] (whose complexity is $R_{pub} + 27$ rounds and $163l + C_{pub} \cdot l + 31l \log l$ multiplications), our Pri-Expo⁺(\cdot) protocol reaches lower round complexity and much lower communication complexity. What's more, we can say that, the larger C_{pub}

is (which implies larger n and k), the greater advantage our protocol has. For systems with relatively more participants, higher security requirements and longer input length (i.e. l), our protocol can be of overwhelming advantage. (See Appendix C (Table 2) for an overview.)

Here is a further discussion. Consider the case where we need X ($1 \leq X \leq l$) invocations of this Bit-Expo⁺(\cdot) protocol (*not* necessarily be parallel) and where the first input to these invocations is the same value $[x]_p$ and the second input is different, e.g. $[\bar{a}_0]_B, [\bar{a}_1]_B, \dots, [\bar{a}_{X-1}]_B$ where $\bar{a}_i \in \mathbb{Z}_p$ for $i \in \{0, 1, \dots, X-1\}$. (This case will appear when we proceed with X invocations of Pri-Expo⁺(\cdot) with the first input be the same and the second be different.) It is easy to see that when proceeding with these X invocations (of Bit-Expo⁺(\cdot)), all the $[A_j]_p$ ($j = 0, 1, \dots, t-1$) computed in Step 1 of Bit-Expo⁺(\cdot) (in Figure 5) can be reused between these invocations. That is to say, in all we need only to run Step 1 (in Figure 5) once for all these X invocations. So the communication complexity of these X invocations is *not* X times of one single invocation (i.e. $X \cdot (C_{pub} \cdot (s+t) + 11l)$) *but* $C_{pub} \cdot (Xs+t) + 11Xl$. Then by the fact that “ $Xs+t \geq 2\sqrt{Xs \cdot t} = 2\sqrt{Xl}$ ” and that “ $Xs+t = 2\sqrt{Xl}$ ” iff $Xs = t = \sqrt{Xl}$ ”, we can see, if we set $s = \sqrt{\frac{l}{X}}$ and $t = \sqrt{Xl}$, we can optimize the efficiency of these X invocations.

See Appendix A for some discussions on extending our techniques (on exponentiation) to \mathbb{Z}_N , and see Appendix E (Figure 6) for a simple hierarchy of the protocols for exponentiation.

5 Further Generalization of BD and Improved Solution for Public Modulo Reduction

In this section, we propose a further generalization of BD and an improved solution for Pub-MRP. The work in this section depends *heavily* on the work in [NX10] which we’d strongly recommend the readers to read before going on.

Given a sharing of secret x , BD allows the parties to extract the shared *base-2 form* of x in constant rounds (See Appendix G.1 for the intuition behind BD). In the work [NX10], Ning and Xu show us a generalization of BD which is named as “Base- m Digit-Decomposition” (or “Base- m Digit-Bit-Decomposition”) and which can extract the shared (or bitwise shared) *base- m form* of x in constant rounds. We note that their generalization can be further generalized to a “Hybrid-base Digit-Decomposition” (or “Hybrid-base Digit-Bit-Decomposition”) protocol which can extract the shared (or bitwise shared) *hybrid-base form* of x ; here *hybrid-base* means the base of every digit can be different. For example, if we denote “9 days 23 hours 59 minutes 59 seconds” (which could be the “Time Left” before the submission deadline of this conference) as $x = \boxed{9} \boxed{23} \boxed{59} \boxed{59}$, then x can be used to represent the *total* seconds (left) and can be viewed as a hybrid-base integer with bases (from left to right) 10, 24, 60, 60. Here the left-most base (i.e. “10”) can be set as we wish, but other bases are fixed.

Below we discuss the relationship between “the value of an integer” and “the bases” in another point of view. Specifically, we list 3 cases below.

1. Getting the base-2 form of $x \in \mathbb{Z}_p$

In this case, we get $l = \lceil \log p \rceil$ bits $x_i \in \{0, 1\}$ for $i \in \{0, 1, \dots, l-1\}$ satisfying

$$x = \sum_{i=0}^{l-1} (x_i \cdot 2^i)$$

2. Getting the base- m form of $x \in \mathbb{Z}_p$

Similarly, in this case, for the given base $m \geq 2$, we get $l^{(m)} = \lceil \log_m p \rceil$ digits $x_i^{(m)} \in \{0, 1, \dots, m-1\}$ for $i \in \{0, 1, \dots, l^{(m)}-1\}$ satisfying

$$x = \sum_{i=0}^{l^{(m)}-1} \left(x_i^{(m)} \cdot m^i \right)$$

3. Getting the hybrid-base form of $x \in \mathbb{Z}_p$

Given an $l^{(M)}$ size “array of bases” $M[\] = [m_{l^{(M)}-1}, \dots, m_1, m_0]$ satisfying $m_i \geq 2$ for $i \in \{0, 1, \dots, l^{(M)}-1\}$ and $\prod_{i=0}^{l^{(M)}-2} m_i < p < \prod_{i=0}^{l^{(M)}-1} m_i$, we get $l^{(M)}$ digits $x_i^{(M)} \in \{0, 1, \dots, m_i-1\}$ for $i \in \{0, 1, \dots, l^{(M)}-1\}$ satisfying the following equation (in which we set $m_{-1} = 1$)

$$x = \sum_{i=0}^{l^{(M)}-1} \left(x_i^{(M)} \cdot \prod_{j=-1}^{i-1} m_j \right)$$

Here, we call $(x_{l^{(M)}-1}^{(M)}, \dots, x_1^{(M)}, x_0^{(M)})$ “the hybrid-base form of x defined by $M[\]$ ”.

It is easy to see that in the hybrid-base case (i.e. Case 3) if we set the “array of bases” $M[\]$ to be $[m, \dots, m, m]$ where $l^{(M)} = l^{(m)}$, then we will get the base- m case (i.e. Case 2); if we set $M[\] = [2, \dots, 2, 2]$ where $l^{(M)} = l$, we will get the base-2 case (i.e. Case 1).

Given a shared secret $[x]_p$ and an “array of bases” $M[\]$, our “Hybrid-base Digit-Decomposition” (or “Hybrid-base Digit-Bit-Decomposition”) protocol, whose asymptotic complexity is $O(1)$ rounds and $O(l^{(M)} \log l^{(M)})$ (or $O(l \log l)$) multiplications, can output the shared (or bitwise shared) hybrid-base form of x defined by $M[\]$ (i.e. the sharings (or bitwise sharings) of all the digits $x_i^{(M)}$ for $i \in \{0, 1, \dots, l^{(M)}-1\}$) which will be referred to as $[x]_D^M$ (or $[x]_{D,B}^M$). That is to say, we have

$$\begin{aligned} [x]_D^M &\leftarrow \text{Hybrid-Base-Digit-Decomposition}([x]_p, M[\]); \\ [x]_{D,B}^M &\leftarrow \text{Hybrid-Base-Digit-Bit-Decomposition}([x]_p, M[\]). \end{aligned}$$

See Appendix F for a concrete example.

The intuition behind our further generalization is similar to that of the generalization of BD in [NX10]. Specifically, as shown in [NX10], for getting the shared (or bitwise shared) base- m form of x , we need to randomize $[x]_p$ using a jointly generated random integer r whose bitwise shared base- m form is known to the parties; that is to say, the parties generate an array of bitwise shared base- m digits to form r ; here, a *base- m digit* is in fact a non-negative integer less than m . (See Appendix G.2 for the intuition.) Similarly, to obtain the shared (or bitwise shared) hybrid-base form of x (which is defined by $M[\]$), we should randomize $[x]_p$ using a (jointly generated random) integer r^+ whose bitwise shared hybrid-base form (*which is also defined by $M[\]$*) is known to the parties (See Appendix G.3 for the intuition). This is the key idea of our further generalization and is also the key difference between the generalization in [NX10] and our further generalization. Our further generalization can be useful in practice because we may often use hybrid-base values in real life, for example, values about time, the license plate numbers (which always include *letters* and *numbers*) and the ID-card numbers in some countries, etc. On the other hand, it may also be useful in theory due to the loose restriction on the bases (i.e. the “array of bases” $M[\]$); we can set $M[\] = [m_{l^{(M)}-1}, \dots, m_1, m_0]$ freely and the only constraint is $m_i \geq 2$ for $i \in \{0, 1, \dots, l^{(M)}-1\}$.

For example, if we set $m_i = i + 2$, then for a given shared secret x , we can get the sharings (or bitwise sharings) of all the digits $x_i^{(M)} \in \{0, 1, \dots, i + 1\}$ for $i \in \{0, 1, \dots, l^{(M)} - 1\}$ satisfying $x = \sum_{i=0}^{l^{(M)}-1} \left(x_i^{(M)} \cdot (i + 1)! \right)$ which may be useful somewhere.

More importantly, as a simplification of our “Hybrid-base Digit-Decomposition” protocol, we can get an *improved* public modulo reduction protocol (denoted as Pub-MR⁺(·) here) which is more efficient than the one in [NX10] (denoted as Pub-MR(·)). Specifically, in [NX10], for solving Pub-MRP (i.e. computing $[x \bmod m]_p$ from $[x]_p$ and $m \in \{2, 3, \dots, p - 1\}$), Ning and Xu view this problem as extracting only (the sharing of) the least significant base- m digit of x , and thus their modulo reduction protocol (i.e. Pub-MR(·)) can be viewed as a simplification of their “Base- m Digit-Decomposition” protocol (which extracts (the sharings of) all the base- m digits of x). In another point of view, we can say that they set $M[\] = [m, \dots, m, m]$ and extract only (the sharing of) $x_0^{(M)}$ (see Case 3 above). (See Appendix G.4 for the intuition.) This is of course correct because in this case we have $x = \sum_{i=0}^{l^{(M)}-1} \left(x_i^{(M)} \cdot m^i \right)$. However, this is not a must, and, enlightened by [NO07] and [GMS10], we find that, by setting $M[\] = [2, \dots, 2, 2, m]$ where $l^{(M)} = \lceil \log \lfloor \frac{p}{m} \rfloor \rceil + 1$, we can also solve Pub-MRP because in this case we have $x = \sum_{i=1}^{l^{(M)}-1} \left(x_i^{(M)} \cdot 2^{i-1} \cdot m \right) + x_0^{(M)}$ and thus $x_0^{(M)} = x \bmod m$. That is to say, in the case where $M[\] = [2, \dots, 2, 2, m]$, if we extract only (the sharing of) the least significant digit of x , which can be viewed as a simplification of our “Hybrid-base Digit-Decomposition”, we can also get $[x \bmod m]_p$; this public modulo reduction protocol is just Pub-MR⁺(·) (See Appendix G.5 for the intuition).

Below we show the advantage of our Pub-MR⁺(·) protocol over Pub-MR(·). Similar to the generalization and further generalization of BD, when computing $[x \bmod m]_p$ from $[x]_p$ and m , both Pub-MR(·) and our Pub-MR⁺(·) need to use a jointly generated random integer to randomize $[x]_p$. The key difference between these two (modulo reduction) protocols lies just in the generation of this random integer. Specifically, in the Pub-MR(·) protocol, the random integer needed, denoted as r here, should be of a “hybrid-base” form defined by $M[\] = [m, \dots, m, m]$ where $l^{(M)} = l^{(m)}$ (see Appendix G.4); whereas in our Pub-MR⁺(·) protocol, the random integer needed, denoted as r^+ , should be of a hybrid-base form defined by $M[\] = [2, \dots, 2, 2, m]$ where $l^{(M)} = \lceil \log \lfloor \frac{p}{m} \rfloor \rceil + 1$ (see Appendix G.5). Then obviously, in Pub-MR(·) when generating r , we need to generate $l^{(M)} = l^{(m)}$ (bitwise shared) base- m digits, whereas in our Pub-MR⁺(·) when generating r^+ , we need only to generate 1 such base- m digit. This is just the advantage of our improved public modulo reduction protocol. Reducing the demand for such base- m digits is very meaningful because the generation of them is a non-trivial work. Specifically, when m is a non-power of 2, roughly speaking the generation of 1 such digit will cost 8 rounds and $64L(m)$ multiplications where $L(m) \triangleq \lceil \log m \rceil$ denotes the bit-length of m [NX10].

Finally, we conclude that, the complexity of our Pub-MR⁺(·) protocol is 22 rounds and (about) $78l + 276L(m)$ multiplications. Compared with Pub-MR(·) (whose complexity is 22 rounds and (about) $326l + 28L(m)$ multiplications), we can see that, for relatively small m (thus $L(m)$ is very small), the communication complexity is reduced considerably. For example, pick $l = 256, m = 100$, then $L(m) = \lceil \log 100 \rceil = 7$; in this case, our Pub-MR⁺(·) protocol involves (about) $78 * 256 + 276 * 7 = 21900$ secure multiplications, whereas Pub-MR(·) (in [NX10]) involves (about) $326 * 256 + 28 * 7 = 83652$ secure multiplications; that is to say the communication complexity is reduced by a factor of $\frac{83652}{21900} \approx 3.8$.

6 Discussions

We note that using the idea in [NO07], the round complexity of our private exponentiation protocols (as well as our public modulo reduction protocol) can be improved; the method is to use preprocessing, i.e. moving all the generation of (shared) random values (e.g. the invocations of $\text{Random-Bit}(\cdot)$) to the beginning of the whole protocol. In the analysis of the round complexity of our protocols, we just ignore this for simplicity.

An interesting point is that the communication complexity of our “Hybrid-base Digit-Decomposition” protocol and “Hybrid-base Digit-Bit-Decomposition” protocol can be reduced to “almost linear” using the techniques of [Tof09]; this is because the techniques proposed in [Tof09] can also be used to fast the dominate part of these two protocols, i.e. computing a prefix- \circ [DFK⁺06, NX10].

All the protocols presented in this paper can be easily translated to the Threshold Homomorphic Cryptosystem (THC) setting (maybe with some minor modifications); however, due to the less efficient realization of the $\text{Random-Bit}(\cdot)$ protocol in the THC setting, the efficiency of protocols in this setting may be lower than their counterpart in the LSSS setting.

7 Conclusion and Future Work

Conclusion. In this paper, we first focus on the open problem whether the *private exponentiation problem* (for computing $[x^a \bmod p]_p$ from $[x]_p$ and $[a]_p$) can be solved without relying on bit-decomposition and show that it can. Specifically, we propose a private exponentiation protocol (i.e. $\text{Pri-Expo}^+(\cdot)$) without bit-decomposition which, under the premise of ensuring perfect security, reaches constant round complexity and linear communication complexity. From Appendix C (Table 2, which is an overview of the new protocols in this paper) we can see that our (private exponentiation) protocol is much more efficient than the previous solution in [DFK⁺06] (i.e. $\text{Pri-Expo-BD}(\cdot)$).

Then, we propose an extension of [NX10] in which Ning and Xu proposed a generalization of bit-decomposition which can output the shared (or bitwise shared) base- m form of a shared secret x (“Base- m Digit-Decomposition” or “Base- m Digit-Bit-Decomposition”) and, as a simplification of their generalization, they also proposed a linear protocol for the public modulo reduction problem (for computing $[x \bmod m]_p$ from $[x]_p$ and a public m) without relying on bit-decomposition (i.e. $\text{Pub-MR}(\cdot)$). Based on their work, we propose a further generalization of bit-decomposition which can output the shared (or bitwise shared) hybrid-base form of x (“Hybrid-base Digit-Decomposition” or “Hybrid-base Digit-Bit-Decomposition”) and as a simplification of our further generalization, we show an improved public modulo reduction protocol ($\text{Pub-MR}^+(\cdot)$).

Future Work. In our private exponentiation protocol, we need an important sub-protocol called *public exponentiation protocol* (i.e. $\text{Pub-Expo}(\cdot)$) for computing $[x^a \bmod p]_p$ from $[x]_p$ and a public a . A problem is, the communication complexity of this sub-protocol is relatively high and more importantly, the communication complexity depends on n and k (see Section 3.1 for the details). We leave it an open problem to construct more efficient protocols for this problem and protocols with communication complexity independent from n and k would be most welcome. What’s more, in our private exponentiation protocol when involving $\text{Pub-Expo}(\cdot)$, the second input (i.e. the public a in Figure 1) is (almost) always a power of 2. So designing more efficient public exponentiation protocols for this special case is also meaningful.

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A Extending to \mathbb{Z}_N

Although we describe our private exponentiation protocol in the secret sharing setting where all the operations take place over \mathbb{Z}_p , our techniques can also be used to solve the private exponentiation problem in the (Paillier) threshold homomorphic setting where operations take place over \mathbb{Z}_N ($N = pq$) [CDN01, DN03]. However, we need some adaptations. For example, in the secret sharing setting, when computing the sharing of $(x^{c+p} \bmod p)$ from the sharing of x and the public $c + p$, we can make use of the equation $x^{c+p} = x^{c+p-\varphi(p)} = x^{c+1} = x^c \cdot x \bmod p$ to simplify the problem and thus omit the invocation of Pub-Expo(\cdot) (See Figure 4, Step 6 and Step 7). However, when we perform a similar operation in the threshold homomorphic setting, i.e. computing the encryption of $(x^{c+N} \bmod N)$ from the encryption of x and the public $c + N$, this simplification can *not* be carried out because the parties do not know $\varphi(N)$ and thus we need to involve Pub-Expo(\cdot). The absence of the knowledge of $\varphi(N)$ also causes the parties, when running the Pub-Expo(\cdot) protocol, unable to get $-a \bmod \varphi(N)$ (see Figure 1) and thus they must involve the Sec-Inver(\cdot) protocol (see Section 2.2). Another problem is when involving our Pri-Expo⁺(\cdot) protocol (as well as Pri-Expo(\cdot)) with the encryptions of $x \in \mathbb{Z}_N - \mathbb{Z}_N^* - \{0\}$ and $a \in \mathbb{Z}_N$ as inputs, it will abort in Step 11 (in Figure 4) because $\tilde{x} = x$ is non-invertible. However, we can simply ignore this because getting such an x is equivalent to factoring N and thus the probability is negligible.

B An Overview of Known Primitives

Protocol Name	Description	Rounds	Multiplications
Sec-Mult(\cdot)	$[xy]_p \leftarrow \text{Sec-Mult}([x]_p, [y]_p)$	1	1
Reveal(\cdot)	$c \leftarrow \text{Reveal}([c]_p)$	0	0
Random-Bit(\cdot)	$[b]_p \leftarrow \text{Random-Bit}()$	2	2
Bit-LessThan(\cdot)	$[x < y]_p \leftarrow \text{Bit-LessThan}([x]_B, [y]_B)$	6	$14l$
Sec-Inver(\cdot)	$[x^{-1} \bmod p]_p \leftarrow \text{Sec-Inver}([x]_p)$	2	2
Sec-Prod*(\cdot)	$\left[\prod_{i=0}^{l-1} A_i \right]_p \leftarrow \text{Sec-Prod}^*([A_0]_p, [A_1]_p, \dots, [A_{l-1}]_p)$	3	$5l$
Equ-Zero(\cdot)	$[x \stackrel{?}{=} 0]_p \leftarrow \text{Equ-Zero}([x]_p)$	8	$81l$
Solved-Bits(\cdot)	$[r]_B \leftarrow \text{Solved-Bits}()$	7	$56l$
BD(\cdot)	$[x]_B \leftarrow \text{BD}([x]_p)$	23	$76l + 31l \log l$

Table 1: Overview of Known Primitives

C An Overview of The New Protocols

The details are presented in Table 2. Below are some notes.

As mentioned in Section 3.1, R_{pub} represents the round complexity of the public exponentiation protocol (i.e. Pub-Expo(\cdot)) and C_{pub} represents the communication complexity (of Pub-Expo(\cdot)), and the values of R_{pub} and C_{pub} are determined by the adversary considered. Specifically, in the semi-honest case, $R_{pub} = R_{pub}^{s-h} = 5$ and $C_{pub} = C_{pub}^{s-h} = 10n + 3$; in the malicious case, $R_{pub} = R_{pub}^{mal} = 7$ and $C_{pub} = C_{pub}^{mal} = 3kn + 10n + 2k + 3$, in which n denotes the number of the participants of the MPC protocol and k is the security parameter for cut-and-choose. Both R_{pub} and C_{pub} can be viewed as *constants* because they are independent from (the input length) l .

What's more, as mentioned in Section 5, $L(m) = \lceil \log m \rceil$ represents the bit-length of m .

Protocol Name	Description	Rounds	Multiplications
Pub-Expo(\cdot)	$[x^a]_p \leftarrow \text{Pub-Expo}([x]_p, a)$	R_{pub}	C_{pub}
Bit-Expo(\cdot)	$[x^a]_p \leftarrow \text{Bit-Expo}([x]_p, [a]_B)$	$R_{pub} + 4$	$C_{pub} \cdot l + 6l$
Bit-Expo ⁺ (\cdot)	$[x^a]_p \leftarrow \text{Bit-Expo}^+([x]_p, [a]_B)$	$R_{pub} + 8$	$C_{pub} \cdot 2\sqrt{l} + 11l$
Pri-Expo-BD(\cdot)	$[x^a]_p \leftarrow \text{Pri-Expo-BD}([x]_p, [a]_p)$	$R_{pub} + 27$	$163l + C_{pub} \cdot l + 31l \log l$
Pri-Expo(\cdot)	$[x^a]_p \leftarrow \text{Pri-Expo}([x]_p, [a]_p)$	$R_{pub} + 15$	$157l + C_{pub} \cdot l + C_{pub} + 5$
Pri-Expo ⁺ (\cdot)	$[x^a]_p \leftarrow \text{Pri-Expo}^+([x]_p, [a]_p)$	$R_{pub} + 19$	$162l + C_{pub} \cdot 2\sqrt{l} + C_{pub} + 5$
Pub-MR(\cdot)	$[x \bmod m]_p \leftarrow \text{Pub-MR}([x]_p, m)$	22	$326l + 28L(m)$
Pub-MR ⁺ (\cdot)	$[x \bmod m]_p \leftarrow \text{Pub-MR}^+([x]_p, m)$	22	$78l + 276L(m)$

Table 2: Overview of The New Protocols

D Further Discussions on the Complexities of Private Exponentiation Protocols

To ensure the fairness of comparison, we assume that all the exponentiation protocols listed here involve the (sub-protocol) $\text{Bit-Expo}(\cdot)$ in Section 3.2, rather than the improved one in Section 4.2 (i.e. $\text{Bit-Expo}^+(\cdot)$). What’s more, the old versions of BD proposed in [DFK⁺06, NO07] are not considered in this section (as well as in this paper) because they are less efficient than the one proposed in [Tof09]. Note that R_{pub} and C_{pub} can be viewed as constants (i.e. they are both independent from l).

Protocol Name	BD involved	Rounds	Multiplications	Security
Pri-Expo-BD(\cdot)	[Tof09] $O(l \log l)$	$R_{pub} + 27$	$163l + C_{pub} \cdot l + 31l \log l$	Perfect
Pri-Expo-BD(\cdot)	[Tof09] $O(l \log^* l)$	$R_{pub} + 37$	$163l + C_{pub} \cdot l + 57l \log^* l$	Perfect
Pri-Expo-BD(\cdot)	[RT09] $O(l)$	$R_{pub} + 16$	$126l + C_{pub} \cdot l$	(at most) Statistical
Pri-Expo(\cdot)	NONE	$R_{pub} + 15$	$157l + C_{pub} \cdot l + C_{pub} + 5$	Perfect

Table 3: Comparison of the Complexities of Private Exponentiation Protocols

From Table 3 we can see that when considering perfect security, which is just the case in this paper, our $\text{Pri-Expo}(\cdot)$ protocol is of great advantage in terms of both communication complexity and round complexity. Moreover, we’d like to stress that, although, when the linear BD in [RT09] is involved, the communication complexity of $\text{Pri-Expo-BD}(\cdot)$ is slightly lower than our $\text{Pri-Expo}(\cdot)$ protocol, this ($\text{Pri-Expo-BD}(\cdot)$) protocol should *not* be viewed as a threat to our $\text{Pri-Expo}(\cdot)$ protocol because it achieves only (at most) *statistical security*, whereas our protocol achieves *perfect security*; what’s more, as mentioned in [RT09], the linear BD in it has some limitations (thus so does the $\text{Pri-Expo-BD}(\cdot)$ protocol involving it), whereas our $\text{Pri-Expo}(\cdot)$ protocol is a *full solution* to the private exponentiation problem and has no such limitations.

E Hierarchy of The Protocols for Exponentiation

The details of the dependency are presented in Figure 6.

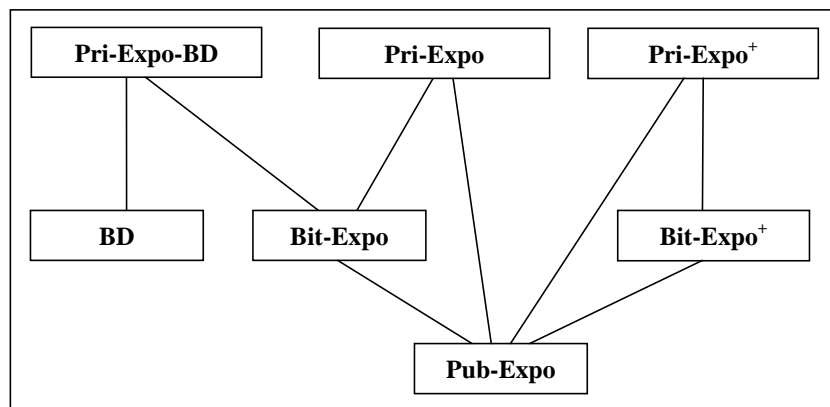


Figure 6: Hierarchy of The Protocols for Exponentiation

F A Concrete Example

As mentioned in the text, if we denote “9 days 23 hours 59 minutes 59 seconds” as $x = \boxed{9} \boxed{23} \boxed{59} \boxed{59}$, then x can be used to represent the *total* seconds and can be viewed as a hybrid-base integer defined by (“array of bases”) $M[\] = [10, 24, 60, 60]$. It can be easily verified that “9 days 23 hours 59 minutes 59 seconds” is “863999 seconds”. That is to say, we have

$$x = \left(\boxed{9} \boxed{23} \boxed{59} \boxed{59} \right)_M = (863999)_{10} = (1101001011101111111)_2$$

Given a sharing of this x , i.e. $[x]_p$, we have the following equations (in which we refer to $[x]_p, [0]_p, [1]_p, \dots$ as $[x], [0], [1], \dots$ for short).

Bit-Decomposition $([x]) \rightarrow [x]_B$

$$= \left([1], [1], [0], [1], [0], [0], [1], [0], [1], [1], [1], [0], [1], [1], [1], [1], [1], [1], [1] \right)$$

Base- m -Digit-Decomposition $([x], 10) \rightarrow [x]_D^{10} = \left([8], [6], [3], [9], [9], [9] \right)$

Base- m -Digit-Bit-Decomposition $([x], 10) \rightarrow [x]_{D,B}^{10} = \left([8]_B, [6]_B, [3]_B, [9]_B, [9]_B, [9]_B \right)$

$$= \left(([1], [0], [0], [0]), ([0], [1], [1], [0]), ([0], [0], [1], [1]), ([1], [0], [0], [1]), ([1], [0], [0], [1]), ([1], [0], [0], [1]) \right)$$

Hybrid-Base-Digit-Decomposition $([x], M[\]) \rightarrow [x]_D^M = \left([9], [23], [59], [59] \right)$

Hybrid-Base-Digit-Bit-Decomposition $([x], M[\]) \rightarrow [x]_{D,B}^M = \left([9]_B, [23]_B, [59]_B, [59]_B \right)$

$$= \left(([1], [0], [0], [1]), ([1], [0], [1], [1], [1]), ([1], [1], [1], [0], [1], [1]), ([1], [1], [1], [0], [1], [1]) \right)$$

G Intuitions behind Protocols

G.1 Intuition behind Bit-Decomposition(BD)

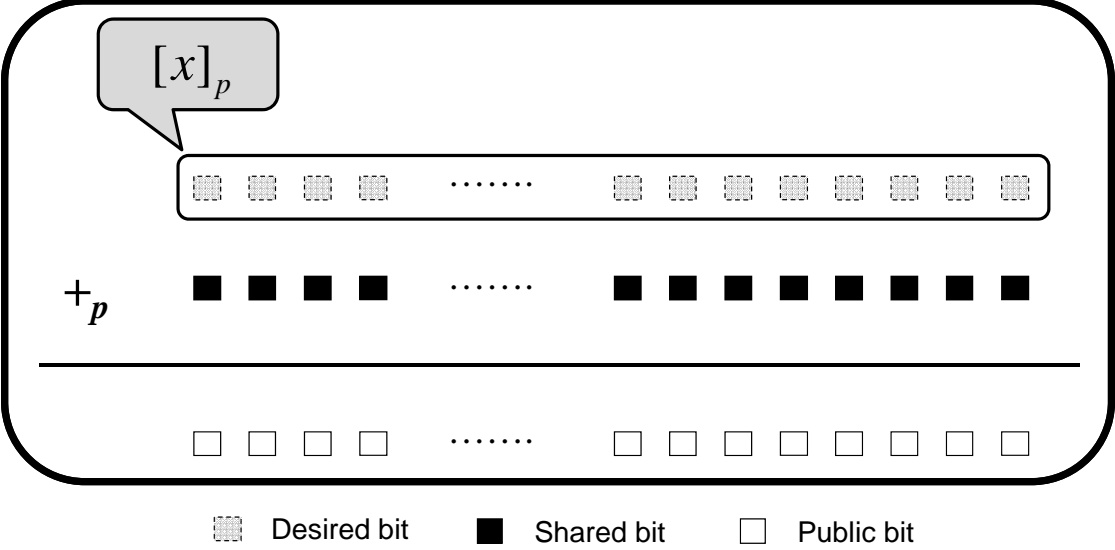


Figure 7: Intuition behind BD

Suppose the input to BD is $[x]_p$, i.e. the parties want to extract the shared base-2 form of x (i.e. $[x]_B$). Figure 7 describes the base-2 addition process (taking place over \mathbb{Z}_p)

$$x + r$$

in which the sum (i.e. $x + r \text{ mod } p$) is made public and in which r is formed by an array of shared random bits (i.e. r is bitwise shared). The public sum and the random r (whose shared base-2 form is known to the parties) will provide the parties with enough information to extract the shared base-2 form of x .

G.2 Intuition behind Base- m Digit-Decomposition & Digit-Bit-Decomposition

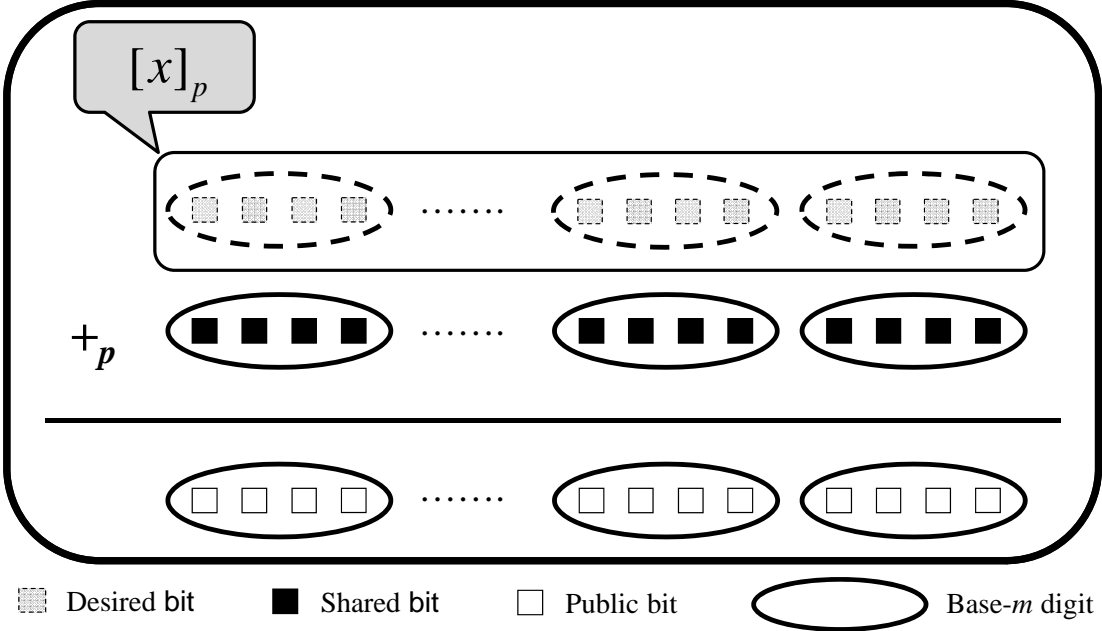


Figure 8: Intuition behind Base- m Digit-Decomposition and Digit-Bit-Decomposition

Suppose the parties want to extract the shared (or bitwise shared) base- m form of x . Figure 8 describes the base- m (e.g. $m = 10$) addition process (taking place over \mathbb{Z}_p)

$$x + r$$

in which the sum (i.e. $x + r \bmod p$) is made public and in which r is formed by an array of *bitwise* shared “base- m digits”. The public sum and the random r (whose *bitwise* shared base- m form is known to the parties) will provide the parties with enough information to extract the shared (or bitwise shared) base- m form of x .

G.3 Intuition behind Hybrid-base Digit-Decomposition & Digit-Bit-Decomposition

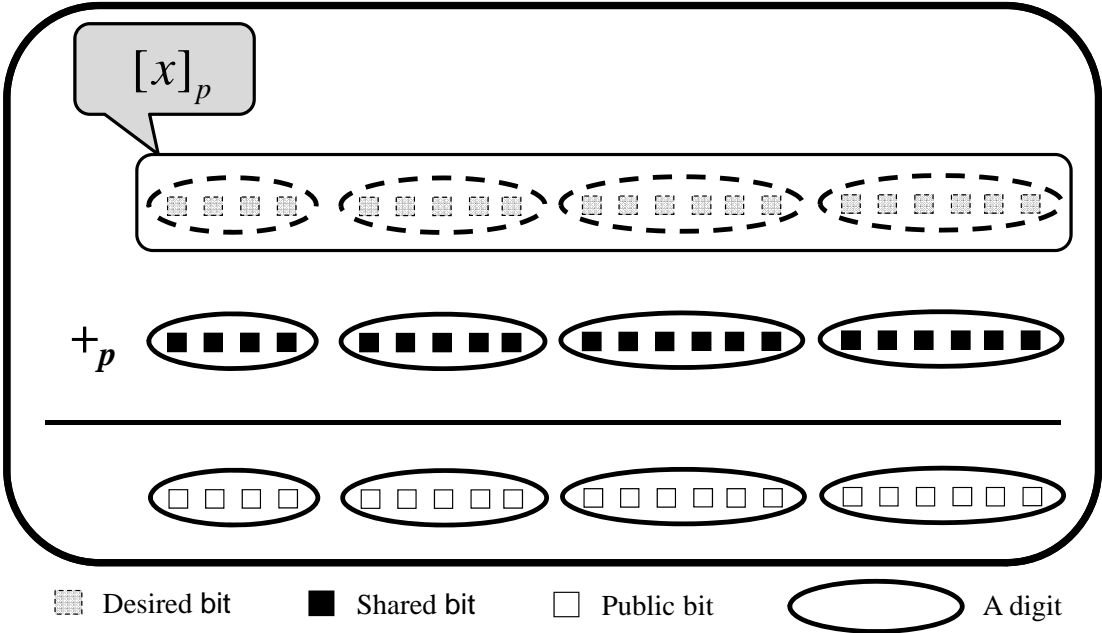


Figure 9: Intuition behind Hybrid-base Digit-Decomposition and Digit-Bit-Decomposition

Suppose the parties want to extract the shared (or bitwise shared) hybrid-base form of x which is defined by $M[\] = [10, 24, 60, 60]$. Figure 9 describes the hybrid-base addition process (taking place over \mathbb{Z}_p)

$$x + r^+$$

in which the sum (i.e. $x + r^+ \pmod p$) is made public and in which r^+ is formed by an array of *bitwise* shared digits which are, from left to right, base-10, base-24, base-60, base-60, respectively (note that for any $m \geq 2$, we need $\lceil \log_2 m \rceil$ bits to represent an (arbitrary) base- m digit). The public sum and the random r^+ (whose *bitwise* shared hybrid-base form (defined by $M[\] = [10, 24, 60, 60]$) is known to the parties) will provide the parties with enough information to extract the shared (or bitwise shared) hybrid-base form of x (defined by $M[\] = [10, 24, 60, 60]$).

G.4 Intuition behind the Pub-MR(\cdot) Protocol in [NX10]

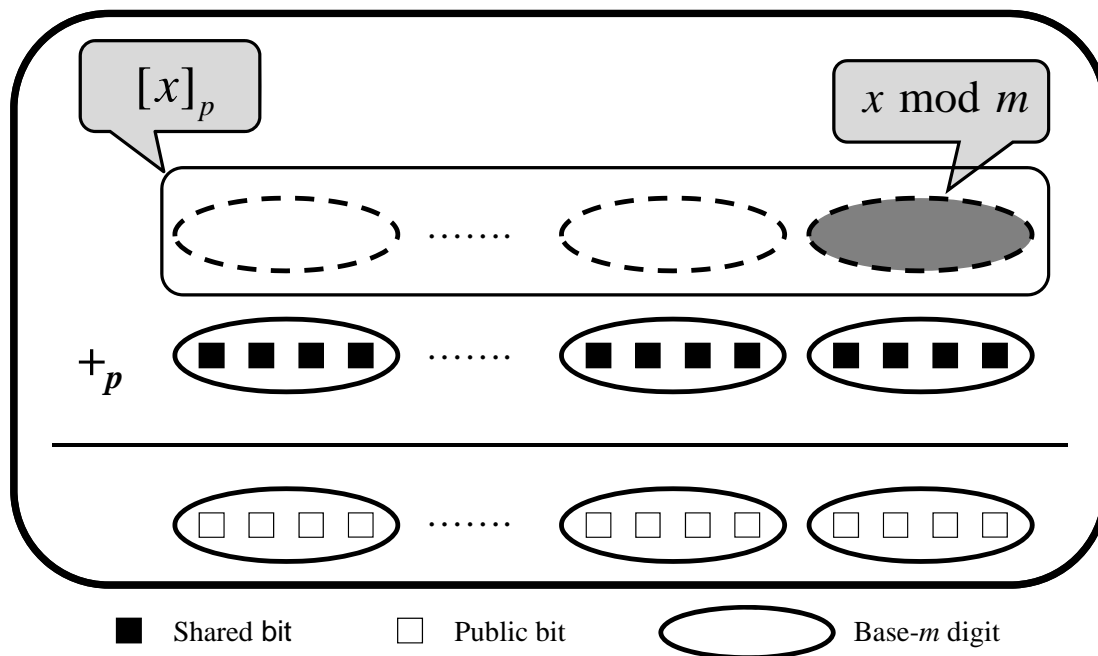


Figure 10: Intuition behind the Pub-MR(\cdot) Protocol in [NX10]

Suppose the parties want to compute $[x \bmod m]_p$ from $[x]_p$ and m . Similar to Figure 8 above, Figure 10 describes the base- m addition process (taking place over \mathbb{Z}_p)

$$x + r$$

in which the sum (i.e. $x + r \bmod p$) is made public and in which r is formed by an array of *bitwise* shared base- m digits. For obtaining $[x \bmod m]_p$, the parties need only to extract the (shared) least significant base- m digit of x (which is marked off in Figure 10); in terms of the communication complexity, the generation of r (which should be of a “hybrid-base” form defined by $M[\] = [m, \dots, m, m]$) is the dominate part of this public modulo reduction protocol, Pub-MR(\cdot).

G.5 Intuition behind Our Pub-MR⁺(·) Protocol

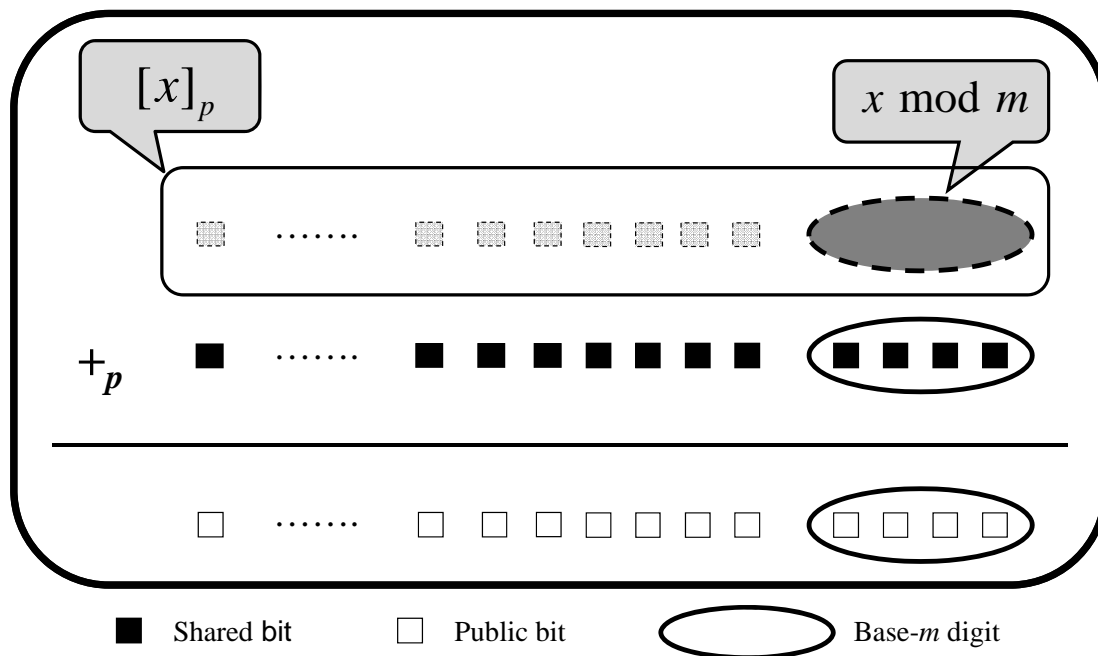


Figure 11: Intuition behind our Pub-MR⁺(·) Protocol

Again, suppose the parties want to compute $[x \bmod m]_p$ from $[x]_p$ and m . Figure 11 describes the hybrid-base addition process (taking place over \mathbb{Z}_p)

$$x + r^+$$

in which the sum (i.e. $x + r^+ \bmod p$) is made public and in which r^+ is formed by an array of shared bits and a *bitwise* shared base- m digit. For obtaining $[x \bmod m]_p$, this time the parties need to extract the (only) base- m digit of x (which is marked off in Figure 11); in terms of the communication complexity, the generation of r^+ (which should be of a hybrid-base form defined by $M[\] = [2, \dots, 2, 2, m]$) is the dominate part of this public modulo reduction protocol, Pub-MR⁺(·).