Fully Encrypted Machine Learning Protocol using Functional Encryption

Seungwan Hong[1], Jiseung Kim[2], Changmin Lee[3], and Minhye Seo[4]

[1] Columbia University / New York Genome Center
shong@nygenome.org
[2] Jeonbuk National University
jiseungkim@jbnu.ac.kr
[3] Korea Institute for Advanced Study
changminlee@kias.re.kr
[4] Duksung Women's University
mhseo@duksung.ac.kr

Abstract. As privacy concerns have arisen in machine learning, privacy-preserving machine learning (PPML) has received significant attention. Fully homomorphic encryption (FHE) and secure multi-party computation (MPC) are representative building blocks for PPML. However, in PPML protocols based on FHE and MPC, interaction between the client (who provides encrypted input data) and the evaluator (who performs the computation) is essential to obtain the final result in plaintext. Functional encryption (FE) is a promising candidate to remove this constraint, but existing FE-based PPML protocols are restricted to evaluating only simple ML models, such as one-layer neural networks, or they support partially encrypted PPML, which makes them vulnerable to information leakage beyond the inference results.

In this paper, we propose a fully encrypted FE-based PPML protocol, which supports the evaluation of arbitrary functions over encrypted data with no information leakage during computation, for the first time. To achieve this, we newly construct a vector functional encryption scheme for quadratic polynomials and combine it with an inner product encryption scheme. This enables multiple compositions of quadratic polynomials to compute arbitrary complex functions in an encrypted manner.

Our FE-based PPML protocol is secure in the malicious model, which means that an adversary cannot obtain any information about the input data even though they intentionally deviate from the protocol. We then show how to use our protocol to build a fully encrypted 2-layer neural network model with quadratic activation functions and present experimental results.

1 Introduction

Machine Learning (ML) has become a vital technology for companies across various industries, as it enables them to provide services that enhance people's quality of life. In traditional machine learning, the data is generally centralized and available to the machine learning algorithm in its raw form. However, when dealing with sensitive data, it is crucial to safeguard the privacy of individuals represented in the data. For example, in the healthcare industry, machine learning models are used to analyze medical data for diagnosis, treatment, and drug discovery. However, medical data is highly sensitive, containing personal information about patients [33, 38]. Similarly, in finance, machine learning models are utilized for fraud detection, risk assessment, and other applications, which often contain sensitive information about individuals' income and spending habits [9, 36]. Additionally, online advertising, which employs machine learning models to personalize ads for individual users, requires the protection of sensitive information such as browsing habits and interests [8, 26]. As ML increasingly permeates various businesses and organizations, privacy issues concerning the underlying data have become more prominent. Privacy-preserving machine learning (PPML) techniques and approaches have been developed to enable machine learning models to provide a useful service while maintaining the data's privacy. In line with this, research on PPML has begun to draw significant attention [17, 18, 28, 45].

The typical approaches to PPML are based on fully homomorphic encryption (FHE) and secure multi-party computation (MPC). However, FHE-based and MPC-based PPML protocols have their own limitations: MPC-based PPML protocols [28,31,35,40] require computations to be performed in the online phase, necessitating the active involvement of the client who owns the data throughout the entire process of evaluating computations. In FHE-based PPML protocols [13,15,20,24], the client does not have to remain online during computations, i.e., he/she encrypts the data prior to the computation and needs not involve in any intermediate computation. However, after the computation of encrypted data, the evaluator (performing the computations) should interact with the client (data owner) to obtain decrypted results. Additionally, this limitation prevents FHE-based PPML protocols from being deployed in certain applications, such as spam filtering. For example, an e-mail server has to classify an encrypted incoming e-mail as spam or not, but filtering cannot be done without the help of the client (e-mail recipient) because the result of the classification is provided encrypted. After all, spam filtering requires constant user involvement, which is not what we expect from spam filtering.

Functional Encryption (FE) is a promising approach for PPML, as it does not require any interaction during computation. FE allows computation on ciphertexts while revealing only the output of the computation and keeping the inputs private. There have been several studies on FE-based PPML. However, they are limited to either partially encrypted or simple structured ML models because efficient functional encryption schemes so far only support linear or quadratic polynomials. Functional encryption for arbitrary functions can be achieved from various cryptographic primitives such as multilinear map or indistinguishability obfuscation (iO) [4,12], but they are so inefficient that these are only theoretically feasible. Thus, to date, FE schemes for inner-product and quadratic functionalities have been taken to construct PPML [11,21,22,25,30,37,44] along with an attempt to speed up using hardware accelerator [6]. But the thing is, most works assume 'partially encrypted setting' which means that the first layer is encrypted, but operations in subsequent layers are visible in clear.

Recently, Carpov *et al.* [7] proposed an attack where adversaries can exploit such cleartexts to partially recover the original input data. That is, the intermediate values could yield information leakage about the original encrypted input data, which leaves a gap with real-world scenarios. Thus, it remains an open problem to achieve fully encrypted machine learning protocol via FE that can support complex computations while maintaining strong privacy guarantees.

1.1 Contributions

In this paper, we propose a fully encrypted PPML protocol for the first time in the literature that

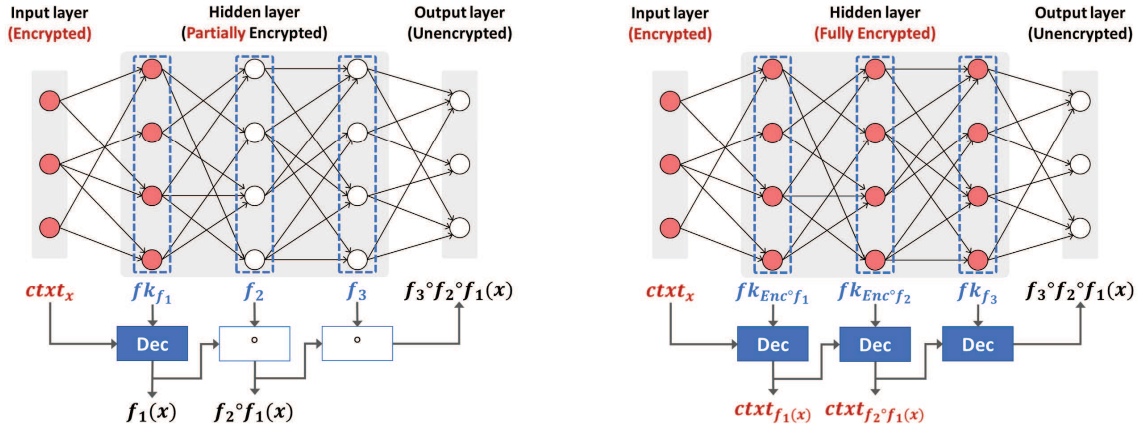
- does not require any involvement of the client (data owner) in the computation process, and
- allows the evaluator to obtain the final inference result without any interaction (since it is output in plaintext in the output layer), and
- has no intermediate leakage while evaluating arbitrary functions over encrypted data.

We construct our PPML protocol using FE, which enables non-interactive computations on encrypted data and reveals only the final output of the computation. However, to make it "fully encrypted" for evaluating "arbitrary" functions (e.g., 3-degree polynomials or higher), we have considered the following technical ideas:

1) Imitate the FHE from FE. Since existing FE could support computation on ciphertext only once, FE-based PPML protocols had to be "partially encrypted". The novel idea of our work is to mimic the FHE in FE framework. To be precise, we consider a composition function of $\mathsf{Enc} \circ f$, where Enc is an encryption of FE scheme. This composition then allows that an output of evaluation still remains encrypted. For ease of understanding, let f_i be the function corresponding to the i-th layer where $i = 1, 2, 3$ as in Fig. 1. In the existing FE-based PPML protocol, the output of the first hidden layer $f_1(x)$ is given in plaintext whereas, in our PPML protocol, the output of the first hidden layer $\mathsf{Enc} \circ f_1(x)$ is presented in ciphertext.

2) Introduce a new compact[¶] FE scheme for function composition. The composition of the encryption algorithm Enc and the function f is infeasible in the existing FE schemes. To make

[¶] There are two definition of the compactness of FE. We here adopt the definition in [3].FE scheme is compact when its encryption time is a polynomial in the security parameter λ, the number of function queries Q, and the size of input message m.



(a) The existing partially encrypted FE-based PPML

(b) Our fully encrypted FE-based PPML

Fig. 1. Privacy-preserving machine learning based on functional encryption

the composition work, the structure of the encryption algorithm should be polynomial so that a composition with functions (i.e., $\mathsf{Enc} \circ f$) is a polynomial as well, which is not supported by the existing FE schemes. Therefore, we newly construct a compact FE scheme for quadratic polynomials (composable-QFE in Section 4), such that its encryption algorithm can be represented by a linear function, thereby resulting in ciphertexts that are vectors. Since the encryption algorithm is a linear function, the composable-QFE supports the functionality $\mathsf{Enc} \circ f$ and our protocol works properly as long as the f is a quadratic polynomial. Since all complex functions can be decomposed into a composition of quadratic polynomials, our protocol covers all polynomial functions.

3) Make the PPML protocol secure in the malicious model. We then construct a secure computation protocol using composable-QFE scheme, called cFE-PPML protocol, and on top of that, we propose a fully encrypted PPML protocol. Since the encryption algorithm of the composable-QFE is a linear function, the composable-QFE is only secure under bounded ciphertext cases. Note that IND-CPA security of the underlying FE scheme could not guarantee the security of the PPML protocol [7]. To make our PPML protocol secure in the malicious model, we introduce random linear functions $h_i, {h_i}^{-1}$ to randomize the functional keys. In addition, we consider random linear functions γ and γ^{-1} to randomize the message. Taken together, our PPML protocol is proven to be secure in the malicious model. This means that the adversary cannot obtain any information about the client's input data from the entire transcript of the protocol except for the final result (in the output layer) even if the evaluator may deviate from the protocol arbitrarily and collude with other clients. See more details in Section 5.

Furthermore, we validate the feasibility of our proposed protocol through experimental results. To be precise, we provide an implementation result using inference of 2-layer neural network classifier on the IRIS and Breast Cancer dataset in the UCI Machine Learning Repository [10]. The source code is available in `https://github.com/swanhong/composable-fe-rs/`.

1.2 Simple description of protocol

A brief description of our protocol is given here. Our protocol involves three parties: the key distributor, the client, and the evaluator. For simplicity, we assume that an evaluator wants to compute a function $f_3 \circ f_2 \circ f_1$ for an input data $\mathbf{x}$ as in Fig. 1. We denote the underlying FE scheme by $\{\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec}\}$. Then the proposed protocol is proceeded as follows:

1. The key distributor runs Setup to obtain $\{\mathsf{pk}, \mathsf{msk}\}$, and samples random pairs $(h_i, {h_i}^{-1})$ for $i = 1, 2$. The key distributor samples random linear functions (γ, γ^{-1}) and sends (pk, γ) to the client.
2. The client encrypts $\gamma(\mathbf{x})$ with a message $\mathbf{x}$ and sends the ciphertext $\mathsf{ct}_{\gamma(\mathbf{x})} \leftarrow \mathsf{Enc}(\mathsf{pk}, \gamma(\mathbf{x}))$ to the evaluator.

Research	Type	ML model	Security model	# parties	# interactions (client)
ABY2.0 [31]	MPC[1]	NN[3]	semi-honest[4]	2	high
SecureML [28]		General	semi-honest	2	
CRYPTGPU [39]		NN	semi-honest	3	
CrypTFlow [19]		NN	semi-honest	3	
Chameleon [34]		NN	semi-honest	3	
SecureNN [40]		NN	malicious[5]	3	
ABY^3 [27]		NN	malicious	3	
AdamInPrivate [5]		NN	malicious	3	
FalconN [41]		General	malicious	3	
BLAZE [32]		General	malicious	3	
CryptoDL [16]	FHE[2]	NN	semi-honest	2	low
CryptoNets [13]		NN	semi-honest	2	
This work	FE	NN	malicious	3	low

Table 1. Comparison with FHE/MPC-based PPML protocol for inference
[1] Secure Multi-Party Computation , [2] Fully Homomorphic Encryption, [3] Neural Network, [4] Adversaries follow the protocol but try to get more information. [5] Adversaries can deviate from the protocol to gain an advantage.

Research	FE type	ML model	Fully Encrypted
Linger et al. [22]	IPFE[1]	ERT[3]	✗
Xu et al. [44]	IPFE	5-layer NN	✗
Sans et al. [11]	QFE[2]	2-layer NN	✗
Ryfell et al. [37]	QFE	2-layer NN	✗
This work	IPFE / QFE	2-layer NN	✓

Table 2. Implementations of FE-based PPML protocol for inference
[1]Inner Product Functional Encryption [2] Quadratic Functional Encryption [3] Extremely Randomized Trees

3. The evaluator sends functions f_1, f_2, f_3 to the key distributor.
4. The key distributor computes functional keys $\mathsf{fk}_1, \mathsf{fk}_2, \mathsf{fk}_3$ and sends them to the evaluator:
   - $\mathsf{fk}_1 \leftarrow \mathsf{KeyGen}(\mathsf{msk}, \mathsf{Enc} \circ F_1)$ where $F_1 = h_1 \circ f_1 \circ \gamma^{-1}$
   - $\mathsf{fk}_2 \leftarrow \mathsf{KeyGen}(\mathsf{msk}, \mathsf{Enc} \circ F_2)$ where $F_2 = h_2 \circ f_2 \circ h_1^{-1}$
   - $\mathsf{fk}_3 \leftarrow \mathsf{KeyGen}(\mathsf{msk}, F_3)$ where $F_3 = f_3 \circ h_2^{-1}$
5. The evaluator computes the following:
   - $\mathsf{ct}_{h_1 \circ f_1(\mathbf{x})} \leftarrow \mathsf{Dec}(\mathsf{fk}_1, \mathsf{ct}_{\gamma(\mathbf{x})})$
   - $\mathsf{ct}_{h_2 \circ f_2 \circ f_1(\mathbf{x})} \leftarrow \mathsf{Dec}(\mathsf{fk}_2, \mathsf{ct}_{h_1 \circ f_1(\mathbf{x})})$
   - $f_3 \circ f_2 \circ f_1(\mathbf{x}) \leftarrow \mathsf{Dec}(\mathsf{fk}_3, \mathsf{ct}_{h_2 \circ f_2 \circ f_1(\mathbf{x})})$

Note that the evaluator can obtain $f_3 \circ f_2 \circ f_1(\mathbf{x})$ in plaintext at the end of the protocol. Since our composable-QFE is constructed in the symmetric-key setting, we additionally adopt an inner-product (public-key) encryption scheme (pkIPFE in Fig. 2) and combine it with the composable-QFE in our protocol. The client encrypts the input data $\mathbf{x}$ using the pkIPFE scheme in the public-key setting, and then the evaluator converts it into the ciphertext of composable-QFE scheme. See Section 5 for more details.

1.3 Comparison and limitation

Compared to the aforementioned protocols - based on MPC, FHE, and FE - our protocol offers the following advantages:

- MPC: Secure computation through MPC is highly regarded for its strong security and efficiency. However, it requires the client to be online and engage in interactions with other parties. In contrast, our protocol only necessitates the client to perform small computations.

- FHE: There is currently no protocol solely based on FHE that provides security in the malicious model. However, our protocol operates securely under the malicious model, even though its structure resembles that of an FHE-based protocol. In fact, when a corrupted party asks the key distributor to decrypt a modified FHE encryption, the party can obtain the master secret key. This is a basic attack against FHE-based protocols inspired by the fact that FHE does not achieve the IND-CCA2 security. In other words, the corrupted party and the key distributor act an adversary and the challenger of IND-CCA2 security game, respectively. Then, since FHE is impossible to achieve the IND-CCA2 security, such protocol is insecure.
- FE: Existing FE-based protocols only offer partially encrypted forms, resulting in information leakage from intermediate values. Defining a comprehensive security model for such FE-based protocols becomes challenging. In contrast, our protocol ensures complete encryption and achieves a comparable level of security to other protocols.

For a detailed comparison of the protocols, please refer to Table 1 and Table 2.

While our algorithm excels in terms of security, it does have a significant drawback in terms of operation time. Generating a ciphertext format necessitates considering a large underlying space. Specifically, we have adopted the decision composite residuosity (DCR) based scheme to accommodate this extensive space. However, computations for the DCR scheme involve exponentiations and multiplications, resulting in significant computational costs. Notably, exponentiation operations are approximately $10^6\times$ more time-consuming than multiplication operations, which is (normally) a basic computation in other protocols. Nevertheless, we believe that our protocol serves as a new approach to secure computations.

2 Preliminaries

2.1 Notation

Throughout this paper, we use bold letters to denote vectors and matrices. Let $\mathbf{O}_n$ be a zero matrix of dimension $n \times n$ and $\mathbf{I}_n$ be an identity matrix of dimension $n \times n$ for any positive integer n. Let $\mathbb{Z}$ be the set of all integers and $\mathbb{N}$ the set of all positive integers.

For any $a, b \in \mathbb{Z}$, we simplify $[a, b] \cap \mathbb{Z}$ as $[a, b]$. We also use other simplified interval notations. For any $N \geq 2$, we identify $\mathbb{Z}_N$ as $[-N/2, N/2) \cap \mathbb{Z}$. For any finite set S, $s \leftarrow S$ is denoted to sampling s from the uniform distribution over S. We denote ϕ by the Euler's totient function.

We describe a composition notation for functions. Let F be $(f_1, \dots, f_k)$, where $f_i : \mathbb{Z}_N^\ell \to \mathbb{Z}_N$ is a quadratic function for every i. Then, a composite function $F \circ h$ is denoted by a function of the form $(f_1 \circ h, \dots, f_k \circ h)$, when the output dimension of the function h is ℓ. Similarly, $h' \circ f \circ h$ is well-defined for a function h' of input dimension k. For every positive integer i and proper input $\mathbf{x}$, $(f_i \circ \dots \circ f_2 \circ f_1)(\mathbf{x})$ is simply denoted by $\bigcirc_{t=1}^{i} f_t(\mathbf{x})$.

Given n-dimensional vector $\mathbf{v} = (v_1, \cdots, v_n)^T$ and group element g, we denote $(g^{v_1}, g^{v_2}, \cdots, g^{v_n})^T$ by $g^{\mathbf{v}}$. Moreover, we use a bracket notation $[a]_g$ to denote g^a. Similarly, for any vector $\mathbf{v}$ and matrix $\mathbf{A}$, $g^{\mathbf{v}}$ and $g^{\mathbf{A}}$ are denoted by $[\mathbf{v}]_g$ and $[\mathbf{A}]_g$, respectively. Given two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$, we define $[\mathbf{v}]_g^{\mathbf{w}}$ as $g^{\mathbf{v}^T \cdot \mathbf{w}} = g^{\langle \mathbf{v}, \mathbf{w} \rangle}$. In addition, we denote their row concatenation as $(\mathbf{v}, \mathbf{w})$ and their column concatenation as $(\mathbf{v} \| \mathbf{w})$. The Kronecker tensor products of vectors $\mathbf{a} \in \mathbb{Z}_N^n$ and $\mathbf{b} \in \mathbb{Z}_N^m$ or matrices $\mathbf{A} \in \mathbb{Z}_N^{n \times m}$ and $\mathbf{B} \in \mathbb{Z}_N^{r \times s}$ are defined by

$$\mathbf{a} \otimes \mathbf{b} = (a_1\mathbf{b}, a_2\mathbf{b}, \dots, a_n\mathbf{b}) \in \mathbb{Z}_N^{nm}, \mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & \dots & a_{1m}\mathbf{B} \\ \vdots & & \vdots \\ a_{n1}\mathbf{B} & \dots & a_{nm}\mathbf{B} \end{pmatrix} \in \mathbb{Z}_N^{nr \times ms}.$$

2.2 Functional Encryption and Its Security

In this section, we describe the definitions of private-key and public-key functional encryption schemes and their security models.

Definition 1 (Private-key functional encryption). *Let $\mathcal{F}$ be a function space, and $\mathcal{M}$ a message space. Then, a private key functional encryption scheme for $\mathcal{F}$ with $\mathcal{M}$ is composed of four probabilistic polynomial-time (PPT) algorithms* $(\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$.

- $\mathsf{Setup}(\lambda, \mathcal{F})$*: For the security parameter* λ*, it outputs the master secret key* msk *and the public parameter* pp*.*
- $\mathsf{KeyGen}(\mathsf{msk}, f \in \mathcal{F}, \mathsf{pp})$ *: For* msk *and a function* f *from* $\mathcal{F}$*, it outputs a functional key* fk_f*.*
- $\mathsf{Enc}(\mathsf{msk}, m \in \mathcal{M}, \mathsf{pp})$ *: For* msk *and a message* m *from* $\mathcal{M}$*, it outputs a ciphertext* ct_m*.*
- $\mathsf{Dec}(\mathsf{ct}_m, \mathsf{fk}_f, \mathsf{pp})$ *: For* ct_m *and* fk_f*, it outputs a value* α*.*

A private-key functional encryption scheme $\mathsf{skFE} = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ *is said to be correct if for every security parameter* λ*,* $(\mathsf{msk}, \mathsf{pp}) \leftarrow \mathsf{Setup}(\lambda, \mathcal{F})$*, for all* $m \in \mathcal{M}$ and $f \in \mathcal{F}$*,* $\mathsf{fk}_f \leftarrow \mathsf{KeyGen}(\mathsf{msk}, f, \mathsf{pp})$*, and* $\mathsf{ct}_m \leftarrow \mathsf{Enc}(\mathsf{msk}, m, \mathsf{pp})$*,* $\mathsf{Dec}(\mathsf{ct}_m, \mathsf{fk}_f, \mathsf{pp}) = f(m)$ *with all but a negligible probability.*

Definition 2 (Semi-adaptive Security of private-key FE [14, 29, 43]). *A private-key functional encryption scheme* $\mathsf{skFE} = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ *for* $\mathcal{F}$ *with* $\mathcal{M}$ *is semi-adaptively secure if for every PPT adversary* $\mathcal{A}$*, there is a negligible function* $\mathsf{neg}(\cdot)$ *such that*

$$\mathsf{Adv}_{\mathcal{A}}(\lambda) = |\Pr[\mathsf{G}(\mathcal{A}, 0) = 1] - \Pr[\mathsf{G}(\mathcal{A}, 1) = 1]| \leq \mathsf{neg}(\lambda) \qquad (1)$$

for $\lambda \in \mathbb{N}$*, where* $\mathsf{G}(\mathcal{A}, b)$ *for* $b \in \{0, 1\}$ *is a semi-adaptive security game between* $\mathcal{A}$ *and a challenger defined as follows.*

1. *(Setup Phase) The challenger samples* $(\mathsf{msk}, \mathsf{pp}) \leftarrow \mathsf{Setup}(\lambda, \mathcal{F})$ *and gives* pp *to* $\mathcal{A}$*.*
2. *(Challenge Phase)* $\mathcal{A}$ *submits* $\big((m_{0,1}, \ldots, m_{0,T}), (m_{1,1}, \ldots, m_{1,T})\big)$*, where* $m_{0,i}$*,* $m_{1,i} \in \mathcal{M}$ *for all* $i \in [1, T]$*, and the challenger returns* $\mathsf{ct}_i \leftarrow \mathsf{Enc}(\mathsf{msk}, m_{b,i})$ *for all* $i \in [1, T]$ *and a randomly chosen* $b \in \{0, 1\}$*.*
3. *(Query Phase)* $\mathcal{A}$ *queries the challenger with* $f \in \mathcal{F}$ *such that* $f(m_{0,i}) = f(m_{1,i})$ *for all* $i \in [1, T]$*. For each* f*, the challenger returns* fk_f*.*
4. *(Output Phase)* $\mathcal{A}$ *outputs a bit* b' *as an output of the game.*

Definition 3 (Public-key functional encryption). *Let* $\mathcal{F}$ *be a function space, and* $\mathcal{M}$ *a message space. Then, a private key functional encryption scheme,* pkFE*, for* $\mathcal{F}$ *with* $\mathcal{M}$ *is composed of four PPT algorithms* $(\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$*.*

- $\mathsf{Setup}(\lambda, \mathcal{F})$*: For the security parameter* λ*, it outputs the master public/secret key pair* $(\mathsf{pk}, \mathsf{msk})$*.*
- $\mathsf{KeyGen}(\mathsf{msk}, f \in \mathcal{F})$ *: For* msk *and a function* f *from* $\mathcal{F}$*, it outputs a functional key* fk_f*.*
- $\mathsf{Enc}(\mathsf{pk}, m \in \mathcal{M})$ *: For* pk *and a message* m *from* $\mathcal{M}$*, it outputs a ciphertext* ct_m*.*
- $\mathsf{Dec}(\mathsf{pk}, \mathsf{ct}_m, \mathsf{fk}_f)$ *: For* ct_m *and* fk_f*, it outputs a value* α*.*

A public-key functional encryption scheme $\mathsf{pkFE} = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ *is said to be correct if for every security parameter* λ*,* $(\mathsf{pk}, \mathsf{msk}) \leftarrow \mathsf{Setup}(\lambda, \mathcal{F})$*, for all* $m \in \mathcal{M}$ and $f \in \mathcal{F}$*,* $\mathsf{fk}_f \leftarrow \mathsf{KeyGen}(\mathsf{msk}, f)$*, and* $\mathsf{ct}_m \leftarrow \mathsf{Enc}(\mathsf{pk}, m)$*,* $\mathsf{Dec}(\mathsf{pk}, \mathsf{ct}_m, \mathsf{fk}_f) = f(m)$ *with all but a negligible probability.*

Definition 4 (Adaptive simulation-based security of public-key FE [1]). *A public-key functional encryption scheme* $\mathsf{pkFE} = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ *for* $\mathcal{F}$ *with* $\mathcal{M}$ *is adaptively simulation secure if for every PPT adversary* $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$*, there exists a PPT simulator* $\mathcal{S} = (\mathsf{Setup}^{\mathcal{S}}, \mathsf{KeyGen}_0^{\mathcal{S}}, \mathsf{Enc}^{\mathcal{S}}, \mathsf{KeyGen}_1^{\mathcal{S}})$ *such that the* Real *and the* Ideal *experiments, defined as follows, are computationally indistinguishable.*

- *In the* Real *experiment:*
  1. $(\mathsf{pk}, \mathsf{msk}) \leftarrow \mathsf{Setup}(\lambda, \mathcal{F})$
  2. $(m^*, st) \leftarrow \mathcal{A}_1^{\mathsf{KeyGen}(\mathsf{msk}, \cdot)}(\mathsf{pk})$
  3. $\mathsf{ct} \leftarrow \mathsf{Enc}(\mathsf{pk}, m^*)$
  4. $\alpha \leftarrow \mathcal{A}_2^{\mathsf{KeyGen}(\mathsf{msk}, \cdot)}(\mathsf{pk}, \mathsf{ct}, st)$
  5. *Output* (m^*, α)
- *In the* Ideal *experiment:*
  1. $(\mathsf{pk}^{\mathcal{S}}, \mathsf{msk}^{\mathcal{S}}) \leftarrow \mathsf{Setup}^{\mathcal{S}}(\lambda, \mathcal{F})$
  2. $(m^*, st) \leftarrow \mathcal{A}_1^{\mathsf{KeyGen}_0^{\mathcal{S}}(\mathsf{msk}^{\mathcal{S}}, \cdot)}(\mathsf{pk}^{\mathcal{S}})$
  3. *Let* $\mathcal{V} = \big\{(f_i, f_i(m^*), fk_{f_i})\big\}_{i=1}^{k}$ *where* $\{f_i \in \mathcal{F}\}_{i=1}^{k}$ *are the functions for which the adversary requests their corresponding keys* $\{fk_{f_i}\}_{i=1}^{k}$

4. $\mathsf{ct}^* \leftarrow \mathsf{Enc}^{\mathcal{S}}(\mathsf{pk}^{\mathcal{S}}, \mathsf{msk}^{\mathcal{S}}, \mathcal{V}, 1^{|m^*|})$
5. $\alpha \leftarrow \mathcal{A}_2^{\mathsf{KeyGen}_1^{\mathcal{S}}(\mathsf{msk}^{\mathcal{S}},\cdot)}(\mathsf{pk}^{\mathcal{S}}, \mathsf{ct}^*, st)$
6. Output (m^*, α)

Definition 5. *We say that a private key* FE *is Q-ciphertext bounded when the scheme is secure for adversaries requesting Q-ciphertexts.*

Definition 6 (Compact FE, Adaptation from [3]). *We say that a Q-ciphertext bounded private key FE scheme* $\mathsf{FE} = (\mathsf{Setup}, \mathsf{KeyGen}, \mathsf{Enc}, \mathsf{Dec})$ *for $\mathcal{F}$ with $\mathcal{M}$ is compact if for all $\lambda \in \mathbb{N}$, the running time of the encryption algorithm* Enc *is a polynomial with respect to parameters λ, Q and m, where $m \in \mathcal{M}$.*

Remark 1. Even though the function space $\mathcal{F}$ is used as the input of the Setup algorithm in the FE definition, we employ more specific notations instead of $\mathcal{F}$.

2.3 Hardness Assumptions

This section provides the hardness assumption used in this paper. We consider groups of order $N \cdot \phi(N)$ where p, q are large primes such that factoring N is hard. In the following section, we assume that the following assumptions hold.

Definition 7 (Bilinear map). *Let $\mathbb{G}_a, \mathbb{G}_b$, and $\mathbb{G}_T$ be groups. We say that $e : \mathbb{G}_a \times \mathbb{G}_b \to \mathbb{G}_T$ is a bilinear map if e satisfies the following:*

1. *$\mathbb{G}_a, \mathbb{G}_b, \mathbb{G}_T$ are groups of the same order that satisfy the discrete logarithm problem is hard on each group.*
2. *For any $g_a \in \mathbb{G}_a, g_b \in \mathbb{G}_b$ and $x_a, x_b \in \mathbb{Z}$, it holds that*
$$e(g_a^{x_a}, g_b^{x_b}) = e(g_a, g_b)^{x_a x_b}.$$
3. *If g_a, g_b are generators of $\mathbb{G}_a, \mathbb{G}_b$ respectively, then $e(g_a, g_b)$ is a generator of $\mathbb{G}_T$.*

Definition 8 (DCR assumption). *Let p, q be prime numbers and $N = pq$. The decision composite residuosity (DCR) assumption is that the following distributions are computationally indistinguishable:*

$$Dist_1 : \{z = z_0^N \bmod N^2 \mid z_0 \leftarrow \mathbb{Z}_N^*\}$$
$$Dist_2 : \{z \leftarrow \mathbb{Z}_{N^2}^*\},$$

where $\mathbb{Z}_N^*$ is a multiplicative group of $\mathbb{Z}_N$.

Definition 9 (DDH assumption). *For $N = pq$ with primes p and q, let $\mathbb{G}$ be a group of order $N \cdot \phi(N)$ and g a generator of $\mathbb{G}$.*

The decisional Diffie-Hellman (DDH) assumption over $\mathbb{G}$ is that the following distributions are computationally indistinguishable:

$$Dist_1 : \{(g, g^x, g^y, g^{xy}) \mid x, y \leftarrow \mathbb{Z}_{N\cdot\phi(N)}\}$$
$$Dist_2 : \{(g, g^x, g^y, g^z) \mid x, y, z \leftarrow \mathbb{Z}_{N\cdot\phi(N)}\}.$$

Definition 10 (χ-MDDH assumption). *For $N = pq$ with primes p and q, let $\mathbb{G}$ be a group of order $N \cdot \phi(N)$ and g a generator of $\mathbb{G}$.*

Let χ be a distribution which returns a vector over $\mathbb{G}^2$. Then, χ-MDDH assumption holds on $\mathbb{G}$ with a generator g if any PPT adversary $\mathcal{A}$ cannot distinguish the following distributions.

$$Dist_1 : \{[\mathbf{a}]_g, [\mathbf{a} \cdot w]_g : \mathbf{A} \leftarrow \chi, w \leftarrow \mathbb{Z}_{N\cdot\phi(N)}\}$$
$$Dist_2 : \{[\mathbf{a}]_g, [\mathbf{u}]_g : \mathbf{a} \leftarrow \chi, \mathbf{u} \leftarrow \mathbb{G}^2\}.$$

There is a simple reduction from DDH to χ-MDDH, so it holds that DDH$\leq$ χ-MDDH. When a bilinear map $e : \mathbb{G}_a \times \mathbb{G}_b \to \mathbb{Z}_{N^2}^*$ is given, the DDH problem over two groups would be considered simultaneously. We describe the problem, the so-called Bilateral 2-LIN assumption, which is used in security proof for bilinear map-based schemes [43].

Definition 11 (Bilateral 2-LIN assumption). *Let $\mathbb{G}_a, \mathbb{G}_b$ be groups of order $N \cdot \phi(N)$ and $e : \mathbb{G}_a \times \mathbb{G}_b \to \mathbb{Z}^*_{N^2}$ a bilinear group. We say bilateral 2-LIN assumption holds on groups $\mathbb{G}_a$ and $\mathbb{G}_b$ if*

$$\{[x]_{g_a}, [y]_{g_a}, [xy]_{g_a}, [x]_{g_b}, [y]_{g_b}, [xy]_{g_b}\} \approx \{[x]_{g_a}, [y]_{g_a}, [z]_{g_a}, [x]_{g_b}, [y]_{g_b}, [z]_{g_b}\},$$

where $x, y, z \leftarrow \mathbb{Z}_{N\cdot\phi(N)}$, $g_a \in \mathbb{G}_a$ and $g_b \in \mathbb{G}_b$.

2.4 DCR-based Inner Product Encryption

This section introduces a DCR-based inner product encryption (IPFE) cryptosystem, which employs inner product functionality to design our secure inference protocol. Whereas there are several forms of DCR-based cryptosystems, we serve a scheme, inspired by a functional encryption scheme [1], to guarantee simulation-based security. We let B_X (resp. $B_{\mathcal{F}}$) denote a size bound of a message (resp. function coefficients).Furthermore, we use a function $\mathsf{DCR.pp}(1^\lambda)$ that outputs a pair (N, p, q, g) such that $N = p \cdot q$, $p = 2p' + 1, q = 2q' + 1 \in \mathbb{Z}$ are primes, where p' and q' are also primes, and g is a group generator of $\mathbb{Z}^*_{N^2}$. This parameter is set to robust against known attacks.

For consistency with the main body, we let pkIPFE denote this DCR-based scheme. The detailed construction of pkIPFE is given by Fig. 2.

- $\mathsf{Setup}_{\mathsf{pkIPFE}}(\lambda, \ell, B_X, B_{\mathcal{F}})$:
  1. $(N, p, q, g) \leftarrow \mathsf{DCR.pp}(1^\lambda)$, such that $\ell B_X B_{\mathcal{F}} < p \cdot q/2$.
  2. For $S = 2^{\lambda+\ell-1} \cdot \left(B_X + \ell(\sqrt{\ell}B_{\mathcal{F}})^\ell\right)^{\ell-1} \cdot \ell N^2$, sample a secret vector $\mathbf{s} = (s_1, \ldots, s_\ell) \leftarrow \mathcal{U}([-S, S] \cap \mathbb{Z})$ and then compute $g_i = g^{s_i} \bmod N^2$.
  3. Return $\mathsf{pk} = (N, g, \{g_i\}_{1\le i\le\ell}, B_X, B_{\mathcal{F}})$ and $\mathsf{msk} = (\{s_i\}_{1\le i\le\ell})$.
- $\mathsf{KeyGen}_{\mathsf{pkIPFE}}(\mathsf{msk}, \mathbf{y})$:
  1. Return $\mathsf{fk}_{\mathbf{y}} := (\langle \mathbf{y}, \mathsf{msk}\rangle, \mathbf{y})$.
- $\mathsf{Enc}_{\mathsf{pkIPFE}}(\mathsf{pk},\mathbf{m})$:
  1. Given $\mathbf{m} \in [-B_X, B_X]^\ell$, sample $r \leftarrow \{0, \ldots, \lfloor N/4 \rfloor\}$ and compute
  $$C_0 = g^r \bmod N^2,$$
  $$C_i = (1 + m_i \cdot N) \cdot g_i^r \bmod N^2 \text{ for } 1 \le i \le \ell.$$
  2. Return $\mathsf{ct}_{\mathbf{m}} = (C_0, \ldots C_\ell)$.
- $\mathsf{Dec}_{\mathsf{pkIPFE}}(\mathsf{fk}_{\mathbf{y}}, \mathsf{ct}_{\mathbf{m}})$:
  1. Given $\mathsf{ct}_{\mathbf{m}} = (C_0, \ldots, C_\ell)$, output
  $$\frac{((\prod_{i=1}^{\ell} C_i^{y_i})/C_0^{\langle \mathbf{y}, \mathsf{msk}\rangle} \bmod N^2) - 1}{N}.$$

Fig. 2. DCR-based pkIPFE [1].

Simulator. For the proof of Theorem 1 and Theorem 3, a simulator of the DCR-based scheme is required. The simulator consists of the following algorithms:

$$\mathsf{Setup}^{\mathcal{S}}, \mathsf{KeyGen}_0^{\mathcal{S}}, \mathsf{Enc}^{\mathcal{S}}, \mathsf{KeyGen}_1^{\mathcal{S}}, \mathsf{Enc}, \mathsf{Dec}.$$

$\mathsf{KeyGen}_0^{\mathcal{S}}$ is only used before the challenge query. $\mathsf{KeyGen}_1^{\mathcal{S}}$ is used in the post-challenge key queries. The message in challenge phase is denoted by $\mathbf{x}^*$. The detailed construction of simulators is given by Fig. 3. The Enc and Dec algorithm exactly coincide with the $\mathsf{Enc}_{\mathsf{pkIPFE}}$ and $\mathsf{Dec}_{\mathsf{pkIPFE}}$. Thus we do not provide an algorithm description.

Theorem 1 ([1]). *The scheme holds semi-adaptive security under the DCR assumption. In particular, it holds $\mathsf{Adv}_{\mathsf{pkIPFE}} \le \mathsf{Adv}_{\mathsf{DCR}}$.*

- $\mathsf{Setup}^{\mathcal{S}}(\lambda, \ell, B_X, B_{\mathcal{F}})$: This step is identical to Setup except that primes p, q are included in the msk. That is, this algorithm returns $\mathsf{pk}^{\mathcal{S}} = (N, g, \{g_i\}_{1 \leq i \leq \ell}, B_X, B_{\mathcal{F}})$ and $\mathsf{msk}^{\mathcal{S}} = (\{s_i\}_{1 \leq i \leq \ell}, p, q)$.
- $\mathsf{KeyGen}_0^{\mathcal{S}}(\mathsf{msk}^{\mathcal{S}}, \mathbf{y})$: For $\mathbf{y} \in [-B_{\mathcal{F}}, B_{\mathcal{F}}]^{\ell}$, it returns $\mathsf{fk}_{\mathbf{y}} = (\langle \mathbf{y}, \mathsf{msk}^{\mathcal{S}} \rangle, \mathbf{y})$.
- $\mathsf{Enc}^{\mathcal{S}}(\mathsf{pk}^{\mathcal{S}}, \{(\mathbf{y}_i, z_i)\}_{i=1}^{k})$: For pre-challenge queries $(\mathbf{y}_i, z_i)$ with $z_i = \langle \mathbf{x}^*, \mathbf{y}_i \rangle$, the algorithm computes $\mathsf{ct}^* = (c_0^*, \{c_i^*\}_{i=1}^{\ell}) \in (\mathbb{Z}_{N^2}^*)^{\ell+1}$ computed as follows:
  1. Compute $\overline{\mathbf{x}} \in \mathbb{Z}^{\ell}$ such that $\langle \overline{\mathbf{x}}, \mathbf{y}_i \rangle = z_i$ for all $i \in [k]$.
  2. Sample $a \leftarrow \mathbb{Z}_N$ and $b \leftarrow \mathbb{Z}_{\phi(N)}$ with $\phi(N) = p' \cdot q'$ and computes
  $$c_0^* = (1 + aN) \cdot g^b \bmod N^2,$$
  $$c_i^* = (1 + \overline{x}_i N) \cdot (c_0^*)^{s_i} \bmod N^2.$$
  3. Return ct^* along with a state $\mathsf{st} = (\overline{\mathbf{x}}, a, \phi(N))$.
- $\mathsf{KeyGen}_1^{\mathcal{S}}(\mathsf{msk}^{\mathcal{S}}, \mathbf{y}, z = \langle \mathbf{y}, \mathbf{x}^* \rangle, \mathsf{st})$:
  1. Compute $\alpha = (a^{-1} \bmod N) \cdot v\phi(N) \bmod N\phi(N)$, where $v = \phi(N)^{-1} \bmod N$.
  2. Return $\mathsf{fk}'_{\mathbf{y}} = \big(\langle \mathsf{msk}^{\mathcal{S}}, \mathbf{y} \rangle - \alpha \cdot (z - \langle \overline{\mathbf{x}}, \mathbf{y} \rangle), \mathbf{y}\big)$.

Fig. 3. Simulator of pkIPFE [1].

Remark 2. For a matrix $\mathbf{Y}$, we define its functional key by $(\mathbf{Y}, \mathbf{Y} \cdot \mathsf{msk})$. Then one can securely compute a matrix multiplication $\mathbf{Y} \cdot \mathbf{x}$ as well. By an abuse of notation, we will denote it as $\mathsf{pkIPFE.KeyGen}(\mathsf{msk},\ \mathbf{Y})$.

3 Composable FE-based Privacy Preserving Machine Learning

The primary objective of this section is to clearly distinguish between FE-PPML protocol security and FE security, including selective security, semi-adaptive security, and adaptive security. This distinction is required because FE schemes that achieve the cryptographic security may still have information leakage when they are used in PPML protocols. Thus, it is essential to consider both types of security when developing FE-PPML protocols.

For example, Ryffel *et al.* [37] proposed an IND-CPA secure QFE scheme for a practical and secure image classification algorithm based on a *partially* encrypted machine learning framework. The term "partially encrypted" indicates that only the first hidden layer in a neural network is encrypted. This framework is based on the FE definition, which ensures that given a ciphertext $\mathbf{x}_b \in \{\mathbf{x}_0, \mathbf{x}_1\}$ and an activation function f of the first hidden layer, the condition $f(\mathbf{x}_0) = f(\mathbf{x}_1)$ always holds. Consequently, this results in the same intermediate values in subsequent layers, preventing the adversary from determining the message of the ciphertext.

However, this partially encrypted framework for PPML leads to significant information leakage [7], implying that IND-CPA security of the underlying FE scheme is insufficient for achieving a secure inference protocol. Since the attack exploits the plain intermediate values of hidden layers, this attack still affects the security of partially encrypted PPML even if the FE scheme of IND-CPA is replaced with that of simulation-based security.

To circumvent this limitation, we propose a new concept called a *composable functional encryption*. Intuitively, this approach allows us to generate a functional key of $\mathsf{Enc} \circ f$, where Enc is an encryption algorithm, and f is a function.

Definition 12 (composable FE). *Let $\mathsf{FE}_i = (\mathsf{Setup}_i, \mathsf{KeyGen}_i, \mathsf{Enc}_i, \mathsf{Dec}_i)$ be a set of functional encryption schemes for a function class $\mathcal{F}_i$, where $1 \leq i \leq \mathcal{E}$. When $\mathcal{F}_i$ includes $\mathsf{Enc}_{i+1} \circ f$ for any function f, we say that $\{\mathsf{FE}_i\}_{1 \leq i \leq \mathcal{E}}$ is an $\mathcal{E}$-composable FE.*

Specifically, if $\mathcal{F}_i$ accommodates $\mathsf{Enc}_{i+1} \circ f$, where f is a quadratic polynomial f, this scheme is referred to as $\mathcal{E}$-composable quadratic FE, or $\mathcal{E}$-cQFE for short.

Later, we adopt a composable FE to design a fully encrypted PPML. The PPML protocol using composable functional encryption (cFE-PPML) should have no intermediate leakages to ensure a secure inference.

3.1 Definition of cFE-PPML

We consider a privacy-preserving machine learning protocol using composable functional encryption (cFE-PPML) as a secure inference on encrypted data. Our cFE-PPML protocol involves three types of entities: a key distributor (KD), an evaluator (E), and a set of clients (C_j for $j \in [0, J]$). The key distributor is a trusted third-party authority to generate a public key (pk) and a master secret key (msk) and providing functional keys (fk) based on the evaluator's query.

Specifically, we assume that an evaluator possesses machine learning models in advance. The client encrypts the data using pk and sends the ciphertext (ct) to the evaluator. The evaluator performs inference on the encrypted data by obtaining a functional key fk_F associated with a pre-trained model F from the key distributor and the ciphertext $\mathsf{ct}_{\mathbf{x}}$ associated with the input data $\mathbf{x}$, and then outputs the computation result $F(\mathbf{x})$ in plaintext. We consider a machine learning model of $\mathcal{E}$ layers as a composition of functions $\bigcirc_{i=1}^{\mathcal{E}} f_i = f_{\mathcal{E}} \circ \cdots \circ f_1$, where the function f_i represents the computation of i-th layer of the model. That is, the inference result $F(\mathbf{x})$ can be represented by $\bigcirc_{i=1}^{\mathcal{E}} f_i(\mathbf{x}) = (f_{\mathcal{E}} \circ \cdots \circ f_1)(\mathbf{x})$.

We now describe the formal definition of cFE-PPML. The notation $A^{B(\cdot)}(Q)$ implies that an algorithm B is executed by an entity A with the input Q. Also, the notation $A^{\left(C^{B(\cdot)}\right)}(Q)$ indicates that A sends a query Q to entity C, C executes $C^{B(\cdot)}(Q)$ and responds to the query.

Definition 13 (cFE-PPML). *Let KD, E, and C denote the (trusted) key distributor, evaluator, and client, respectively, which are participants in the protocol. Let $cFE = (Setup, KeyGen, Enc, Dec)$ denote the public-key composable functional encryption scheme. We define the cFE-PPML protocol $\mathcal{P}$ as follows:*

1. $(\mathsf{pk}, \mathsf{msk}) \leftarrow KD^{\mathsf{Setup}(\cdot)}(1^\lambda)$
2. $\mathsf{ct}_{\mathbf{x}_j} \leftarrow C^{\mathsf{Enc}(\mathsf{pk},\cdot)}(\mathbf{x}_j)$
3. $\mathsf{fk}_{F_l} \leftarrow E^{KD^{\mathsf{KeyGen}(\mathsf{msk},\cdot)}}(F_l)$
4. $F_l(\mathbf{x}_j) \leftarrow E^{\mathsf{Dec}(\cdot,\cdot)}(\mathsf{ct}_{\mathbf{x}_j}, \mathsf{fk}_{F_l})$

where λ is a security parameter, $\mathbf{x}_j$ is an input data of C_j, and $F_l = \bigcirc_{i=1}^{\mathcal{E}} f_{l,i}$ is a machine learning model owned by E. The functional key fk_{F_l} is a set of $\{\mathsf{fk}_{f_{l,i}}\}_{i \in [1,\mathcal{E}]}$ where $\mathsf{fk}_{f_{l,i}} \leftarrow \mathsf{KeyGen}(\mathsf{msk}, \mathsf{Enc} \circ f_{l,i})$ for $i \in [1, \mathcal{E} - 1]$ and $\mathsf{fk}_{f_{l,\mathcal{E}}} \leftarrow \mathsf{KeyGen}(\mathsf{msk}, f_{l,\mathcal{E}})$.

3.2 General framework for cFE-PPML protocol

We construct a cFE-PPML protocol for multiple clients by adopting $\mathcal{E}$-composable functional encryption scheme for general circuits as in Fig. 4.

The problem is that previously known functional encryption (FE) schemes for general circuits are infeasible, and the same is true for composable FE. Therefore, we aim to demonstrate how to construct a secure cFE-PPML protocol using a composable FE scheme for quadratic functions (composable-QFE), which is truly feasible for real-world implementations. In brief, we first convert a pre-trained model F_l for $l \in [0, \zeta]$ into compositions of quadratic functions $F_l = \bigcirc_{i=1}^{\mathcal{E}} f_{l,i}$, and then apply composable-QFE iteratively. In Section 4, we construct a (ciphertext-bounded) composable-QFE scheme. In composable-QFE, an encryption algorithm (denoted by Enc) itself can be represented as a linear function. This allows us to generate a functional key for the composition of encryption algorithm and arbitrary quadratic function f (i.e., $\mathsf{Enc} \circ f$). Note that existing FE schemes cannot support this property, and therefore the composition of Enc and a function f is not possible. A full description of cFE-PPML protocol is given in Fig. 9.

3.3 Security of cFE-PPML

We revisit the security definition for PPML protocol in the malicious model by adapting the security by Lindell [23]. In the cFE-PPML, the following parties are involved in the protocol.

Participants: Clients ($\{C_j\}_{j\in[0,J]}$), key distributor (KD), and evaluator (E)
Protocol:

Protocol Setup by KD

1. KD samples $(\mathsf{pk}, \mathsf{msk}) \leftarrow \mathsf{Setup}(\lambda, \zeta)$ for the security parameter λ and pre-determined parameter ζ and sends pk to every client.

Encryption by C

1. Let $\mathbf{x}_j$ be the input data of C_j for $j \in [0, J]$. Every C_j encrypts a message to generate $\mathsf{ct}_{\mathbf{x}_j} \leftarrow \mathsf{Enc}(\mathsf{pk}, \mathbf{x}_j)$ and sends the ciphertext to E.

Functional Key Generation between KD and E

1. Let $\{F_l\}_{l\in[0,\zeta]}$ be a set of (pre-trained) machine learning models of the form $F_l = f_{l,\mathcal{E}} \circ \cdots \circ f_{l,1}$ for any function $f_{l,i}$. E sends the family of models $\{F_l\}_{l\in[0,\zeta]}$ to KD.
2. KD computes a set of functional keys $\{\mathsf{fk}_{\mathsf{Enc}\circ f_{l,i}}\}_{l\in[0,\zeta],i\in[1,\mathcal{E}-1]}$ and $\{\mathsf{fk}_{f_{l,\mathcal{E}}}\}_{l\in[0,\zeta]}$ defined as below. Then, KD sends them to E.

$$\mathsf{fk}_{\mathsf{Enc}\circ f_{l,i}} \leftarrow \mathsf{KeyGen}(\mathsf{msk}, \mathsf{Enc} \circ f_{l,i})$$
$$\mathsf{fk}_{f_{l,\mathcal{E}}} \leftarrow \mathsf{KeyGen}(\mathsf{msk}, f_{l,\mathcal{E}})$$

Model Evaluation by E

1. For each $j \in [0, J]$, E computes $\mathsf{Enc}(f_{l,1}(\mathbf{x}_j)) \leftarrow \mathsf{Dec}(\mathsf{pk}, \mathsf{fk}_{\mathsf{Enc}\circ f_{l,1}}, \mathsf{ct}_{\mathbf{x}_j})$ for every $l \in [0, \zeta]$. Sequentially, E computes $\mathsf{Enc}(\bigcirc_{i=1}^{\mathcal{E}-1} f_{l,i}(\mathbf{x}_j))$ for each $j \in [0, J]$.
2. Using $\mathsf{fk}_{f_{l,\mathcal{E}}}$ for every $l \in [0, \zeta]$, E obtains $F_l(\mathbf{x}_j)$ for each $j \in [0, J]$.

$$F_l(\mathbf{x}_j) \leftarrow \mathsf{Dec}(\mathsf{pk}, \mathsf{fk}_{f_{l,\mathcal{E}}}, \mathsf{ct}_{\bigcirc_{i=1}^{\mathcal{E}-1} f_{l,i}(\mathbf{x}_j)})$$

Fig. 4. General framework for cFE-PPML.

- the key distributor, denoted by KD.
- the honest client C_0 and other (malicious) clients $\{C_j\}_{j\in[1,J]}$ that have a message $\mathbf{x}_j$ for $j \in [0, J]$.
- the adversary $\mathcal{A}$
- the evaluator E which has several machine learning models $\{F_l\}$ where $\{F_l\}_{l\in[0,\zeta]}$ is a set of machine learning models of the form

$$F_l = \bigcirc_{i=1}^{\mathcal{E}} f_{l,i} = f_{l,\mathcal{E}} \circ \ldots \circ f_{l,1}$$

  for some functions $f_{l,i}$.

We note that there exist a bunch of intermediate values, which can be used to get information associated with $\mathbf{x}_0$, in the PPML protocol such as $\mathsf{Enc}((f_{l,i} \circ f_{l,i-1} \circ \ldots \circ f_{l,1})(\mathbf{x}_j))$ for any indices l, i, j. We say that cFE-PPML is secure when $\mathcal{A}$ cannot learn any information of $\mathbf{x}_0$ even if $\mathcal{A}$ interacts with clients $\{C_j\}_{j\in[1,J]}$ and the evaluator E. More formally, it can be defined as follows:

Definition 14. *We say that the cFE-PPML protocol $\mathcal{P}$ securely computes $\{F_l(\mathbf{x}_0)\}$ for $l \in [0, \zeta]$ in the presence of static malicious adversaries if for every probabilistic polynomial-time adversary $\mathcal{A}$ in the real-world, there exists a probabilistic polynomial-time algorithm $\mathcal{S}$ in the ideal-world such that for every input data $\mathbf{x}_j$ and machine learning model F_l, we have that the following two distribution ensembles (over the security parameter λ) are computationally indistinguishable:*

$$\{\mathsf{REAL}_{\mathcal{P},\mathcal{A}}(\{\mathbf{x}_j\}_{j\in[0,J]}, \{F_l\}_{l\in[0,\zeta]}, \lambda, \zeta)\} \overset{c}{\approx} \{\mathsf{IDEAL}_{\mathcal{S}}(\{\mathbf{x}_j\}_{j\in[0,J]}, \{F_l\}_{l\in[0,\zeta]}, \lambda, \zeta)\},$$

where $\{\mathsf{REAL}_{\mathcal{P},\mathcal{A}}(\{\mathbf{x}_j\}_{j\in[0,J]}, \{F_l\}_{l\in[0,\zeta]}, \lambda, \zeta)\}$ denotes the view of the corrupted clients and the adversary $\mathcal{A}$ from the real execution of $\mathcal{P}$ on inputs $(\mathbf{x}_j, F_l, \lambda, \zeta)$ for $j \in [0, J]$ and $l \in [0, \zeta]$, and $\{\mathsf{IDEAL}_{\mathcal{S}}(\{\mathbf{x}_j\}_{j\in[0,J]}, \{F_l\}_{l\in[0,\zeta]}, \lambda, \zeta)\}$ denotes the (simulated) view of corrupted clients and the simulator $\mathcal{S}$ from the ideal execution of $\mathcal{I}$ on inputs $(\mathbf{x}_j, F_l, \lambda, \zeta)$ for $j \in [0, J]$ and $l \in [0, \zeta]$. Both views are defined as the outputs of the following experiments:

- *In the* $\mathsf{REAL}_{\mathcal{P},\mathcal{A}}(\{\mathbf{x}_j\}_{j\in[0,J]}, \{F_l\}_{l\in[0,\zeta]}, \lambda, \zeta)$*:*
  1. $\mathsf{pk} \leftarrow KD^{\mathsf{Setup}(1^\lambda,\zeta)}$
  2. $f_{l,i} \leftarrow E$ *for* $l \in [0,\zeta]$, $i \in [1,\mathcal{E}]$
  3. $\mathsf{ct}_{\mathbf{x}_0} \leftarrow C_0^{\mathsf{Enc}(\mathsf{pk},\mathbf{x}_0)}$
  4. $(\mathsf{ct}_{\mathbf{x}_j}, \mathbf{x}_j) \leftarrow C_j^{\mathsf{Enc}(\mathsf{pk},\mathbf{x}_j)}$ *for* $j \in [1,J]$
  5. $\mathsf{fk}_{\mathsf{Enc}\circ f_{l,i}} \leftarrow KD^{\mathsf{KeyGen}(\mathsf{msk},\mathsf{Enc}\circ f_{l,i})}$ *for* $l \in [0,\zeta]$, $i \in [1,\mathcal{E}]$
  6. $\mathsf{Enc}(\bigcirc_{t=1}^{i} f_{l,t}(\mathbf{x}_j)) \leftarrow E^{\mathsf{Dec}(\mathsf{pk},\mathsf{fk}_{\mathsf{Enc}\circ f_{l,t}},\mathsf{ct}_{\bigcirc_{t=1}^{i-1} f_{l,t}(\mathbf{x}_j)})}$
  7. $F_l(\mathbf{x}_j) \leftarrow E^{\mathsf{Dec}(\mathsf{pk},\mathsf{fk}_{f_{l,\mathcal{E}}},\mathsf{Enc}(\bigcirc_{i=1}^{\mathcal{E}-1} f_{l,i}(\mathbf{x}_j))}$
- *In the* $\mathsf{IDEAL}_{\mathcal{S}}(\{\mathbf{x}_j\}_{j\in[0,J]}, \{F_l\}_{l\in[0,\zeta]}, \lambda, \zeta)$*:*
  1. $\mathsf{pk}^{\mathcal{S}} \leftarrow KD^{\mathsf{Setup}^{\mathcal{S}}(1^\lambda,\zeta)}$
  2. $f_{l,i} \leftarrow E$ *for* $l \in [0,\zeta]$, $i \in [1,\mathcal{E}]$
  3. $\mathsf{ct}_{\mathbf{x}_0} \leftarrow C_0^{\mathsf{Enc}^{\mathcal{S}}(\mathsf{pk}^{\mathcal{S}},F_l(\mathbf{x}_0))}$
  4. $(\mathsf{ct}_{\mathbf{x}_j}, \mathbf{x}_j) \leftarrow C_j^{\mathsf{Enc}(\mathsf{pk}^{\mathcal{S}},\mathbf{x}_j)}$ *for* $j \in [1,J]$
  5. $\mathsf{fk}_{\mathsf{Enc}\circ f_{l,i}} \leftarrow KD^{\mathsf{KeyGen}^{\mathcal{S}}(\mathsf{msk},\mathsf{Enc}\circ f_{l,i})}$ *for* $l \in [0,\zeta]$, $i \in [1,\mathcal{E}]$
  6. $\mathsf{Enc}(\bigcirc_{t=1}^{i} f_{l,t}(\mathbf{x}_j)) \leftarrow E^{\mathsf{Dec}^{\mathcal{S}}(\mathsf{pk}^{\mathcal{S}},\mathsf{fk}_{\mathsf{Enc}\circ f_{l,t}},\mathsf{ct}_{\bigcirc_{t=1}^{i-1} f_{l,t}(\mathbf{x}_j)})}$
  7. $F_l(\mathbf{x}_j) \leftarrow E^{\mathsf{Dec}^{\mathcal{S}}(\mathsf{pk}^{\mathcal{S}},\mathsf{fk}_{f_{l,\mathcal{E}}},\mathsf{Enc}(\bigcirc_{i=1}^{\mathcal{E}-1} f_{l,i}(\mathbf{x}_j))}$

In Section 6.1, we will prove that the protocol in Fig. 9 achieves the security. Here, we consider the evaluator E as the malicious party.

4 Candidate for Composable Functional Encryption Scheme

4.1 Technical Overview

Our primary technical insight involves incorporating the encryption algorithm into the function itself and considering a key generation algorithm on the composition of the encryption and the function. This approach enables us to obtain a functional key $\mathsf{fk}_{\mathsf{Enc}\circ f}$ which corresponds to the key generation algorithm applied to $\mathsf{Enc} \circ f$. Using the decryption algorithm on $(\mathsf{fk}_{\mathsf{Enc}\circ f}, \mathsf{ct}_{\mathbf{m}})$, we can produce an evaluated value in the form of an encrypted ciphertext $\mathsf{ct}_{f(\mathbf{m})}$. Since the output is still encrypted, we can proceed with the decryption iteratively. Specifically, we compute the decryption algorithm as

$$\mathsf{Dec}(\mathsf{fk}_{\mathsf{Enc}\circ f_{i+1}}, \mathsf{ct}_{\bigcirc_{t=1}^{i} f_t(\mathbf{m})})$$

for $i = 1, \dots, \mathcal{E}$, where $\mathcal{E}$ is the number of functions to be computed. This framework is illustrated in Fig. 1.(b).

One major challenge in computing the functional key $\mathsf{fk}_{\mathsf{Enc}\circ f}$ is that existing functional encryption schemes for arbitrary circuits are truly infeasible. To address this limitation, we restrict functionalities to quadratic polynomials, and instantiate the scheme by modifying the Musciagna FE scheme [29] so that the Enc can be represented by a linear function.

To provide an intuition, we briefly describe the FE scheme in terms of an IPFE. Let $F : \mathbb{Z}^\ell \to \mathbb{Z}^m$ be a linear multivariate function and $\mathbf{M}_F \in \mathbb{Z}^{m\times\ell}$ be its matrix representation satisfying $F(\mathbf{x}) = \mathbf{M}_F \cdot \mathbf{x}$. Consider a ciphertext as an encryption of a message $\mathbf{m}$ from a message space $\mathcal{M} = \mathbb{Z}^\ell$ using inner product encryption. The functional key is represented by $\mathsf{fk}_F = \mathbf{M}_F \cdot \mathbf{D}_0$, and the ciphertext is of the form $\mathsf{ct}_{\mathbf{m}} = [\mathbf{D_0}^{-1} \cdot \mathbf{m}]_g$, where g is a public group element and $\mathbf{D}_0$ is an invertible matrix of size $\ell \times \ell$ computed with the master secret key. Then, the decryption algorithm for this scheme can be computed as

$$\log_g(\mathsf{ct}_{\mathbf{m}}^{\mathsf{fk}_F}) = \log_g([\mathbf{M}_F \cdot \mathbf{D}_0 \cdot \mathbf{D}_0^{-1} \cdot \mathbf{m}]_g) = F(\mathbf{m}),$$

where $\log_g$ is a discrete logarithm over a group with base g.

We now introduce a concept to develop a desired functional encryption algorithm. To be precise, we consider a functional key for a composition function $\mathsf{Enc} \circ F$ using another secret invertible matrix $\mathbf{D}_1$:

$$\mathsf{fk}_{\mathsf{Enc}\circ F} = [\mathbf{D}_1^{-1} \cdot \mathbf{M}_F \cdot \mathbf{D}_0]_g.$$

It is clear that the decryption algorithm on a pair $(\mathsf{fk}_{\mathsf{Enc}\circ F}, \mathsf{ct}_{\mathbf{m}} = \mathbf{D}_0^{-1} \cdot \mathbf{m})$ outputs a value

$$\log_g((\mathsf{fk}_{\mathsf{Enc}\circ F})^{\mathsf{ct}_{\mathbf{m}}}) = \mathbf{D}_1^{-1} \cdot \mathbf{M}_F \cdot \mathbf{m} = \mathsf{ct}_{F(\mathbf{m})}.$$

Hence, the decryption output is a new ciphertext representing the evaluation of F and the original message $\mathbf{m}$ using another secret.

Another challenge is efficiently solving the discrete log problem $(\log_g(\cdot))$ in the decryption algorithm. While an elliptic curve-based FE scheme is efficient, it should have a relatively small message space since the decryption algorithm requires to solve the discrete log problem. This small message space would imply the security issue that the size of every intermediate ciphertext obtained by decryption algorithm of $(\mathsf{fk}_{\mathsf{Enc}\circ F}, \mathsf{ct}_{\mathbf{m}})$ is small. Thus, we need a large message space with an efficient decryption process. To address this, we use a decision composite residuosity (DCR)-based FE scheme that enables efficient solving of the discrete log problem for a relatively large message set. Therefore, we had to consider a group-based FE in which DCR groups could be used, so we modified the DCR-based Musciagna scheme. The detailed discussion of the security of the modified FE scheme and original scheme description is given in Appendix B.

4.2 Ciphertext-bounded composable-QFE

We present a composable private-key *ciphertext-bounded* quadratic functional encryption (QFE) scheme, for short composable-QFE, of which ciphertexts are vectors. We note that it is sufficient to generate the composable-QFE because any polynomial can be represented as a composite of quadratic polynomials. This scheme can be employed to instantiate a secure computation protocol described in Section 3.

We first clarify a set of quadratic functionalities $\mathcal{F}$ of our composable-QFE. The functionality $\mathcal{F}$ corresponds to a set of functions of the form $((\mathbf{x}\|1) \otimes (\mathbf{x}\|1))^T \cdot \mathbf{c}_f$ for some $\mathbf{x} \in \mathbb{Z}^\ell$, and $\mathbf{c}_f \in \mathbb{Z}^{(\ell+1)^2}$ such that $\|\mathbf{x}\|_\infty \le B_X$ and $\|\mathbf{c}_f\|_\infty \le B_{\mathcal{F}}$. Here, $\mathbf{c}_f$ is a coefficient vector corresponding to a quadratic function f. This section provides only a scheme description of ciphertext-bounded composable-QFE. Its security proof is deferred to Appendix B when the composable-QFE allows only a bounded number of ciphertexts.

Notation. We introduce notations to describe the composable-QFE. Thus, the underlying group size is $N \cdot \phi(N)$ while the message space is $\mathbb{Z}_N$. Throughout this paper, we denote $N \cdot \phi(N)$ by Δ for simplicity. Since our algorithm is based on the DCR scheme, we employ the $\mathsf{DCR.pp}$ function described in the Section 2.4

As a building block of our scheme, given a wide matrix $\mathbf{D} \in \mathbb{Z}_\Delta^{m\times n}$ $(m < n)$, we define sets of (the right) kernel and inverse as following.

- $\mathsf{Ker}(\mathbf{D}) = \{\mathbf{D}^\perp \in \mathbb{Z}_\Delta^{n\times(n-m)} : \mathbf{D}\cdot\mathbf{D}^\perp = \mathbf{O} \bmod \Delta\}$
- $\mathsf{Rinv}(\mathbf{D}) = \{\mathbf{D}^{-1} \in \mathbb{Z}_\Delta^{n\times m} : \mathbf{D}\cdot\mathbf{D}^{-1} = \mathbf{I}_m \bmod \Delta\}$

Composable-IPFE. To construct the composable-QFE, we first describe an inner products encryption scheme (composable-IPFE) as an ingredient The goal of composable-IPFE is to compute the inner product $[\mathbf{y}^T \cdot \mathbf{x}]_g$ for two inputs; a message $\mathbf{x} \in \mathbb{Z}_N^\ell$ and a function $\mathbf{y} \in \mathbb{Z}_N^\ell$ by running the decryption algorithm. The detailed construction of composable-IPFE is given by Fig. 5.

Correctness (of composable-IPFE in Fig. 5). To this end, it suffices to show that $\mathsf{Dec}_{\mathsf{IPFE}}(\mathsf{pp}, \mathsf{fk}_{\mathbf{y}}, \mathsf{ct}_{\mathbf{x}}) = [\mathbf{y}^T \cdot \mathbf{x}]_g$. It can be easily proven by examining the decryption procedure. Specifically, we can observe that:

$$\begin{aligned}
\mathsf{fk}_{\mathbf{y}}^{\mathsf{ct}_{\mathbf{x}}} &= \left[\begin{pmatrix} -\mathbf{U}^T \cdot \mathbf{y} \\ \mathbf{y} \end{pmatrix}^T \cdot \mathbf{D} \cdot \mathsf{ct}_{\mathbf{x}}\right]_g \\
&= \left[\begin{pmatrix} -\mathbf{U}^T \cdot \mathbf{y} \\ \mathbf{y} \end{pmatrix}^T \cdot \mathbf{D} \cdot \mathbf{D}^\perp \cdot \widetilde{\mathbf{r}} + \begin{pmatrix} -\mathbf{U}^T \cdot \mathbf{y} \\ \mathbf{y} \end{pmatrix}^T \mathbf{D} \cdot \mathbf{D}^{-1} \cdot \begin{pmatrix} r \cdot \mathbf{a} \\ \mathbf{x} + r \cdot \mathbf{U} \cdot \mathbf{a} \end{pmatrix}\right]_g \\
&= \left[\begin{pmatrix} -\mathbf{U}^T \cdot \mathbf{y} \\ \mathbf{y} \end{pmatrix}^T \cdot \begin{pmatrix} r \cdot \mathbf{a} \\ \mathbf{x} + r \cdot \mathbf{U} \cdot \mathbf{a} \end{pmatrix}\right]_g
\end{aligned}$$

- $\mathsf{Setup}_{\mathsf{IPFE}}(\lambda, \ell, Q)$:
  1. Choose $(N, p, q, g) \leftarrow \mathsf{DCR.pp}(\lambda)$ and define $\Delta = N \cdot \phi(N)$.
  2. Sample $\mathbf{a} \leftarrow \mathbb{Z}_{\Delta}^{2}$, $\mathbf{U} \leftarrow \mathbb{Z}_{\Delta}^{\ell \times 2}$, and $\mathbf{D} \leftarrow \mathbb{Z}_{\Delta}^{(\ell+2)\times(\ell+2+Q)}$.
  3. Sample $\mathbf{D}^{\perp} \leftarrow \mathsf{Ker}(\mathbf{D})$ and $\mathbf{D}^{-1} \leftarrow \mathsf{Rinv}(\mathbf{D})$
  4. Return
  $$\mathsf{msk} = \{\mathbf{a}, \mathbf{U}, \mathbf{D}, \mathbf{D}^{\perp}, \mathbf{D}^{-1}\} \text{ and } \mathsf{pp} = \{g, N, \Delta\}.$$
- $\mathsf{KeyGen}_{\mathsf{IPFE}}(\mathsf{msk}, \mathsf{pp}, \mathbf{y} \in \mathbb{Z}_N^{\ell})$:
  1. Return
  $$\mathsf{fk}_{\mathbf{y}} = \left[\begin{pmatrix} -\mathbf{U}^T \cdot \mathbf{y} \\ \mathbf{y} \end{pmatrix}^T \cdot \mathbf{D} \right]_g .$$
- $\mathsf{Enc}_{\mathsf{IPFE}}(\mathsf{msk}, \mathsf{pp}, \mathbf{x} \in \mathbb{Z}_N^{\ell})$:
  1. Sample $r \leftarrow \mathbb{Z}_{\Delta}$ and $\widetilde{\mathbf{r}} \leftarrow \mathbb{Z}_{\Delta}^{Q}$.
  2. Return
  $$\mathsf{ct}_{\mathbf{x}} = \mathbf{D}^{\perp} \cdot \widetilde{\mathbf{r}} + \mathbf{D}^{-1} \cdot \begin{pmatrix} r \cdot \mathbf{a} \\ \mathbf{x} + r \cdot \mathbf{U} \cdot \mathbf{a} \end{pmatrix} \in \mathbb{Z}_{\Delta}^{\ell+2+Q}.$$
- $\mathsf{Dec}_{\mathsf{IPFE}}(\mathsf{pp}, \mathsf{fk}_{\mathbf{y}}, \mathsf{ct}_{\mathbf{x}})$:
  1. Return $\mathsf{fk}_{\mathbf{y}}^{\mathsf{ct}_{\mathbf{x}}}$.

Fig. 5. composable-IPFE.

$$= \left[(-\mathbf{U}^T \cdot \mathbf{y})^T \cdot r \cdot \mathbf{a} + \mathbf{y}^T \cdot \mathbf{x} + r \cdot \mathbf{y}^T \cdot \mathbf{U} \cdot \mathbf{a}\right]_g = \left[\mathbf{y}^T \cdot \mathbf{x}\right]_g$$

This completes the correctness.

Composable-QFE. We now introduce the composable-QFE, built from composable-IPFE. The detailed construction of composable-QFE is then given by Fig. 6.

Correctness (of composable-QFE in Fig. 6). We first observe the term $\mathbf{e}_1 = [(\mathbf{D}_0 \otimes \mathbf{D}_1)^T \cdot \mathbf{c}_f]_g^{(\mathsf{ct}_0 \otimes \mathsf{ct}_1)}$:

$$\begin{aligned}
\mathbf{e}_1 &= [(\mathbf{D}_0 \otimes \mathbf{D}_1)^T \cdot \mathbf{c}_f]_g^{(\mathsf{ct}_0 \otimes \mathsf{ct}_1)} = [\langle (\mathbf{D}_0 \otimes \mathbf{D}_1)^T \cdot \mathbf{c}_f, (\mathsf{ct}_0 \otimes \mathsf{ct}_1) \rangle]_g \\
&= [\mathbf{c}_f^T \cdot ((\mathbf{D}_0 \cdot \mathsf{ct}_0) \otimes (\mathbf{D}_1 \cdot \mathsf{ct}_1))]_g = \left[\mathbf{c}_f^T \cdot \big((c \cdot \overline{\mathbf{x}} + \mathbf{V} \cdot \mathbf{r}_0) \otimes (\overline{\mathbf{x}} + \mathbf{W} \cdot \mathbf{r}_1)\big)\right]_g \\
&= \left[\mathbf{c}_f^T \cdot \big(c \cdot \overline{\mathbf{x}} \otimes \overline{\mathbf{x}} + c \cdot \overline{\mathbf{x}} \otimes \mathbf{W} \cdot \mathbf{r}_1 + \mathbf{V} \cdot \mathbf{r}_0 \otimes \overline{\mathbf{x}} + \mathbf{V} \cdot \mathbf{r}_0 \otimes \mathbf{W} \cdot \mathbf{r}_1\big)\right]_g \\
&= \left[\mathbf{c}_f^T \cdot \big(c \cdot \overline{\mathbf{x}} \otimes \overline{\mathbf{x}} + (\mathbf{V} \cdot \mathbf{r}_0) \otimes \overline{\mathbf{x}} + (c \cdot \overline{\mathbf{x}} + \mathbf{V} \cdot \mathbf{r}_0) \otimes (\mathbf{W} \cdot \mathbf{r}_1)\big)\right]_g \\
&= \left[\mathbf{c}_f^T \cdot \big(c \cdot \overline{\mathbf{x}} \otimes \overline{\mathbf{x}} + (\mathbf{V} \otimes \mathbf{I}_{\ell+1}) \cdot (\mathbf{r}_0 \otimes \overline{\mathbf{x}}) + (\mathbf{I}_{\ell+1} \otimes \mathbf{W}) \cdot ((c \cdot \overline{\mathbf{x}} + \mathbf{V} \cdot \mathbf{r}_0) \otimes \mathbf{r}_1)\big)\right]_g \\
&= \left[c \cdot \mathbf{c}_f^T \cdot (\overline{\mathbf{x}} \otimes \overline{\mathbf{x}}) + \underbrace{\begin{pmatrix} \mathbf{r}_0 \otimes \overline{\mathbf{x}} \\ (c \cdot \overline{\mathbf{x}} + \mathbf{V} \cdot \mathbf{r}_0) \otimes \mathbf{r}_1 \end{pmatrix}^T}_{\mathbf{h}^T} \underbrace{\begin{pmatrix} \mathbf{V}^T \otimes \mathbf{I}_{\ell+1} \\ \mathbf{I}_{\ell+1} \otimes \mathbf{W}^T \end{pmatrix} \cdot \mathbf{c}_f}_{\mathsf{fk}_0 \cdot \mathbf{c}_f} \right]_g .
\end{aligned}$$

On the other hand, by the correctness of $\mathsf{Dec}_{\mathsf{IPFE}}$, we obtain that

$$\mathbf{e}_2 = [\mathsf{Dec}_{\mathsf{IPFE}}(\mathsf{pp}_{\mathsf{IPFE}}, \mathsf{fk}_{\mathsf{IPFE}}, \mathsf{ct}_{\mathsf{IPFE}})]_g = [\mathbf{h}^T \cdot \mathsf{fk}_0 \cdot \mathbf{c}_f]_g.$$

The ratio $\mathbf{e}_1/\mathbf{e}_2$ is then identical to

$$[c \cdot \mathbf{c}_f^T \cdot (\overline{\mathbf{x}} \otimes \overline{\mathbf{x}})]_g = (1+N)^{\mathbf{c}_f^T \cdot (\overline{\mathbf{x}} \otimes \overline{\mathbf{x}})} = (1+N)^{f(\mathbf{x})}.$$

Hence, the $\log_{(1+N)}(\mathbf{e}_1/\mathbf{e}_2)$ outputs $f(\mathbf{x})$ mod N. It directly implies that $\mathsf{Dec}_{\mathsf{QFE}}(\mathsf{pp}, \mathsf{fk}_f, \mathsf{ct}) = f(\mathbf{x})$ as long as $|f(\mathbf{x})| < N/2$.

- $\mathsf{Setup}_{\mathsf{QFE}}(\lambda, \ell, Q)$:
  1. Sample $(\mathsf{msk}_{\mathsf{IPFE}}, \mathsf{pp}_{\mathsf{IPFE}} = \{g, N, \varDelta\}) \leftarrow \mathsf{Setup}_{\mathsf{IPFE}}(\lambda, 4\ell + 4, Q)$.
  2. Let c_0 be a inverse of $\frac{(g^{\phi(N)} \bmod N^2)-1}{N}$ modulo N and $c = c_0 \cdot \phi(N)$ so that it satisfies $g^c = 1 + N \bmod N^2$.
  3. Sample $\mathbf{V} \leftarrow \mathbb{Z}_{\varDelta}^{(\ell+1)\times 2}$, $\mathbf{W} \leftarrow \mathbb{Z}_{\varDelta}^{(\ell+1)\times 2}$.
  4. Sample $\mathbf{D}_b \leftarrow \mathbb{Z}_{\varDelta}^{(\ell+1)\times(\ell+1+Q)}$ for $b \in \{0, 1\}$.
  5. Sample $\mathbf{D}_b^{\perp} \leftarrow \mathsf{Ker}(\mathbf{D}_b)$ and $\mathbf{D}_b^{-1} \leftarrow \mathsf{Rinv}(\mathbf{D}_b)$ for $b \in \{0, 1\}$.
  6. Return $(\mathsf{pp}, \mathsf{msk})$ where $\mathsf{pp} = \mathsf{pp}_{\mathsf{IPFE}}$ and
$$\mathsf{msk} = \left\{\mathbf{V}, \mathbf{W}, \{\mathbf{D}_b, \mathbf{D}_b^{\perp}, \mathbf{D}_b^{-1}\}_{b\in\{0,1\}}, c, \mathsf{msk}_{\mathsf{IPFE}}\right\}.$$
- $\mathsf{KeyGen}_{\mathsf{QFE}}(\mathsf{msk}, \mathsf{pp}, \mathbf{c}_f \in \mathbb{Z}^{(\ell+1)^2})$:
  1. Sample $\mathsf{fk}_{\mathsf{IPFE}} \leftarrow \mathsf{KeyGen}_{\mathsf{IPFE}}(\mathsf{msk}_{\mathsf{IPFE}}, \mathsf{pp}_{\mathsf{IPFE}}, \mathsf{fk}_0 \cdot \mathbf{c}_f)$ where
$$\mathsf{fk}_0 = \begin{pmatrix} \mathbf{V}^T \otimes \mathbf{I}_{\ell+1} \\ \mathbf{I}_{\ell+1} \otimes \mathbf{W}^T \end{pmatrix} \in \mathbb{Z}_{\varDelta}^{4(\ell+1)\times(\ell+1)^2}.$$
  2. Compute $\mathsf{fk}_1 = \left[(\mathbf{D}_0 \otimes \mathbf{D}_1)^T \cdot \mathbf{c}_f\right]_g \in \mathbb{G}^{(\ell+1+Q)^2}$
  3. Return $\mathsf{fk}_f = \{\mathsf{fk}_{\mathsf{IPFE}}, \mathsf{fk}_1\}$.
- $\mathsf{Enc}_{\mathsf{QFE}}(\mathsf{msk}, \mathsf{pp}, \mathbf{x} \in \mathbb{Z}^{\ell})$:
  1. Set $\overline{\mathbf{x}} = (\mathbf{x}\|1)$.
  2. Sample $\mathbf{r}_b \leftarrow \mathbb{Z}_{\varDelta}^2$ for $b \in \{0, 1\}$.
  3. Compute ct_b for $b \in \{0, 1\}$ as follows.
$$\begin{aligned} \mathsf{ct}_0 &= \mathbf{D}_0^{\perp} \cdot \widetilde{\mathbf{r}}_0 + \mathbf{D}_0^{-1} \cdot (c \cdot \overline{\mathbf{x}} + \mathbf{V} \cdot \mathbf{r}_0) \in \mathbb{Z}_{\varDelta}^{\ell+1+Q} \\ \mathsf{ct}_1 &= \mathbf{D}_1^{\perp} \cdot \widetilde{\mathbf{r}}_1 + \mathbf{D}_1^{-1} \cdot (\overline{\mathbf{x}} + \mathbf{W} \cdot \mathbf{r}_1) \in \mathbb{Z}_{\varDelta}^{\ell+1+Q}. \end{aligned}$$
  4. Sample $\mathsf{ct}_{\mathsf{IPFE}} \leftarrow \mathsf{Enc}_{\mathsf{IPFE}}(\mathsf{msk}_{\mathsf{IPFE}}, \mathbf{h})$ for
$$\mathbf{h} = \big((\mathbf{r}_0 \otimes \overline{\mathbf{x}}) \,\|\, ((c \cdot \overline{\mathbf{x}} + \mathbf{V} \cdot \mathbf{r}_0) \otimes \mathbf{r}_1)\big) \in \mathbb{Z}_{\varDelta}^{4(\ell+1)}.$$
  5. Return ct defined as follows:
$$\mathsf{ct} = \left\{\{\mathsf{ct}_b\}_{b\in\{0,1\}}, \mathsf{ct}_{\mathsf{IPFE}}\right\}$$
- $\mathsf{Dec}_{\mathsf{QFE}}(\mathsf{pp}, \mathsf{fk}_f, \mathsf{ct})$:
  1. Compute $\mathbf{e}_1$ and $\mathbf{e}_2$ as follows.
$$\begin{aligned} \mathbf{e}_1 &= \mathsf{fk}_1^{\mathsf{ct}_0 \otimes \mathsf{ct}_1}, \\ \mathbf{e}_2 &= \left[\mathsf{Dec}_{\mathsf{IPFE}}(\mathsf{pp}_{\mathsf{IPFE}}, \mathsf{fk}_{\mathsf{IPFE}}, \mathsf{ct}_{\mathsf{IPFE}})\right]_g. \end{aligned}$$
  2. Return $\log_{(1+N)}(\mathbf{e}_1/\mathbf{e}_2)$.

Fig. 6. composable-QFE.

Remark 3. The operations $\mathsf{KeyGen}_{\mathsf{IPFE,QFE}}$ (resp. $\mathsf{Enc}_{\mathsf{IPFE,QFE}}$) were performed on a matrix $\mathbf{M}$ in a column-by-column manner. To illustrate, the function $\mathsf{KeyGen}_{\mathsf{IPFE}}(\mathsf{msk}, \mathsf{pp}, \mathbf{M})$ is defined by the expression $(\mathsf{KeyGen}_{\mathsf{IPFE}}(\mathsf{msk}, \mathsf{pp}, \mathbf{M}[i]))_{i\in[1,\mathsf{col}]}$, where the notation $(\mathbf{M}[i])_{i\in[1,\mathsf{col}]}$ represents the set of all column vectors of length col comprising the matrix $\mathbf{M}$.

Composite evaluation via composable QFE. We emphasize that for fixed randomness in the encryption of the QFE scheme, $\mathsf{EncQFE}(\mathsf{msk}_{\mathsf{QFE}}, \mathsf{pp}_{\mathsf{QFE}}, \cdot)$ is a linear function for a message. This implies that, rather than requesting the vector quadratic function $F : \mathbb{Z}^{\ell} \to \mathbb{Z}^m$ during the key generation process, as in traditional FE, one can query the coefficient vector of the composite function $\mathsf{Enc}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE}}, \mathsf{pp}, \cdot) \circ F$.

To enable this, we exploit the $\mathsf{Enc}_{\mathsf{QFE}}$ algorithm for a linear or quadratic function input F with a matrix representation $\mathbf{M}_F$. Specifically, $F(\mathbf{x}) = \mathbf{M}_F \cdot ((\mathbf{x}\|1) \otimes (\mathbf{x}\|1))$ (or $F(\mathbf{x}) = \mathbf{M}_F \cdot (\mathbf{x}\|1)$

if F is linear) and outputs a composition of encryption and the function evaluation $\mathbf{M}_{\mathsf{Enc}\circ F}$. We denote this algorithm by cEnc, representing the composition of encryption and a function.

For further composite evaluations, we incorporate two subroutines into the description of the algorithm and will now detail them. For these evaluations, it is necessary to get an encryption of $(F(\mathbf{x})\|1)$. Therefore, we define the expanded matrix representation of F, $\overline{\mathbf{M}}_F$, by concatenation of the unit vector $(0,\ldots,0,1)$ as the last row of the matrix $\mathbf{M}_F$. It is clear that $\overline{\mathbf{M}}_F\cdot((\mathbf{x}\|1)\otimes(\mathbf{x}\|1)) = (F(\mathbf{x})\|1)$.

We then implement a functional key of $\mathsf{Enc}_{\mathsf{QFE}}\circ F$ and its decryption process by using the expanded matrix representation. The detailed algorithms are given by Fig. 8. In the description, we denote $\overline{\mathbf{M}}_F[i]$ as the i-th column vector of $\overline{\mathbf{M}}_F$.

In order to compute composite evaluations, it is necessary to impose size restrictions on the ciphertexts of the composable-QFE since the correctness of composable-QFE only works when an evaluated value is less than $N/2$. For this purpose, given $L\in\mathbb{Z}$, we set $Y=\lfloor \varDelta^{1/L}\rceil$. In other words, it holds that $Y^{L-1}<\varDelta<Y^L$. To restrict the size of ciphertexts, we then define two functions:

$$\begin{aligned}\mathsf{Decomp}_L:\mathbb{Z}_\varDelta &\mapsto \mathbb{Z}^L\\ v &\to (v_0,v_1,\ldots,v_{L-1})\\ \mathsf{Power}_L:\mathbb{Z}^L &\mapsto \mathbb{Z}\\ (w_0,w_1,\ldots,w_{L-1}) &\to \sum_{i=0}^{L-1} w_i\cdot Y^i,\end{aligned}$$

where the $\{v_i\}$ holds that $\sum_{i=0}^{L-1} v_i\cdot Y^i = v$. By the definition of both functions, it is clear that

$$\mathsf{Power}_L(\mathsf{Decomp}_L(v)) = v.$$

In the case where the input is a matrix, we apply the Decomp_Y function to each entry. The power map is then properly defined as an inverse map of the decomposition function on matrices. We now show the details of the algorithm in Fig. 7.

Correctness (of Fig. 7 and Fig. 8). Due to the decryption correctness of composable-QFE, $\mathsf{cDec}_{\mathsf{QFE}}$ returns a vector of the form $\mathsf{Power}_L(\mathbf{t})$ for $\mathbf{t}=\left(\mathbf{M}^{\mathsf{decomp}}_{\mathsf{Enc}\circ F}\cdot\mathbf{y}\right)$, where $\mathbf{y}$ is $(\mathbf{x}\|1)\otimes(\mathbf{x}\|1)$. Let B_X be a bound of $\mathbf{x}$. Since each entry of $\mathbf{M}^{\mathsf{decomp}}_{\mathsf{Enc}\circ F}$ is less than $Y=\lfloor\varDelta^{1/L}\rceil$, each entry of the product $\mathbf{M}^{\mathsf{decomp}}_{\mathsf{Enc}\circ F}\cdot((\mathbf{x}\|1)\otimes(\mathbf{x}\|1))$ is less than $Y\cdot B_X\cdot(\ell+1)^2$. Thus, if we choose large enough L to satisfy $Y\cdot B_X\cdot(\ell+1)^2<N/2$, then $\mathsf{cDec}_{\mathsf{QFE}}$ in Fig. 8 will output an exact decryption value $\mathbf{t}_{\mathsf{idx}}$. Similarly, $\mathsf{cDec}_{\mathsf{pkIPFE}}$ yields an exact decryption value $\mathbf{t}_{\mathsf{idx}}$ as long as $Y\cdot B_X\cdot(\ell+1)<N/2$.

Hence, from the linear property, $\mathsf{Decomp}_L(\mathbf{t})$ is represented by a vector $(\mathsf{ct}_0,\mathsf{ct}_1,\mathsf{ct}_{\mathsf{IPFE}})$ of the form

$$\begin{aligned}\mathsf{ct}_0 &= \mathbf{D}_0^{\perp}\cdot\widetilde{\mathbf{r}}_0+\mathbf{D}_0^{-1}\cdot(c\cdot(F(\mathbf{x})\|1)+\mathbf{V}\cdot\mathbf{r}_0) \bmod \varDelta\\ \mathsf{ct}_1 &= \mathbf{D}_1^{\perp}\cdot\widetilde{\mathbf{r}}_1+\mathbf{D}_1^{-1}\cdot((F(\mathbf{x})\|1)+\mathbf{W}\cdot\mathbf{r}_1) \bmod \varDelta\\ \mathsf{ct}_{\mathsf{IPFE}} &= \mathsf{Enc}_{\mathsf{IPFE}}(\mathsf{msk}_{\mathsf{IPFE}},\mathbf{h})\end{aligned}$$

with $\mathbf{h}=\big(\mathbf{r}_0\otimes F(\mathbf{x})\|(c\cdot F(\mathbf{x})+\mathbf{V}\cdot\mathbf{r}_0)\otimes\mathbf{r}_1\big)$ for some $\widetilde{\mathbf{r}}_0$ and $\widetilde{\mathbf{r}}_1$. Thus, the output of both cDec can be regarded as a ciphertext of message $F(\mathbf{x})$.

Remark 4. The randomness $(\mathbf{r}_0,\mathbf{r}_1)$ in $\mathsf{cDec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE}},\mathsf{fk}_{\mathsf{Enc}\circ F},\mathsf{ct},L)$ is shared with that of $\mathsf{fk}_{\mathsf{Enc}\circ F}$. Therefore, it cannot be guaranteed that the ciphertext is secure. Accordingly, when instantiating cFE-PPML using the current composable-QFE, an additional factor is required to ensure sufficient randomness. This additional factor will be discussed in greater detail in Section 5.

4.3 Security proof of composable FE

This section demonstrates that the composable-QFE satisfies the semi-adaptive security when the number of ciphertexts is Q-bounded.

Theorem 2. *The composable-QFE described in Section 4 is semi-adaptively secure under the MDDH and bilateral 2-LIN assumptions for Q-bounded ciphertexts. In particular, it holds that*

$$\mathsf{Adv}_{\mathsf{cQFE}} \leq \mathsf{Adv}_{\mathsf{MDDH}} + 2 \cdot \mathsf{Adv}_{\mathsf{2\text{-}Lin}}.$$

To prove this theorem, our strategy is to provide a polynomial time reduction from the quadratic functional encryption scheme, $\mathsf{Mus.QFE}$ with Q ciphertexts, which is suggested by Musciagna [29] to the composable-QFE scheme.

On the other hand, [29] proved that the $\mathsf{Mus.QFE}$ scheme achieves the semi-adaptive simulation security under MDDH and 2-Lin assumptions. When these conditions are met, the following can be concluded:

$$\mathsf{Adv}_{\mathsf{cQFE}} \leq \mathsf{Adv}_{\mathsf{Mus.QFE}} \leq \mathsf{Adv}_{\mathsf{MDDH}} + 2 \cdot \mathsf{Adv}_{\mathsf{2\text{-}Lin}}.$$

We provide both the description and reduction of the $\mathsf{Mus.QFE}$ in the Appendix A. The proof of Theorem 2 will be given by Appendix B.

5 Secure Protocol for Function Compositions

In this section, we propose a secure protocol for function compositions using functional encryption for quadratic polynomials. We will demonstrate the application of our protocol through privacy-preserving quadratic neural networks, as detailed in Section 7.3. Our approach effectively addresses the pervasive issue of intermediate leakage, which has been identified as a significant challenge in existing FE-based PPML solutions.

The protocol $\mathcal{P}$ is constructed using a (public-key) inner product encryption scheme, pkIPFE in Fig. 2, and a composable FE scheme for quadratic polynomials, composable-QFE in Fig. 6. The

- $\mathsf{cEnc}(\mathsf{msk}_{\mathsf{QFE}}, \mathsf{pp}_{\mathsf{QFE}}, \mathbf{M}_F, L)$
  1. Sample $\mathbf{r}_b \leftarrow \mathbb{Z}_{\Delta}^2$ for $b \in \{0,1\}$.
  2. Let m and col be the number of rows and columns of $\mathbf{M}_F$, respectively, and define $\overline{\mathbf{M}}_F = \begin{pmatrix} \mathbf{M}_F \\ \mathbf{e}_{\mathsf{col}} \end{pmatrix}$, where $\mathbf{e}_{\mathsf{col}}$ is a unit vector $(0, \ldots, 0, 1)$ of length col.
  3. For $1 \leq i < \mathsf{col}$, sample $\widetilde{\mathbf{r}}_{b,i} \leftarrow \mathbb{Z}_{\Delta}^{Q}$ for $b \in \{0,1\}$ and compute the following:
$$\begin{aligned} \mathsf{ct}_{0,i} &= \mathbf{D}_0^{\perp} \cdot \widetilde{\mathbf{r}}_{0,i} + \mathbf{D}_0^{-1} \cdot c \cdot \overline{\mathbf{M}}_F[i] \in \mathbb{Z}_{\Delta}^{m+1+Q} \\ \mathsf{ct}_{1,i} &= \mathbf{D}_1^{\perp} \cdot \widetilde{\mathbf{r}}_{1,i} + \mathbf{D}_1^{-1} \cdot \overline{\mathbf{M}}_F[i] \in \mathbb{Z}_{\Delta}^{m+1+Q} \\ \mathsf{ct}_{\mathsf{IPFE},i} &\leftarrow \mathsf{Enc}_{\mathsf{IPFE}}(\mathsf{msk}_{\mathsf{IPFE}}, \mathsf{pp}_{\mathsf{IPFE}}, \mathbf{h}_i), \end{aligned}$$
     where $\mathbf{h}_i = \big((\mathbf{r}_0 \otimes \overline{\mathbf{M}}_F[i]) \| (c \cdot \overline{\mathbf{M}}_F[i] \otimes \mathbf{r}_1)\big) \in \mathbb{Z}_{\Delta}^{4(m+1)}$.
  4. Sample $\widetilde{\mathbf{r}}_{b,\mathsf{col}} \leftarrow \mathbb{Z}_{\Delta}^{Q}$ for $b \in \{0,1\}$ and compute the following:
$$\begin{aligned} \mathsf{ct}_{0,\mathsf{col}} &= \mathbf{D}_0^{\perp} \cdot \widetilde{\mathbf{r}}_{0,\mathsf{col}} + \mathbf{D}_0^{-1} \cdot (c \cdot \overline{\mathbf{M}}_F[\mathsf{col}] + \mathbf{V} \cdot \mathbf{r}_0) \bmod \Delta \\ \mathsf{ct}_{1,\mathsf{col}} &= \mathbf{D}_1^{\perp} \cdot \widetilde{\mathbf{r}}_{1,\mathsf{col}} + \mathbf{D}_1^{-1} \cdot (\overline{\mathbf{M}}_F[\mathsf{col}] + \mathbf{W} \cdot \mathbf{r}_1) \bmod \Delta \\ \mathsf{ct}_{\mathsf{IPFE},\mathsf{col}} &\leftarrow \mathsf{Enc}_{\mathsf{IPFE}}(\mathsf{msk}_{\mathsf{IPFE}}, \mathbf{h}_{\mathsf{col}}), \end{aligned}$$
     where $\mathbf{h}_{\mathsf{col}} = \big(\mathbf{r}_0 \otimes \overline{\mathbf{M}}_F[\mathsf{col}] \| (c \cdot \overline{\mathbf{M}}_F[\mathsf{col}] + \mathbf{V} \cdot \mathbf{r}_0) \otimes \mathbf{r}_1\big) \in \mathbb{Z}_{\Delta}^{4(m+1)}$.
  5. Set a matrix $\mathbf{M}_{\mathsf{Enc} \circ F}$ of which i-th column vector is the concatenation of three ciphertexts, i.e. for $1 \leq i \leq \mathsf{col}$, define
$$\mathbf{M}_{\mathsf{Enc} \circ F}[i] \leftarrow (\mathsf{ct}_{0,i} \| \mathsf{ct}_{1,i} \| \mathsf{ct}_{\mathsf{IPFE},i}) \in \mathbb{Z}_{\Delta}^{6m+3Q+8}.$$
  6. Decompose $\mathbf{M}_{\mathsf{Enc} \circ F}$ over columns to get $\mathbf{M}_{\mathsf{Enc} \circ F}^{\mathsf{decomp}} \in \mathbb{Z}_{\Delta}^{L(6m+3Q+8) \times \mathsf{col}}$
  7. Return $\mathbf{M}_{\mathsf{Enc} \circ F}^{\mathsf{decomp}}$.

Fig. 7. Composition algorithm of $\mathsf{Enc}_{\mathsf{QFE}}$ and a matrix $\mathbf{M}_F$, a matrix representation of a function F. Recall that $\overline{\mathbf{M}}_F[i]$ is ith column of the matrix $\overline{\mathbf{M}}_F$.

- $\mathsf{cKeyGen}_{\mathsf{pkIPFE}}(\mathsf{msk}_{\mathsf{pkIPFE}}, \mathsf{pk}_{\mathsf{pkIPFE}}, \mathsf{msk}_{\mathsf{QFE}}, \mathsf{pp}_{\mathsf{QFE}}, f : \mathbb{Z}^{\ell} \to \mathbb{Z}^{m}, L)$
  1. For given a linear function f, Let $\mathbf{M}_f \in \mathbb{Z}^{m \times \ell}$ be a matrix representation of f such that $f(\mathbf{x}) = \mathbf{M}_f \cdot \mathbf{x}$.
  2. Compute $\mathbf{M}^{\mathsf{decomp}}_{\mathsf{Enc}\circ f} \leftarrow \mathsf{cEnc}(\mathsf{msk}_{\mathsf{QFE}}, \mathsf{pp}_{\mathsf{QFE}}, \mathbf{M}_f, L)$
  3. For every $1 \le \mathsf{idx} \le L(6m + 3Q + 8)$, compute
  $$\mathsf{fk}_{\mathsf{Enc}\circ F}[\mathsf{idx}] \leftarrow \mathsf{KeyGen}_{\mathsf{pkIPFE}}(\mathsf{msk}_{\mathsf{pkIPFE}}, \mathsf{pk}_{\mathsf{pkIPFE}}, (\mathbf{M}^{\mathsf{decomp}}_{\mathsf{Enc}\circ f})^{T}[\mathsf{idx}]),$$
  where $\mathbf{M}^{T}_{\mathsf{Enc}\circ f}[\mathsf{idx}]$ is the idx-th column vector of $\mathbf{M}^{T}_{\mathsf{Enc}\circ f}$,
  4. Return a set of functional keys $\mathsf{fk}_{\mathsf{Enc}\circ f} = \{\mathsf{fk}_{\mathsf{Enc}\circ f}[\mathsf{idx}]\}_{\mathsf{idx}=1}^{L(6m+3Q+8)}$.
- $\mathsf{cDec}_{\mathsf{pkIPFE}}(\mathsf{pk}_{\mathsf{pkIPFE}}, \mathsf{fk}_{\mathsf{Enc}\circ f}, \mathsf{ct}, L)$
  1. Parse $\mathsf{fk}_{\mathsf{Enc}\circ f} = \{\mathsf{fk}_{\mathsf{Enc}\circ f}[\mathsf{idx}]\}_{\mathsf{idx}}$. Compute $\mathbf{t} \in \mathbb{Z}^{L(6m+3Q+8)}$ where idx-th entry of $\mathbf{t}$ is
  $$\mathbf{t}_{\mathsf{idx}} \leftarrow \mathsf{Dec}_{\mathsf{pkIPFE}}(\mathsf{pk}_{\mathsf{pkIPFE}}, \mathsf{fk}_{\mathsf{Enc}\circ f}[\mathsf{idx}], \mathsf{ct}).$$
  2. Return a vector $\mathsf{Power}_L(\mathbf{t})$.
- $\mathsf{cKeyGen}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE}}, \mathsf{pp}_{\mathsf{QFE}}, \mathsf{msk}_{\mathsf{QFE}'}, \mathsf{pp}_{\mathsf{QFE}'}, F : \mathbb{Z}^{\ell} \to \mathbb{Z}^{m}, L)$
  1. For given quadratic function F, let $\mathbf{M}_F \in \mathbb{Z}^{m \times (\ell+1)^2}$ be a matrix representation of F s.t. $F(\mathbf{x}) = \mathbf{M}_F \cdot (\overline{\mathbf{x}} \otimes \overline{\mathbf{x}})$, where $\overline{\mathbf{x}} = (\mathbf{x} \| 1)$.
  2. Compute $\mathbf{M}^{\mathsf{decomp}}_{\mathsf{Enc}\circ F} \leftarrow \mathsf{cEnc}(\mathsf{msk}_{\mathsf{QFE}'}, \mathsf{pp}_{\mathsf{QFE}'}, \mathbf{M}_F, L)$
  3. For every $1 \le \mathsf{idx} \le L(6m + 3Q + 8)$, compute
  $$\mathsf{fk}_{\mathsf{Enc}\circ F}[\mathsf{idx}] \leftarrow \mathsf{KeyGen}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE}}, \mathsf{pp}_{\mathsf{QFE}}, (\mathbf{M}^{\mathsf{decomp}}_{\mathsf{Enc}\circ F})^{T}[\mathsf{idx}]),$$
  where $\mathbf{M}^{T}_{\mathsf{Enc}\circ F}[\mathsf{idx}]$ is the idx-th column vector of $\mathbf{M}^{T}_{\mathsf{Enc}\circ F}$.
  4. Return a set of functional keys $\mathsf{fk}_{\mathsf{Enc}\circ F} = \{\mathsf{fk}_{\mathsf{Enc}\circ F}[\mathsf{idx}]\}_{\mathsf{idx}=1}^{L(6m+3Q+8)}$.
- $\mathsf{cDec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE}}, \mathsf{fk}_{\mathsf{Enc}\circ F}, \mathsf{ct}, L)$
  1. Parse $\mathsf{fk}_{\mathsf{Enc}\circ F} = \{\mathsf{fk}_{\mathsf{Enc}\circ F[\mathsf{idx}]}\}_{\mathsf{idx}}$. Compute $\mathbf{t} \in \mathbb{Z}^{L(6m+3Q+8)}$ where idx-th entry of $\mathbf{t}$ is
  $$\mathbf{t}_{\mathsf{idx}} \leftarrow \mathsf{Dec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE}}, \mathsf{fk}_{\mathsf{Enc}\circ F}[\mathsf{idx}], \mathsf{ct}).$$
  2. Return a vector $\mathsf{Power}_L(\mathbf{t})$.

Fig. 8. Composite evaluation.

final result of the protocol is the composition of quadratic polynomials $\bigcirc_{i=1}^{\mathcal{E}} f_i(\mathbf{x})$, where each $f_i : \mathbb{Z}^{\ell_i} \to \mathbb{Z}^{\ell_{i+1}}$ is a vector quadratic polynomial. Throughout this paper, we set $\ell_i = \ell_{i+1}$ for all indices for simplicity. To ensure the correctness and the security of $\mathcal{P}$, specific requirements and conditions must be met by composable-QFE. Subsequent sections will provide a detailed analysis of these aspects, with a view to evaluating the reliability and security of the protocol.

5.1 Building block: integrating linear functions for security

Our main idea is to express higher-order polynomial operations required in ML as compositions of quadratic polynomials as $F = \bigcirc_{i=1}^{\mathcal{E}} f_i$, and to perform each quadratic polynomial using composable-QFE. To ensure security, instead of directly generating functional keys of f_i, we compute functional keys for $\mathsf{Enc}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE}}, \mathsf{pp}_{\mathsf{QFE}}, \cdot) \circ f_i$. Consequently, the decryption algorithm $\mathsf{Dec}_{\mathsf{QFE}}$ using such a functional key outputs the encrypted ciphertext of intermediate evaluations. However, this approach renders the encryption deterministic, thereby compromising security. To reintroduce randomness for security, we compose linear functions during key generation. Specifically, for a function $f_i : \mathbb{Z}^{\ell} \to \mathbb{Z}^{\ell}$, we define a function $G_i : \mathbb{Z}_N^{\ell+k} \to \mathbb{Z}_N^{\ell+k}$ for $i \in [1, \mathcal{E} - 1]$ by $G_i = h_i \circ f_i \circ h'_{i-1}$. Here, h_i and h'_i are linear functions that serve to calculate the composition order and substantiate the security of our protocol.

Given positive integers ℓ, N, k and $B_{\mathcal{H}}$, we sample a matrix $\mathbf{H} \in \mathbb{Z}^{(\ell+k)\times\ell}$ and its left-inverse $\mathbf{H}' \in \mathbb{Z}^{\ell\times(\ell+k)}$, and then define linear functions $h(\mathbf{x}) = \mathbf{H} \cdot \mathbf{x}$ and $h'(\mathbf{x}) = \mathbf{H}' \cdot \mathbf{x}$. The explicit algorithm is described in Alg. 1. Furthermore, in order to guarantee the security of the protocol, we consider a matrix $\mathbf{\Gamma} \in \{0,1\}^{2\ell \times \ell}$ and its left-inverse $\mathbf{\Gamma}^{-1} \in \mathbb{Z}_N^{\ell \times 2\ell}$ to randomize the messages in the protocol. We denote that γ and γ^{-1} are linear functions that correspond to $\mathbf{\Gamma}$ and $\mathbf{\Gamma}^{-1}$, respectively.

Algorithm 1 Algorithm for generating linear functions h and h' for randomness

```
function genLin($\ell, N, k, B_{\mathcal{H}}$)
    Sample a matrix $\mathbf{U} \leftarrow \{-1,0,1\}^{(\ell+k)\times k}$ such that $\mathbf{U}[i]^T \cdot \mathbf{U}[j] = 0$ for all $i \neq j$
    Compute a kernel matrix $\mathbf{V} \in \mathbb{Z}^{\ell \times (\ell+k)}$ such that $\mathbf{V} \cdot \mathbf{U} = \mathbf{O}$
    $\mathbf{H} \leftarrow [-B_{\mathcal{H}}, B_{\mathcal{H}}]^{(\ell+k)\times \ell}$ until $\mathbf{V} \cdot \mathbf{H}$ is left-invertible over $\mathbb{Z}_N$
    Define $\mathbf{T} \in \mathbb{Z}_N^{\ell \times \ell}$ as the left inverse of $\mathbf{V} \cdot \mathbf{H}$ over $\mathbb{Z}_N$
    Define $h(\mathbf{x}) = \mathbf{H} \cdot \mathbf{x}, h'(\mathbf{x}) = \mathbf{H}' \cdot \mathbf{x}$ for $\mathbf{H}' = \mathbf{T} \cdot \mathbf{V} \bmod N$
    return $h, h'$
end function
```

For the sake of security, the parameter for k requires a condition to ensure the sufficient number of possible $\mathbf{H}$'s for sampling.

Lemma 1. *Given $(h, h') \leftarrow$ genLin$(\ell, N, k, B_{\mathcal{H}})$, there exist at least $(2 \cdot B_{\mathcal{H}})^{\ell k}$-matrices $\overline{\mathbf{H}} \in [-B_{\mathcal{H}}, B_{\mathcal{H}}]^{(2\ell+k)\times \ell}$ such that $\mathbf{H}' \cdot \overline{\mathbf{H}} = \mathbf{I}_\ell$.*

Proof. Given the definitions of matrices $\mathbf{U}$ and $\mathbf{V}$, it holds that

$$\mathbf{H}' \cdot (\mathbf{H} + \mathbf{U} \cdot \mathbf{B}) = \mathbf{I}_\ell \bmod \mathbb{Z}_N$$

for any matrix $\mathbf{B} \in \mathbb{Z}^{k \times \ell}$. This requires us to enumerate the matrices $\mathbf{B}$ such that $(\mathbf{H} + \mathbf{U} \cdot \mathbf{B}) \in [-B_{\mathcal{H}}, B_{\mathcal{H}}]^{(\ell+k)\times \ell}$.

Define $\mathbf{B}[i]$ and $\mathbf{H}[i]$ as the i-th column vector of $\mathbf{B}$ and $\mathbf{H}$, respectively. The task then is to determine the cardinality of the set:

$$\mathcal{S}_i := \{\mathbf{B}[i] \mid \mathbf{H}[i] + \mathbf{U} \cdot \mathbf{B}[i] \in [-B_{\mathcal{H}}, B_{\mathcal{H}}]\}.$$

The total number of suitable matrices $\mathbf{B}$ is given by the product $\prod_{i=1}^{\ell} |\mathcal{S}_i|$.

According to the Alg. 1, the orthogonality of $\mathbf{U}$'s column vectors implies that the entries of $\mathbf{B}[i]$ do not interfere with each other's magnitudes. This allows for the assessment of each entry's potential size in $\mathbf{B}[i]$ and the computation of the set $\mathcal{S}_i$'s size.

Only the possible coefficients for the first entry need to be counted for each $\mathbf{B}[i]$ due to symmetry. Therefore, the problem simplifies to counting the number of c_1 such that each entry size of $\mathbf{H}[i] + \mathbf{U}[1] \cdot c_1$ remains below $B_{\mathcal{H}}$. Initially, counting occurrences where $\mathbf{U}[1] \cdot c_1$ fits within the set $[-B_{\mathcal{H}}, B_{\mathcal{H}}]^{\ell+k}$ indicates $c_1 \in [-B_{\mathcal{H}}, B_{\mathcal{H}}]$, yielding exactly $2 \cdot B_{\mathcal{H}} + 1$ possibilities. The focus is on shifted instances from these cases. Decomposing the $\mathbf{H}[i]$ into $c_1' \cdot \mathbf{H}[i] + \mathbf{H}[i]^{\perp}$, where $\mathbf{H}[i]^{\perp}$ is an orthogonal vector to $\mathbf{H}[i]$, it is required to $c_1 + c_1' \in [-B_{\mathcal{H}}, B_{\mathcal{H}}]$ so that $\mathbf{H}[i] + c_1 \cdot \mathbf{U}[1] \in [-B_{\mathcal{H}}, B_{\mathcal{H}}]^{\ell+k}$. It guarantees at least $2 \cdot B_{\mathcal{H}}$ cases for c_1. As a result, for all $\mathbf{U}[i]$, $(2 \cdot B_{\mathcal{H}})^k$ cases are possible and $|\mathcal{S}_i| \geq (2 \cdot B_{\mathcal{H}})^k$.

Combining all indices, we have $\prod_{i=1}^{\ell} |\mathcal{S}_i| \geq (2 \cdot B_{\mathcal{H}})^{\ell k}$, which completes the proof. □

From Lemma 1, we set the parameters k and $B_{\mathcal{H}}$ to satisfy the following condition.

$$(2 \cdot B_{\mathcal{H}})^{\ell k} \geq 2^{\lambda} \tag{2}$$

This ensures a sufficiently large number of possible $\mathbf{H}$ configurations for security.

5.2 Protocol description

This section provides a secure protocol $\mathcal{P}$ for function compositions. The participants of our protocol consist of three types of entities: clients $\{C_j\}_{j \in [0,J]}$, a key distributor KD, and an evaluator E. As discussed, instead of generating functional keys for f_i directly, we compose linear functions to add randomness. In the protocol, the key distributor generates a pair of linear function

(γ, γ^{-1}) as described in Section 5.1 and $\mathcal{E}$ distinct pairs of linear functions $\{(h_i, h'_i)\}$ using Alg. 1 for $i \in [0, \mathcal{E}-1]$, then define G_i by $h_i \circ f_i \circ h'_{i-1}$ for each $i \in [1, \mathcal{E}-1]$ and $G_{\mathcal{E}}$ by $f_{\mathcal{E}} \circ h'_{\mathcal{E}-1}$. Then, we observe that for any $i \in [1, \mathcal{E}-1]$ and a vector $\mathbf{x}$,

$$\begin{aligned}
&G_{i+1}(h_i(\mathbf{x})) = (h_{i+1} \circ f_{i+1} \circ h'_i)(h_i(\mathbf{x})) = h_{i+1}(f_{i+1}(\mathbf{x})) \bmod N, \\
&\left(\bigcirc_{t=1}^{i} G_t\right)(h_0(\mathbf{x})) = h_i(\bigcirc_{t=1}^{i} f_t(\mathbf{x})) \bmod N, \\
&\left(\bigcirc_{t=1}^{\mathcal{E}} G_t\right)(h_0(\mathbf{x})) = \bigcirc_{t=1}^{\mathcal{E}} f_t(\mathbf{x}) \bmod N.
\end{aligned}$$

The whole progress of our protocol is described in Fig. 9. In the protocol, there are two additional subscripts, denoted by l and j. The index l indicates several machine learning model F_l that is compositions of quadratic polynomials $f_{l,i}$, and the other index j indicates a client C_j.

Correctness (of secure protocol in Fig. 9). By definition of $\mathsf{fk}_{\mathsf{E}_1 \circ G_{l,0,j}}$, it holds that

$$\begin{aligned}
M_{l,0,j} &= \mathsf{cDec}_{\mathsf{pkIPFE}}(\mathsf{pk}_{\mathsf{pkIPFE}}, \mathsf{fk}_{\mathsf{E}_1 \circ G_{l,0,j}}, \mathsf{ct}_{l,j}, L) \\
&= (\mathsf{E}_1 \circ h_{l,0,j} \circ \gamma_{l,j}^{-1})(\gamma_{l,j}(\mathbf{x}_j)) = \mathsf{E}_1(h_{l,0,j}(\mathbf{x}_j)).
\end{aligned}$$

Furthermore, we claim that $M_{l,i,j} = \mathsf{E}_{i+1}(h_{l,i,j} \bigcirc_{t=1}^{i} f_{l,t}(\mathbf{x}_j))$ for each $i \in [1, \mathcal{E}-1]$. Using the fact that $h'_{l,i,j} \circ h_{l,i,j}$ is an identity function for each i, it can be checked inductively as follows:

$$\begin{aligned}
M_{l,i,j} &= \mathsf{cDec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE},i}, \mathsf{fk}_{\mathsf{E}_{i+1} \circ G_{l,i,j}}, M_{l,i-1,j}, L) \\
&= (\mathsf{E}_{i+1} \circ h_{l,i,j} \circ f_{l,i} \circ h'_{l,i-1,j}) \left((h_{l,i-1,j} \bigcirc_{t=1}^{i-1} f_{l,t}(\mathbf{x}_j)\right) \\
&= (\mathsf{E}_{i+1} \circ h_{l,i,j})(\bigcirc_{t=1}^{i} f_{l,t}(\mathbf{x}_j)).
\end{aligned}$$

Therefore, it satisfies

$$M_{l,\mathcal{E}-1,j} = (\mathsf{E}_{\mathcal{E}} \circ h_{l,\mathcal{E}-1,j})(\bigcirc_{t=1}^{\mathcal{E}-1} f_{l,t}(\mathbf{x}_j)).$$

Then the final result is:

$$\begin{aligned}
M_{l,\mathcal{E},j} &= \mathsf{Dec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE},\mathcal{E}}, \mathsf{fk}_{G_{l,\mathcal{E},j}}, M_{l,\mathcal{E}-1,j}) \\
&= (f_{l,\mathcal{E}} \circ h'_{l,\mathcal{E}-1,j}) \left((h_{l,\mathcal{E}-1,j} \bigcirc_{t=1}^{\mathcal{E}-1} f_{l,t}(\mathbf{x}_j)\right) \\
&= \bigcirc_{i=1}^{\mathcal{E}} f_{l,i}(\mathbf{x}_j) = F_l(\mathbf{x}_j).
\end{aligned}$$

Time complexity. Let $\{f_{l,i}\}$ be a family of quadratic functions and $\{h_{l,i,j}\}$ be a family of invertible linear functions in Section 5.1. It is clear that the underlying algorithms including $\mathsf{cDec}, \mathsf{cKeygen}, \mathsf{Enc}$ terminates in polynomial time in input parameters. Then Fig. 9 terminates in polynomial time in input parameters as well.

6 Security Proof

In this section, we show that the protocol $\mathcal{P}$ described in Fig. 9 does not reveal information about an unknown message under a malicious model. We recall the malicious security of the protocol based on Definition 14. Let KD, E, and $\{C_j\}_{j \in [0,J]}$ denote the (trusted) key distributor, evaluator, and clients, respectively, who are participants in the protocol. We here note that C_0 is only one honest client, and the evaluator E has $\zeta + 1$ machine learning models. Each model consists of a series of quadratic polynomials. Let $F_0 = \bigcirc_{i=1}^{\mathcal{E}} f_{0,i}$ be a machine learning model requested by C_0. The other models, denoted as $\{F_l\}_{l \in [1,\zeta]} := \{\bigcirc_{i=1}^{\mathcal{E}} f_{l,i}\}_{l \in [1,\zeta]}$, are not requested by C_0.

The objective is to demonstrate that an adversary $\mathcal{A}$ cannot distinguish between a protocol $\mathcal{P}$ and an ideal protocol from its view. Then, it ensures the privacy of any intermediate values of the honest client C_0 during machine learning model computations because $\mathcal{A}$ cannot distinguish where the intermediate value comes from. This indistinguishability holds even if $\mathcal{A}$ interacts with other clients $\{C_j\}_{j \in [1,J]}$ and an evaluator E. To elucidate the adversary's view, we describe both real-world and ideal-world protocols.

Real-World. $\mathcal{A}$ interacting with the corrupted clients and an evaluator can be described as follows:

Participants: Clients ($\{C_j\}_{j\in[0,J]}$), key distributor (KD), and evaluator (E)
Protocol:

Protocol Setup by KD

1. KD samples keys of pkIPFE:

$$\{\mathsf{msk}_{\mathsf{pkIPFE}}, \mathsf{pk}_{\mathsf{pkIPFE}}\} \leftarrow \mathsf{Setup}_{\mathsf{pkIPFE}}(\lambda, 2\ell, B_X, B_F)$$

2. KD samples keys of composable-QFE $\mathcal{E}$ times for $i \in [1, \mathcal{E}]$:

$$\{\mathsf{msk}_{\mathsf{QFE},i}, \mathsf{pp}_{\mathsf{QFE},i}\} \leftarrow \mathsf{Setup}_{\mathsf{QFE}}(\lambda, \ell, 2\ell + 1)$$

3. KD sets the smallest integer k as Eq. (2) and samples pairs of linear functions $(h_{l,i,j}, h'_{l,i,j})$ as Alg. 1 for each $l \in [0, \zeta], i \in [0, \mathcal{E} - 1]$, and $j \in [0, J]$.

$$(h_{l,i,j}, h'_{l,i,j}) \leftarrow \textsc{genLin}(\ell, N, k, B_{\mathcal{H}})$$

4. For every l, j, KD samples $\mathbf{\Gamma}_{l,j} \leftarrow \{0, 1\}^{2\ell\times\ell}$ until there exists its left-inverse $\mathbf{\Gamma}^{-1}_{l,j}$. Define $\gamma_{l,j}(\mathbf{x}) = \mathbf{\Gamma}_{l,j} \cdot \mathbf{x}$ and $\gamma^{-1}_{l,j}(\mathbf{x}) = \mathbf{\Gamma}^{-1}_{l,j} \cdot \mathbf{x}$ for any $\mathbf{x}$.
5. Set $\{\mathsf{msk}_j\}_{j\in[0,J]}$ and pk as follows:

$$\mathsf{msk}_j = \{\mathsf{msk}_{\mathsf{pkIPFE}}, \{\mathsf{msk}_{\mathsf{QFE},i}\}_{i\in[1,\mathcal{E}]}, \{h_{l,i,j}, h'_{l,i,j}\}_{l\in[0,\zeta],i\in[0,\mathcal{E}-1]}\}$$
$$\mathsf{pk} = \{\mathsf{pk}_{\mathsf{pkIPFE}}, \{\mathsf{pp}_{\mathsf{QFE},i}\}_{i\in[1,\mathcal{E}]}\}.$$

Encryption between C_j and KD

1. KD sends a tuple $(\mathsf{pk}_{\mathsf{pkIPFE}}, \{h_{l,0,j} \circ \gamma^{-1}_{l,j}\}_{l\in[0,\zeta]}, \gamma_{l,j})$ to C_j.
2. C_j encrypts $\gamma_{l,j}(\mathbf{x}_j)$ to generate $\mathsf{ct}_{l,j}$:

$$\mathsf{ct}_{l,j} \leftarrow \mathsf{Enc}_{\mathsf{pkIPFE}}(\mathsf{pk}_{\mathsf{pkIPFE}}, \gamma_{l,j}(\mathbf{x}_j))$$

3. C_j sends $\mathsf{ct}_{l,j}$ to the E.

Functional key generation between KD and E

1. E sends a model $F_l = \bigcirc_{i=1}^{\mathcal{E}} f_{l,i}$ for a certain l to KD.
2. With an encryption map $\mathsf{E}_i : \mathbf{x} \mapsto \mathsf{Enc}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE},i}, \mathsf{pp}_{\mathsf{QFE},i}, \mathbf{x})$, KD sends a set of functional keys $\mathsf{F}_{l,j} := \{\mathsf{fk}_{\mathsf{E}_{i+1}\circ G_{l,i,j}}\}_{i\in[0,\mathcal{E}-1],j\in[0,J]} \cup \{\mathsf{fk}_{G_{l,\mathcal{E},j}}\}$ to E. For every $i \in [1, \mathcal{E} - 1]$, $j \in [0, J]$, and a decomposition parameter L, define functional keys as follows:
   - $\mathsf{fk}_{\mathsf{E}_1\circ G_{l,0,j}} \leftarrow \mathsf{cKeyGen}_{\mathsf{pkIPFE}}(\mathsf{msk}_{\mathsf{pkIPFE}}, \mathsf{pk}_{\mathsf{pkIPFE}}, \mathsf{msk}_{\mathsf{QFE},1}, \mathsf{pp}_{\mathsf{QFE},1}, \underbrace{h_{l,0,j} \circ \gamma^{-1}_{l,j}}_{:=G_{l,0,j}}, L)$
   - $\mathsf{fk}_{\mathsf{E}_{i+1}\circ G_{l,i,j}} \leftarrow \mathsf{cKeyGen}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE},i}, \mathsf{pp}_{\mathsf{QFE},i}, \mathsf{msk}_{\mathsf{QFE},i+1}, \mathsf{pp}_{\mathsf{QFE},i+1}, \underbrace{h_{l,i,j} \circ f_{l,i} \circ h'_{l,i-1,j}}_{:=G_{l,i,j}}, L)$
   - $\mathsf{fk}_{G_{l,\mathcal{E},j}} \leftarrow \mathsf{KeyGen}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE},\mathcal{E}}, \mathsf{pp}_{\mathsf{QFE},\mathcal{E}}, \underbrace{f_{l,\mathcal{E}} \circ h'_{l,\mathcal{E}-1,j}}_{:=G_{l,\mathcal{E},j}}, L)$

Model Evaluation by E

1. E inductively computes the following:
   - $M_{l,0,j} \leftarrow \mathsf{cDec}_{\mathsf{pkIPFE}}(\mathsf{pk}_{\mathsf{pkIPFE}}, \mathsf{fk}_{\mathsf{E}_1\circ G_{l,0,j}}, \mathsf{ct}_{l,j}, L)$
   - $M_{l,i,j} \leftarrow \mathsf{cDec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE},i}, \mathsf{fk}_{\mathsf{E}_i\circ G_{l,i,j}}, M_{l,i-1,j}, L)$ for $i \in [1, \mathcal{E} - 1]$
   - $M_{l,\mathcal{E},j} \leftarrow \mathsf{Dec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE},i}, \mathsf{fk}_{G_{l,\mathcal{E},j}}, M_{l,\mathcal{E}-1,j})$
2. E obtains $M_{l,\mathcal{E},j}$.

Fig. 9. Secure computation protocol $\mathcal{P}$ for function compositions $\bigcirc_{i=1}^{\mathcal{E}} f_{l,i}(\mathbf{x})$.

- Functional Keys: E queries several models $\{F_l\}_{l\in[0,\zeta]}$ corresponding to all clients to KD and transmits $\{\mathsf{F}_{l,j}\}_{l\in[0,\zeta],j\in[0,J]}$ with $\mathsf{F}_{l,j} = \{\mathsf{fk}_{\mathsf{E}_{i+1}\circ G_{l,i,j}}\}_{i\in[0,\mathcal{E}-1]} \cup \mathsf{fk}_{G_{l,\mathcal{E},j}}$ to $\mathcal{A}$, where $G_{l,i,j} = h_{l,i,j} \circ f_{l,i} \circ h'_{l,i-1,j}$ as in the Fig. 9.
- Target ciphertext: The honest client C_0 selects a message $\mathbf{x}_0$ and makes a ciphertext $\mathsf{ct}_0 := \mathsf{ct}_{0,0} = \mathsf{Enc}_{\mathsf{pkIPFE}}(\mathsf{pk}_{\mathsf{pkIPFE}}, \gamma_{0,0}(\mathbf{x}_0))$, given a linear function $\gamma_{0,0}$. Then, C_0 sends it to E.
- Additional Ciphertexts: For $l \in [0,\zeta]$ and $j \in [1,J]$, each client C_j also generates a ciphertext $\mathsf{ct}_{l,j} := \mathsf{Enc}_{\mathsf{pkIPFE}}(\mathsf{pk}_{\mathsf{pkIPFE}}, \gamma_{l,j}(\mathbf{x}_j))$ for some message $\mathbf{x}_j$ and a linear function $\gamma_{l,j}$, and sends $\mathsf{ct}_{l,j}$ to E and $(\mathsf{ct}_{l,j}, \mathbf{x}_j)$ to $\mathcal{A}$.
- Evaluation: For every $i \in [1,\mathcal{E}-1], l \in [0,\zeta]$ and $j \in [0,J]$, E computes intermediate values $(\mathsf{E}_i(h_{l,i-1,j} \bigcirc_{t=1}^{i-1} f_{l,t}(\mathbf{x}_j)))$ from ciphertexts and functional keys, and final results of the form $F_l(\mathbf{x}_j)$. E sends them to $\mathcal{A}$.

Based on the description of $\mathcal{P}$, we define $\mathsf{REAL}_{\mathcal{P}}(\mathbf{x}_0, \{\mathbf{x}_j\}_{j\in[1,J]}, \{F_l\}_{l\in[0,\zeta]}, \lambda, \zeta)$ to be the view of $\mathcal{A}$ in the real world.

$$\mathsf{REAL}_{\mathcal{P},\mathcal{A}} = \Big\{ \{\mathbf{x}_j\}_{j\in[1,J]}, \{F_l\}_{l\in[0,\zeta]}, \{F_l(\mathbf{x}_j)\}_{l\in[0,\zeta],j\in[0,J]}, \{\mathsf{F}_{l,j}\}_{l\in[0,\zeta],j\in[0,J]}$$
$$\{\mathsf{ct}_{l,j}\}_{l\in[0,\zeta],j\in[0,J]}, \{\mathsf{E}_i(h_{i-1,j} \bigcirc_{t=1}^{i-1} f_{l,t}(\mathbf{x}_j))\}_{i\in[1,\mathcal{E}-1],l\in[0,\zeta],j\in[0,J]} \Big\}$$

Ideal-World. In an ideal-world, there exists a simulator $\mathcal{S}$ for a client C_0, which mimics the KeyGen and $\mathsf{Enc}_{\mathsf{pkIPFE}}$. An ideal-world interaction then coincides with the real-world except for terms related to the honest client C_0; the target ciphertext and functional key:

- Functional Keys for client C_0: E queries function $\{F_l\}_{l\in[0,\zeta]}$ corresponding to all clients to KD. Then KD transmits functional keys $\{\mathsf{F}^{\mathcal{S}}_{l,0}\}_{l\in[0,\zeta]}$ with $\mathsf{F}^{\mathcal{S}}_{l,0} = \{\mathsf{fk}_{\mathsf{E}_{i+1}\circ G^{\mathcal{S}}_{l,i,0}}\}_{i\in[0,\mathcal{E}-1]} \cup \mathsf{fk}_{G^{\mathcal{S}}_{l,\mathcal{E},j}}$ to $\mathcal{A}$.
- Target ciphertext: The honest client C_0 sends a ciphertext
$$\mathsf{ct}^*_0 := \mathsf{Enc}^{\mathcal{S}}_{\mathsf{pkIPFE}}(\mathsf{pk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \{h_{0,0,0} \circ {\gamma_{0,0}}^{-1}, h_{0,0,0}(\mathbf{x}_0)\})$$
to E.
- Additional ciphertexts: These ciphertexts are generated by $\mathsf{ct}_{l,j} := \mathsf{Enc}_{\mathsf{pkIPFE}}(\mathsf{pk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \gamma_{l,j}(\mathbf{x}_j))$.

Analogues to the real-world, we define $\mathsf{IDEAL}_{\mathcal{S}}(\mathbf{x}_0, \{\mathbf{x}_j\}_{j\in[1,J]}, \{F_l\}_{l\in[0,\zeta]}, \lambda, \zeta)$ to be the view of $\mathcal{A}$ in the ideal-world.

$$\mathsf{IDEAL}_{\mathcal{S}} = \Big\{ \{\mathbf{x}_j\}_{j\in[1,J]}, \{F_l\}_{l\in[0,\zeta]}, \{F_l(\mathbf{x}_j)\}_{l\in[0,\zeta],j\in[0,J]}, \{\mathsf{F}^{\mathcal{S}}_{l,j}\}_{l\in[0,\zeta],j\in[0,J]}$$
$$\mathsf{ct}^*_0, \{\mathsf{ct}_{l,j}\}_{l\in[0,\zeta],j\in[0,J]}, \{\mathsf{E}_i(h_{i-1,j} \bigcirc_{t=1}^{i-1} f_{0,t}(\mathbf{x}_j))\}_{i\in[1,\mathcal{E}-1],l\in[0,\zeta],j\in[0,J]} \Big\}$$

We then aim to show that the following two distributions are computationally indistinguishable:

$$\mathsf{REAL}_{\mathcal{P},\mathcal{A}}(\mathbf{x}_0, \{\mathbf{x}_j\}_{j\in[1,J]}, \{F_l\}_{l\in[0,\zeta]}, \lambda, \zeta) \overset{c}{\approx} \mathsf{IDEAL}_{\mathcal{S}}(\{F_l(\mathbf{x}_0)\}, \{\mathbf{x}_j\}_{j\in[1,J]}, \{F_l\}_{l\in[0,\zeta]}, \lambda, \zeta). \quad (3)$$

As a high-level idea for proof, a collection of linear functions $\{h_{l,i,j}\}$ (represented by $\{\mathbf{H}_{l,i,j}\}$ for each l, i and j) plays a significant role in ensuring security. This matrix allows composite operations only when each l, j is coincided. Hence, the matrix $\{\mathbf{H}_{l,i,0}\}$ are independent to other matrices. In other words, the diversity of $\{h_{l,i,0}\}$ compensates for the lack of randomness of an encryption function corresponding to E_i. Given that the evaluator lacks knowledge of $h_{l,i,0}$, $\mathsf{E}_i(h_{l,i-1,j}(\bigcirc_{t=1}^{i-1} f_{l,t}(\mathbf{x}_j)))$ for any $i \in [1,\mathcal{E}-1]$ seems to be an encryption of random value in the adversary's view. In addition, other clients' ciphertext is not helpful. In the following, we show that there are at least 2^λ-ensembles $(h_{l,i-1,0,k}, \mathbf{x}_{0,k})$ $\mathsf{E}_i(h_{l,i-1,0}(\bigcirc_{t=1}^{i-1} f_{l,t}(\mathbf{x}_0))) = \mathsf{E}_i(h_{l,i-1,0,k}(\mathbf{x}_{0,k}))$. Consequently, the adversary is unable to distinguish $f_{l,i-1}(\mathbf{x}_0)$ and $\mathbf{x}_{0,k}$, even if $\mathcal{A}$ is already familiar with the function F_l.

6.1 Security proof of the protocol

The primary purpose of this section is to prove the following theorem, which implies the security of a protocol $\mathcal{P}$ of Fig. 9.

Theorem 3. *Let λ be the security parameter and $(k, B_{\mathcal{H}})$ be integers such that $(2 \cdot B_{\mathcal{H}})^{\ell k} \geq 2^{\lambda}$. The protocol in Fig. 9 securely computes $F_0 = \bigcirc_{t=1}^{\mathcal{E}} f_{0,t}$ over $\mathbb{Z}^{\ell}$, when pkIPFE is adaptively simulation secure and composable-QFE is $(2\ell+1)$-ciphertext bounded semi-adaptively secure. More precisely, there exists a simulator $\mathcal{S}$ such that any adversary $\mathcal{A}$ cannot distinguish the Eq. (3) except for the following advantages:*

$$\mathsf{Adv}_{\mathsf{pkIPFE}} + \mathcal{E} \cdot \mathsf{Adv}_{\mathsf{cQFE}} + \frac{\mathcal{E}}{2^{\lambda}}$$

where $\mathsf{Adv}_{\mathsf{pkIPFE}}, \mathsf{Adv}_{\mathsf{cQFE}}$ *are the advantages of pkIPFE and composable-QFE, respectively.*

Proof of Theorem 3. In the security of the protocol $\mathcal{P}$ in Fig. 9, the adversary is provided with a distribution

$$\mathcal{D} = \mathsf{REAL}_{\mathcal{P},\mathcal{A}}(\mathbf{x}_0, \{\mathbf{x}_j\}_{j=1}^{J}, \{F_l\}_{l=0}^{\zeta}, \lambda).$$

We then consider $(\mathcal{E}+1)$-modified protocols denoting $\mathcal{P}_1$ and $\mathcal{P}_{2,i}$ for $i \in [1, \mathcal{E}]$ such that

$$\mathcal{P}_{2,\varepsilon} = \mathsf{IDEAL}_{\mathcal{S}}(\mathbf{x}_0, \{\mathbf{x}_j\}_{j=1}^{J}, \{F_l\}_{l=0}^{\zeta}, \lambda).$$

These modified protocols coincide with $\mathcal{P}$ for clients $\{C_j\}_{j=1}^{J}$ except for C_0. We therefore only describe the protocols for the client C_0. For ease of exposition, we denote that $\mathcal{P}_1$ (resp. $\mathcal{P}_{2,i}$) yields a distribution $\mathcal{D}_1$ (resp. $\mathcal{D}_{2,i}$).

The main objective is to demonstrate that the $(\mathcal{E}+1)$-protocols adhere to the following relation

$$\mathcal{P} \overset{\mathsf{Adv}_{\mathsf{pkIPFE}}}{\approx} \mathcal{P}_1 \overset{\mathsf{Adv}_{\mathsf{cQFE}}}{\approx} \mathcal{P}_{2,1} \overset{\mathsf{Adv}_{\mathsf{cQFE}}}{\approx} \dots \overset{\mathsf{Adv}_{\mathsf{cQFE}}}{\approx} \mathcal{P}_{2,\mathcal{E}},$$

where the notation $\overset{\mathsf{Adv}}{\approx}$ indicates that the distributions cannot be distinguished with the advantage Adv. It immediately says that the advantage for distinguishing between $\mathcal{D}$ and $\mathcal{D}_{2,\mathcal{E}}$ is less than

$$\mathsf{Adv}_{\mathsf{pkIPFE}} + \mathcal{E} \cdot \mathsf{Adv}_{\mathsf{cQFE}},$$

where $\mathsf{Adv}_{\mathsf{pkIPFE}}, \mathsf{Adv}_{\mathsf{cQFE}}$ are the advantages corresponding to pkIPFE and composable-QFE, respectively. As the last step, we show that the $\mathcal{D}_{2,\mathcal{E}}$ does not leak information about $\mathbf{x}_0$. By combining the two statements, we conclude the proof. In the following, we will prove that each statement is correct.

Protocol $\mathcal{P}_1$. First of all, we introduce a protocol $\mathcal{P}_1$. Intuitively, the protocol is identical to the original protocol $\mathcal{P}$ except for C_0, and the only difference between $\mathcal{P}$ and $\mathcal{P}_1$ is how to implement pkIPFE algorithm. We thus mainly introduce how protocol $\mathcal{P}_1$ is executed for an honest client C_0. We let

$$\mathsf{pkIPFE}^{\mathcal{S}} = \{\mathsf{Setup}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \mathsf{KeyGen}^{\mathcal{S}}_{\mathsf{pkIPFE},0}, \mathsf{KeyGen}^{\mathcal{S}}_{\mathsf{pkIPFE},1}, \mathsf{Enc}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \mathsf{Dec}_{\mathsf{pkIPFE}}\}$$

denote the simulator.

The protocol $\mathcal{P}_1$ is exactly the same as $\mathcal{P}$ except for the algorithm related to pkIPFE. The detailed description of $\mathcal{P}_1$ is given in Fig. 10.

Let $\mathcal{P}_0$ be the origin protocol $\mathcal{P}$, and $\mathcal{G}_i$ be a game in which the challenger interacts with the adversary of $\mathcal{P}_i$ with $i \in \{0,1\}$. At the start of the game, the challenger randomly chooses $b \leftarrow \{0,1\}$ and interacts with the adversary in $\mathcal{G}_b$. At the end of the game, $\mathcal{A}$ returns $b' \in \{0,1\}$. We define $\mathsf{Adv}_{01}(\mathcal{A})$ as $|\Pr[b' = b] - 1/2|$. We directly obtain Lemma 2 from the results in [1].

Lemma 2. *$\mathcal{P}$ and $\mathcal{P}_1$ are computationally indistinguishable under the assumption that* pkIPFE *is adaptively simulation secure. More precisely, it holds that* $\mathsf{Adv}_{01}(\mathcal{A}) \leq \mathsf{Adv}_{\mathsf{pkIPFE}}$.

Proof of Lemma 2. The only difference between $\mathcal{P}_0$ and $\mathcal{P}_1$ is the usage of pkIPFE. $\mathcal{P}_0$ exploits a real pkIPFE and $\mathcal{P}_1$ uses a simulator of pkIPFE. Thus, under the assumption that pkIPFE achieves simulation-based security proved by [1], this modification cannot affect $\mathcal{A}$'s view, which yields that $\mathsf{Adv}_{01}(\mathcal{A}) \leq \mathsf{Adv}_{\mathsf{pkIPFE}}$. □

Participants: Client (C_0), key distributor (KD), and evaluator (E)
Protocol:

Protocol Setup by KD

1. KD samples keys of pkIPFE:

$$\{\mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \mathsf{pk}^{\mathcal{S}}_{\mathsf{pkIPFE}}\} \leftarrow \mathsf{Setup}^{\mathcal{S}}_{\mathsf{pkIPFE}}(\lambda, 2\ell, B_X, B_F)$$

2. KD samples keys of composable-QFE $\mathcal{E}$ times for $i \in [1, \mathcal{E}]$:

$$\{\mathsf{msk}_{\mathsf{QFE},i}, \mathsf{pp}_{\mathsf{QFE},i}\} \leftarrow \mathsf{Setup}_{\mathsf{QFE}}(\lambda, \ell, 2\ell + 1)$$

3. KD sets the smallest integer k as Eq. (2) and samples pairs of linear functions $(h_{l,i,0}, h'_{l,i,0})$ as Alg. 1 for each $l \in [0, \zeta]$ and $i \in [0, \mathcal{E} - 1]$.

$$(h_{l,i,0}, h'_{l,i,0}) \leftarrow \textsc{genLin}(\ell, N, k, B_{\mathcal{H}})$$

4. KD samples $\mathbf{\Gamma}_{0,0} \leftarrow \{0,1\}^{2\ell \times \ell}$ until there exists its left-inverse $\mathbf{\Gamma}^{-1}_{0,0}$. Define $\gamma_{0,0}(\mathbf{x}) = \mathbf{\Gamma}_{0,0} \cdot \mathbf{x}$ and $\gamma^{-1}_{0,0}(\mathbf{x}) = \mathbf{\Gamma}^{-1}_{0,0} \cdot \mathbf{x}$.
5. Set msk_0 and pk as follows:

$$\mathsf{msk}_0 = \{\mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \{\mathsf{msk}_{\mathsf{QFE},i}\}_{i\in[1,\mathcal{E}]}, \{h_{l,i,0}, h'_{l,i,0}\}_{l\in[0,\zeta], i\in[0,\mathcal{E}-1]}\}$$
$$\mathsf{pk} = \{\mathsf{pk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \{\mathsf{pp}_{\mathsf{QFE},i}\}_{i\in[1,\mathcal{E}]}\}.$$

Encryption between C_0 and KD

1. KD sends a tuple $(\mathsf{pk}_{\mathsf{pkIPFE}}, h_{0,0,0} \circ \gamma^{-1}_{0,0}, \gamma_{0,0})$ to C_0.
2. C_0 encrypts $\gamma_{0,0}(\mathbf{x}_0)$ to generate ct^*_0:

$$\mathsf{ct}^*_0 \leftarrow \mathsf{Enc}^{\mathcal{S}}_{\mathsf{pkIPFE}}(\mathsf{pk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, h_{0,0,0}(\mathbf{x}_0))$$

3. C_0 sends ct^*_0 to the E.

Functional key generation between KD and E

1. E sends a model $F_0 = \bigcirc^{\mathcal{E}}_{i=1} f_{0,i}$ to KD.
2. With an encryption map $\mathsf{E}_i : \mathbf{x} \mapsto \mathsf{Enc}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE},i}, \mathsf{pp}_{\mathsf{QFE},i}, \mathbf{x})$, KD sends a set of functional keys $\mathsf{F}_{0,0} := \{\mathsf{fk}_{\mathsf{E}_{i+1} \circ G_{0,i,0}}\}_{i\in[0,\mathcal{E}]} \cup \mathsf{fk}_{G_{0,\mathcal{E},0}}$ to E. For every $i \in [1, \mathcal{E} - 1]$, and a decomposition parameter L, define functional keys as follows:
   - $\mathsf{fk}_{\mathsf{E}_1 \circ G_{0,0,0}} \leftarrow \mathsf{cKeyGen}_{\mathsf{pkIPFE}}(\mathsf{msk}_{\mathsf{pkIPFE}}, \mathsf{pk}_{\mathsf{pkIPFE}}, \mathsf{msk}_{\mathsf{QFE},1}, \mathsf{pp}_{\mathsf{QFE},1}, \underbrace{h_{0,0,0} \circ \gamma^{-1}_{0,0}}_{:=G_{0,0,0}}, L)$
   - $\mathsf{fk}_{\mathsf{E}_{i+1} \circ G_{0,i,0}} \leftarrow \mathsf{cKeyGen}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE},i}, \mathsf{pp}_{\mathsf{QFE},i}, \mathsf{msk}_{\mathsf{QFE},i+1}, \mathsf{pp}_{\mathsf{QFE},i+1}, \underbrace{h_{0,i,0} \circ f_{0,i} \circ h'_{0,i-1,0}}_{:=G_{0,i,0}}, L)$
   - $\mathsf{fk}_{G_{0,\mathcal{E},0}} \leftarrow \mathsf{KeyGen}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE},\mathcal{E}}, \mathsf{pp}_{\mathsf{QFE},\mathcal{E}}, \underbrace{f_{0,\mathcal{E}} \circ h'_{0,\mathcal{E}-1,0}}_{:=G_{0,\mathcal{E},0}}, L)$

Model Evaluation by E

1. E inductively computes the following:
   - $M_{0,0,0} \leftarrow \mathsf{cDec}_{\mathsf{pkIPFE}}(\mathsf{pk}_{\mathsf{pkIPFE}}, \mathsf{fk}_{\mathsf{E}_1 \circ G_{0,0,0}}, \mathsf{ct}^*_0, L)$
   - $M_{0,i,0} \leftarrow \mathsf{cDec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE},i}, \mathsf{fk}_{\mathsf{E}_i \circ G_{0,i,0}}, M_{0,i-1,0})$ for $i \in [1, \mathcal{E} - 1]$
   - $M_{0,\mathcal{E},0} \leftarrow \mathsf{Dec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE},i}, \mathsf{fk}_{G_{0,\mathcal{E},0}}, M_{0,\mathcal{E}-1,0})$
2. E obtains $M_{0,\mathcal{E},0}$.

Fig. 10. Protocol $\mathcal{P}_1$ for a client C_0.

Protocol $\mathcal{P}_{2,1}$. Let $\mathcal{P}_{2,1}$ be a protocol identical to $\mathcal{P}_1$ except for generating a functional key $\mathsf{fk}_{G_{0,0,0}}$.To this end, we define the function $\mathsf{cKeyGen}^{\mathcal{S}}_{\mathsf{pkIPFE},0}$, which mirrors the $\mathsf{cKeyGen}$ function except in its execution of KeyGen. Instead of the standard KeyGen component, it executes $\mathsf{KeyGen}^{\mathcal{S}}_0$. The new functional key $\mathsf{fk}_{\mathsf{E}_1 \circ G^{\mathcal{S}}_{0,0,0}}$ in $\mathcal{P}_{2,1}$ is then generated by

$$\mathsf{fk}_{\mathsf{E}_1 \circ G^{\mathcal{S}}_{0,0,0}} \leftarrow \mathsf{cKeyGen}^{\mathcal{S}}_{\mathsf{pkIPFE},0}(\mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \mathsf{pk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \mathsf{msk}_{\mathsf{QFE},1}, \mathsf{pp}_{\mathsf{QFE},1}, G^{\mathcal{S}}_{0,0,0}, L)$$

where $G^{\mathcal{S}}_{0,0,0} = \overline{h}_{0,0,0} \circ \gamma^{-1}_{0,0}$ is a linear function that has a representation matrix $\overline{\mathbf{H}}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0}$ such that

$$\mathbf{H}'_{0,0,0} \cdot \overline{\mathbf{H}}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0} = \mathbf{H}'_{0,0,0} \cdot \mathbf{H}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0} \bmod \Delta. \tag{4}$$

The matrix $\overline{\mathbf{H}}_{0,0,0}$ satisfying Eq. (4) always exists from the Lemma 1. Then, the following lemma holds.

Lemma 3. *$\mathcal{P}_1$ and $\mathcal{P}_{2,1}$ are computationally indistinguishable under the assumption that E_1 is $(2\ell + 1)$ ciphertext-bounded semi-adaptively secure. The advantage of distinguishing between $\mathcal{D}_1$ and $\mathcal{D}_{2,1}$ is smaller than $\mathsf{Adv}_{\mathsf{cQFE}}$.*

Furthermore, no one computationally obtains any information about $\mathbf{x}_0$ from $\mathsf{E}_1(h_{0,0,0}(\mathbf{x}_0))$.

Proof of Lemma 3. We first note that E_1 is also a linear function. That is, there exists a matrix $\mathbf{C}_{2,1}$ such that $\mathsf{E}_1(\mathbf{x}) = \mathbf{C}_{2,1} \cdot \overline{\mathbf{x}}$ with $\overline{\mathbf{x}} = (\mathbf{x} \| 1)$. It also holds that $\mathsf{E}_1(\mathbf{H}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0}) = \mathbf{C}_{2,1} \cdot \overline{\mathbf{H}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0}}$ with $\overline{\mathbf{H}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0}} = \begin{pmatrix} \mathbf{H}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0} \\ \mathbf{1}_{\mathsf{col}} \end{pmatrix}$, where col is the number columns of $\mathbf{H}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0}$ and $\mathbf{1}_{\mathsf{col}}$ is a vector $(1, \ldots, 1, 1)$ of length col.

The functional key $\mathsf{fk}_{\mathsf{E}_1 \circ G_{0,0,0}}$ in $\mathcal{P}_1$ can be represented as $(\mathbf{A} \cdot \mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \mathbf{A})$, where $\mathbf{A} = \mathbf{C}_{2,1} \cdot \overline{\mathbf{H}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0}}$ and $\mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}$ is the secret key of $\mathsf{pkIPFE}^{\mathcal{S}}$. By definition, we also observe that

$$(\mathbf{A} \cdot \overline{\mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}}, \mathbf{A}) = (\mathsf{E}_1(\mathbf{H}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0} \cdot \mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}), \mathsf{E}_1(\mathbf{H}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0})),$$

where $\overline{\mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}} = (\mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}} \| 1)$ We note that this $\mathsf{fk}_{\mathsf{E}_1 \circ G_{0,0,0}}$ exactly corresponds to $(2\ell + 1)$-ciphertexts of E_1 algorithm. For the sake of simplicity, we let $\mathsf{fk}_{\mathsf{E}_1 \circ G_{0,0,0}}[\mathsf{idx}]$ denote the idx-th column vector of $(\mathbf{A} \cdot \overline{\mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}}, \mathbf{A})$ in this proof. Then, each $\mathsf{fk}_{\mathsf{E}_1 \circ G_{0,0,0}}[\mathsf{idx}]$ is a ciphertext of the form

$$\mathsf{fk}_{\mathsf{E}_1 \circ G_{0,0,0}}[\mathsf{idx}] = \begin{cases} \mathsf{E}_1(\mathbf{H}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0} \cdot \mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}) & \text{if } \mathsf{idx} = 1 \\ \mathsf{E}_1(\mathbf{H}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0}[\mathsf{idx}]) & \text{if } 2 \leq \mathsf{idx} \leq 2\ell + 1. \end{cases}$$

On the other hand, $\mathsf{fk}_{\mathsf{E}_1 \circ G^{\mathcal{S}}_{0,0,0}}$ in $\mathcal{P}_{2,1}$ is of the form

$$(\mathsf{E}_1(\overline{\mathbf{H}}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0} \cdot \mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}), \mathsf{E}_1(\overline{\mathbf{H}}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0})),$$

which corresponds to $(2\ell+1)$-ciphertexts of the form $(\mathsf{E}_1(\overline{\mathbf{H}}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0} \cdot \mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}), \mathsf{E}_1(\overline{\mathbf{H}}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0}))$. Similarly, $\mathsf{fk}_{\mathsf{E}_1 \circ G^{\mathcal{S}}_{0,0,0}}[\mathsf{idx}]$ is denoted by the idx-th column vector. From the constraint in Eq. (4) and definition of Fig. 10, it implies that for $\mathsf{idx} \in [1, 2\ell]$, we have

$$\begin{aligned} \mathsf{cDec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE},2}, \mathsf{fk}_{\mathsf{E}_2 \circ G_{0,2,0}}, \mathsf{ct}_{0,1,0}[\mathsf{idx}]) &= \mathsf{cDec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE},2}, \mathsf{fk}_{\mathsf{E}_2 \circ G_{0,2,0}}, \mathsf{E}_1(\mathbf{H}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0}[\mathsf{idx}])) \\ &= \mathsf{cDec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE},2}, \mathsf{fk}_{\mathsf{E}_2 \circ G_{0,2,0}}, \mathsf{E}_1(\overline{\mathbf{H}}_{0,0,0} \cdot \mathbf{\Gamma}^{-1}_{0,0}[\mathsf{idx}])) \\ &= \mathsf{cDec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE},2}, \mathsf{fk}_{\mathsf{E}_2 \circ G_{0,2,0}}, \mathsf{fk}_{\mathsf{E}_1 \circ G^{\mathcal{S}}_{0,0,0}}[\mathsf{idx}]) \end{aligned} \tag{5}$$

where $\mathsf{fk}_{\mathsf{E}_2 \circ G_{0,2,0}}$ is defined as Fig. 10, and $\mathsf{ct}_{0,1,0}[\mathsf{idx}] = \mathsf{fk}_{\mathsf{E}_1 \circ G_{0,0,0}}[\mathsf{idx}]$. We briefly remark that the second equality holds due to Eq. (4).

In summary, we observe that $\mathsf{fk}_{\mathsf{E}_1 \circ G_{0,0,0}}$ and $\mathsf{fk}_{\mathsf{E}_1 \circ G^{\mathcal{S}}_{0,0,0}}$ are both considered as $(2\ell+1)$ ciphertexts of E_1. Moreover, they provide the same evaluated values given functional keys $\mathsf{fk}_{\mathsf{E}_2 \circ G_{0,2,0}}$ due to Eq. (5).

Any adversary $\mathcal{A}$ in distinguishing between $\mathcal{P}_1$ and $\mathcal{P}_{2,1}$ can obtain $(2\ell+1)$ ciphertexts $\mathsf{fk}_{\mathsf{E}_1 \circ G_{0,0,0}}$ or $\mathsf{fk}_{\mathsf{E}_1 \circ G^{\mathcal{S}}_{0,0,0}}$ depending on a protocol. However, $\mathcal{A}$ cannot distinguish between $\mathcal{P}_1$ and $\mathcal{P}_{2,1}$ using this information since E_1 is $(2\ell+1)$ ciphertext-bounded semi-adaptively secure.

We further argue that the information of $\mathbf{x}_0$ computationally is hidden. We point out that an adversary can obtain $\mathsf{E}_1(\overline{h}_{0,0,0}(\mathbf{x}_0))$ under the protocol $\mathcal{P}_{2,1}$.

We may assume that the adversary learns exactly $\overline{\mathbf{H}}_{0,0,0} \cdot \mathbf{x}_0$ from the $\mathsf{E}_1(\overline{h}_{0,0,0}(\mathbf{x}_0))$. Since $\overline{\mathbf{H}}_{0,0,0}$ is a tall matrix, determining $\overline{\mathbf{H}}_{0,0,0}$ immediately recovers $\mathbf{x}_0$. That is the number of possible candidates of $\mathbf{x}_0$ is obtained from that of $\overline{\mathbf{H}}_{0,0,0}$. As discussed in Lemma 1, the possible number of $\overline{\mathbf{H}}_{0,0,0}$ is larger than 2^{λ} by the setup. Hence, the message $\mathbf{x}_0$ can be recovered with probability at most $\frac{1}{2^{\lambda}}$. □

Protocol $\mathcal{P}_{2,i}$. For $i \geq 1$, $\mathcal{P}_{2,i+1}$ is the same protocol as $\mathcal{P}_{2,i}$ except for the i-th functional key of composable-QFE. As $\mathcal{P}_{2,1}$ is defined in the above, $\mathcal{P}_{2,i}$ with $i \geq 2$ could be well defined. To this end, we define some notations. For each $i \in [1, \mathcal{E})$, let $\overline{\mathbf{H}}_{0,i,0}$ be a matrix such that

$$\mathbf{H}'_{0,i,0} \cdot \overline{\mathbf{H}}_{0,i,0} = \mathbf{H}'_{0,i,0} \cdot \mathbf{H}_{0,i,0} \bmod \Delta \tag{6}$$

and $\overline{h}_{0,i,0}$ is a linear function that corresponds to $\overline{\mathbf{H}}_{0,i,0}$ for each $1 \leq i < \mathcal{E}$. Given functions $G^{\mathcal{S}}_{0,i,0} = \overline{h}_{0,i,0} \circ f_{0,i} \circ h'_{0,i-1,0}$, the modified functional keys $\mathsf{fk}_{\mathsf{E}_{i+1} \circ G^{\mathcal{S}}_{0,i,0}}$ of $\mathcal{P}_{2,i+1}$ are then generated by

$$\mathsf{fk}_{\mathsf{E}_{i+1} \circ G^{\mathcal{S}}_{0,i,0}} \leftarrow \mathsf{cKeyGen}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE},i}, \mathsf{pp}_{\mathsf{QFE},i}, \mathsf{msk}_{\mathsf{QFE},i+1}, \mathsf{pp}_{\mathsf{QFE},i+1}, G^{\mathcal{S}}_{0,i,0}, L).$$

Similarly to Lemma 3, we remark that $\mathsf{fk}_{\mathsf{E}_{i+1} \circ G_{0,i,0}}$ is $(2\ell+1)$ ciphertexts of the form

$$\mathsf{fk}_{\mathsf{E}_{i+1} \circ G_{0,i,0}}[\mathsf{idx}] = \begin{cases} \mathsf{E}_{i+1}(\mathbf{H}_{0,i,0} \cdot \mathbf{M}_{f_{0,i}} \cdot (\mathbf{H}'_{0,i-1,0} \otimes \mathbf{H}'_{0,i-1,0}) \cdot \overline{\mathsf{sk}_{\mathsf{QFE},i}}) & \text{if } \mathsf{idx} = 1 \\ \mathsf{E}_{i+1}(\mathbf{H}_{0,i,0} \cdot \mathbf{M}_{f_{0,i}} \cdot (\mathbf{H}'_{0,i-1,0} \otimes \mathbf{H}'_{0,i-1,0})[\mathsf{idx}]) & \text{o.w} \end{cases}$$

where $\overline{\mathsf{sk}_{\mathsf{QFE},i}} = (\mathsf{sk}_{\mathsf{QFE},i} \| 1)$ with $\mathsf{sk}_{\mathsf{QFE},i}$, a certain secret vector and $\mathbf{M}_{f_{0,i}}$ is a matrix representation of $f_{0,i}$. Then, with the same argument of Eq. (5) and the constraint in Eq. (6), it holds that

$$\mathsf{cDec}(\mathsf{pp}_{\mathsf{QFE},i+1}, \mathsf{fk}_{\mathsf{E}_{i+2} \circ G_{0,i+1,0}}, \mathsf{ct}_{0,i,0}[\mathsf{idx}], L) = \mathsf{cDec}_{\mathsf{QFE}}(\mathsf{pp}_{\mathsf{QFE},i+1}, \mathsf{fk}_{\mathsf{E}_{i+2} \circ G_{0,i+1,0}}, \overline{\mathsf{ct}}_{0,i,0}[\mathsf{idx}], L).$$

where $\mathsf{ct}_{0,i,0}[\mathsf{idx}] = \mathsf{fk}_{\mathsf{E}_{i+1} \circ G_{0,i,0}}[\mathsf{idx}]$ and $\overline{\mathsf{ct}}_{0,i,0}[\mathsf{idx}] = \mathsf{fk}_{\mathsf{E}_{i+1} \circ G^{\mathcal{S}}_{0,i,0}}[\mathsf{idx}]$ respectively.

According to the hardness proof of the composable-QFE, an adversary $\mathcal{A}$ cannot distinguish between $\mathcal{P}_{2,i}$ and $\mathcal{P}_{2,i+1}$. For more details, we leave this proof in the Lemma 5. For a $\mathcal{P}_{2,i+1}$, a matrix for a randomness is converted into $\overline{\mathbf{H}}_{0,i,0}$. In the same vein as the above, thus $\mathcal{A}$ cannot obtain information related to $\bigcirc_{t=1}^{i} f_{l,t}(\mathbf{x}_0)$ from an intermediate value $\mathsf{E}_{i+1}(h_i \bigcirc_{t=1}^{i} f_{l,t}(\mathbf{x}_0))$ for a protocol $\mathcal{P}_{2,i+1}$.

In summary, we demonstrate that any adversary $\mathcal{A}$ only distinguishes between $(\mathcal{D}, \mathcal{D}_{2,\mathcal{E}})$ with at most advantage $\mathsf{Adv} \leq \mathsf{Adv}_{\mathsf{pkIPFE}} + \mathcal{E} \cdot \mathsf{Adv}_{\mathsf{cQFE}}$. Although, $\mathcal{D}_{2,\mathcal{E}})$ for $0 \leq i < \mathcal{E}$ gives every intermediate values $\mathsf{E}_{i+1}(h_{0,i,0}(\bigcirc_{t=1}^{i} f_{l,t}(\mathbf{x}_0)))$. the message $\bigcirc_{t=1}^{i} f_{l,t}(\mathbf{x}_0)$ for some i can be recovered with probability at most $\frac{\mathcal{E}}{2^{\lambda}}$. Putting it together, the advantage of adversary $\mathcal{A}$ in revealing any information about the message $\mathbf{x}_0$ is at most $\mathsf{Adv}_{\mathsf{pkIPFE}} + \mathcal{E} \cdot \mathsf{Adv}_{\mathsf{cQFE}} + \frac{\mathcal{E}}{2^{\lambda}}$.

□

7 Benchmarks and Applications to ML Classification

In this section, we present the results of our protocol's performance and demonstrate its application in secure inference for classification. All benchmarks were performed on Linux with Intel(R) Xeon(R) Silver 4208 CPU @ 2.10GHz with 30GB RAMS.

7.1 Benchmarks

First, we provide the benchmarks of our protocol with toy parameters. Our implementation utilizes a public key IPFE [1] based on the Decisional composite residuosity (DCR) assumption. To achieve

128-bit security, we preselect p and q by 3072-bit primes for both DCR-based IPFE and our protocol. These values are hard-coded into our implementation.

Under the above settings, we execute our protocol Fig. 9 for two-layer model across various dimensions ℓ, where ℓ ranges from 1 to 6. As an input, we randomly set a set of vectors $\{\mathbf{c}_i\}_{i\in[1,\ell]} \subset \mathbb{Z}^\ell$ and $\mathbf{d} \in \mathbb{Z}^\ell$. Then, we consider a composition of two quadratic functions as an evaluation model: a multinomial polynomial $f_1 : \mathbb{Z}^\ell \to \mathbb{Z}^\ell$ and $f_2 : \mathbb{Z}^\ell \to \mathbb{Z}$, defined as the following formula.

$$f_1(\mathbf{x}) = \left((\langle \mathbf{c}_i, \mathbf{x}\rangle)^2\right)_{1\le i\le \ell}, f_2(\mathbf{x}) = (\langle \mathbf{d}, \mathbf{x}\rangle)^2$$

Consequently, the protocol outputs the evaluation of two quartic functions $f_2 \circ f_1(\mathbf{x})$ for an input $\mathbf{x}$. It is important to note that the time cost of the protocol is independent of the size of an input vector and the coefficients of the functions. Therefore, these parameters are randomly chosen from small integers.

The result is in the Table 3. The time cost is summed up by the Setup, KeyGen, Enc, and Dec categories. In details, using the notation of protocol in Fig. 9, we elaborate as following.

- "Setup" in the Key Distributor involves "Protocol Setup" step.
- "KeyGen" in the Key Distributor implies "Functional Key Generation" step.
- "Enc" in the Client implies "Encryption" step.
- "Dec" in the Evaluator implies "Model Evaluation" step.

Note that every Setup and KeyGen algorithms in Fig. 9 are computed by a key distributor, and both the client and evaluator are only required to compute encryption and decryption, respectively.

We note that the KeyGen and Dec steps in our experimental results significantly dominate the total time costs. This is because both steps involve power operations on a group, specifically computing the power of an integer sampled in a 12288-bit space. Additionally, since we use the DCR group in our implementation, the discrete logarithm with base $N+1$ can be performed using simple arithmetic operations, which does not require much time. In addition, other parameters in our protocol are set to $Q_i = 2\ell_i + 1$ and $k = 1$. The details of our implementation can be found in the github repository[‖].

Dim (ℓ)	Key Distributor		Evaluator	Client
	Setup	KeyGen	Dec	Enc
1	0.69s	0.55h	0.58h	0.70s
2	0.70s	0.60h	0.64h	0.75s
3	0.87s	1.08h	1.13h	0.77s
4	1.17s	1.79h	1.84h	0.97s
5	1.46s	2.75h	2.78h	1.02s
6	1.74s	4.04h	4.02h	1.11s

Table 3. Benchmarks for protocol under 128-bit security level. The protocol runs for two-layer model with input and intermediate dimension ℓ and outputs a scalar value. Note that we precompute 3072-bit prime in advance so the time to select 3072-bit prime is not included during the Setup.

7.2 Complexity Analysis

In this section, we provide a detailed complexity analysis of our proposed method for evaluating multi-layer neural networks. Our analysis focuses on the computational complexities of dominant operations. In our case, assuming that we only deal with small dimensions in neural networks and a group for a 128-bit security parameter, the dominant operation is the exponentiation over $\mathbb{Z}^*_{N^2}$ with $N = p \cdot q$, the product of two 3072-bit primes p, q. We let ℓ_i denote the message dimension of the i-th layer in neural networks and L be a decomposition parameter.

Adapting algorithms and parameters from Fig. 9, the complexity for each step in our algorithm is as below. Note that we set $k = O(1)$, $L = O(\log \ell_{i+1})$, and $Q_i = O(\ell_i)$ for the QFE of i-th layer.

[‖] https://github.com/swanhong/composable-fe-rs

- Encryption (Enc): The encryption step computes pkIPFE encryption for a vector of dimension $2\ell_1$, which costs $O(\ell_1)$ exponentiations.
- Functional Key Generation (Keygen): In the functional key generation step, the key distributor generates one pkIPFE functional key of dimension $\ell_1 + k$ and a QFE key of dimension $\ell_i + k$ for ℓ_{i+1} times for each $i \in [1, \mathcal{E} - 1]$. For the last layer, a QFE key is generated for dimension $\ell_{\mathcal{E}}$. Hence, the total number of exponentiations is $\sum_{i=1}^{\mathcal{E}-1} O(\ell_i^2 \cdot \ell_{i+1}) + O(\ell_{\mathcal{E}})$, where the last $O(\ell_{\mathcal{E}})$ term can be omitted.
- Evaluation (First Layer): For the first layer, the evaluator computes pkIPFE decryption, which requires $L \cdot O(\ell_1) = O(\ell_1 \log \ell_2)$ exponentiations.
- Evaluation (Intermediate Layers): For each intermediate layer i (where $i \in [2, \mathcal{E} - 1]$), the evaluator performs QFE decryption, which evaluates the quadratic function of input dimension ℓ_i and output dimension ℓ_{i+1}. This requires $L \cdot O(\ell_{i+1}) = O(\ell_{i+1} \log \ell_{i+1})$ number of QFE decryption operations of input dimension ℓ_i, totaling $O(\ell_i^2 \ell_{i+1} \log \ell_{i+1})$ exponentiations.
- Evaluation (Last Layer): For the last layer, a single QFE decryption from dimension $\ell_{\mathcal{E}-1}$ to $\ell_{\mathcal{E}}$ costs $O(\ell_{\mathcal{E}-1}^2 \ell_{\mathcal{E}})$ exponentiations.

Table 4 summarizes the complexity analysis and shows the comparison with [37].

	Enc	Keygen	First Layer	Intermediate Layer	Last Layer
[37]	$O(\ell_1)(E_1 + E_2)$	E_2	$O(\ell_1^2)(E_1 + P)$	$*$	$*$
ours	$O(\ell_1)E$	$\left(\sum_{i=1}^{\mathcal{E}-1} O(\ell_i^2 \cdot \ell_{i+1})\right) E$	$O(\ell_1 \log \ell_2)E$	$O(\ell_i^2 \ell_{i+1} \log \ell_{i+1})E$	$O(\ell_{\mathcal{E}-1}^2 \ell_{\mathcal{E}})E$

Table 4. Comparison of the complexity analysis for evaluating multi-layer neural networks. [37] utilizes a pairing map $G_1 \times G_2 \to G_T$ over the groups G_1, G_2, and G_T, whereas our method employs a group $G = \mathbb{Z}_{N^2}$. We denote the exponentiation complexities over G_1, G_2, and G by E_1, E_2, and E, respectively, and the pairing complexity by P. In our environment, each operation costs on average $E_1 = 16$ms, $E_2 = 16$ms, and $P = 22$ms using the Charm library [2], and $E = 321$ms in our implementation. An asterisk($*$) indicates that all computations in the layer are performed in plaintext.

7.3 Application to Quadratic Neural Networks with UCI Datasets

In this section, we briefly introduce Quadratic Neural Networks (QNN) and their application in secure inference for image classification.

Quadratic Neural Networks. Quadratic Neural Networks (QNN) utilize a neural network model with a *quadratic* activation function, which is mainly used for data classification. The input data consists of pairs $(\mathbf{a}_k, b_k)$, where $\mathbf{a}_k \in \mathbb{R}^\ell$ represents the input features, and $b_k \in [d]$ is a discrete class label, where d is the number of classes. The primary goal of a neural network for classification is to predict the probability that an input belongs to a specific class.

We consider a two-layer neural network, so that we assume the presence of u units in the hidden layer and d distinct classes. The model involves two steps of computation:

1. In the hidden layer, each unit computes $f_h(\mathbf{x}) = \left((\langle \mathbf{W}_{h,i}, \mathbf{x}\rangle + \beta_{h,i})^2\right)_{1 \le i \le u}$, where $\mathbf{W}_h = (\mathbf{W}_{h,1}, \dots, \mathbf{W}_{h,u}) \in \mathbb{R}^{u \times \ell}$ is a weight matrix and $\beta_h \in \mathbb{R}^u$ is a bias.
2. In the output layer, each unit calculates $f_o(\mathbf{x}) = \left((\langle \mathbf{W}_{o,j}, \mathbf{x}\rangle + \beta_{o,j})^2\right)_{1 \le j \le d}$, where $\mathbf{W}_o = (\mathbf{W}_{o,1}, \dots, \mathbf{W}_{o,d}) \in \mathbb{R}^{d \times u}$ is a weight matrix and $\beta_o \in \mathbb{R}^d$ is a bias.

After training phase, we find the models $f_h = (f_{h,1}, f_{h,2}, \dots, f_{h,u})$ and $f_o = (f_{o,1}, f_{o,2}, \dots, f_{o,d})$ for some univariate functions $\{f_{h,i}\}_{i=1}^u$ and $\{f_{o,j}\}_{h=1}^d$, and then the inference phase can get a prediction vector $\mathbf{b}$ of a new input data $\mathbf{a}$ by computing following multinomial quadratic formulas:

$$\mathbf{a}^* = (f_{h,1}(\mathbf{a}), f_{h,2}(\mathbf{a}), \dots, f_{h,u}(\mathbf{a})), \mathbf{b} = (f_{o,1}(\mathbf{a}^*), f_{o,2}(\mathbf{a}^*), \dots, f_{o,d}(\mathbf{a}^*))$$

Application to UCI Datasets. In this section, we describe the implementation of the inference step of QNN with real datasets using our protocol. We employed the Iris and Breast Cancer datasets from the UCI Machine Learning Repository [10]. For both datasets, we implemented a 2-layer neural network model with quadratic activation functions and utilized cross-entropy as the loss function. The model's architecture includes an input layer of dimension $\ell = 4$ for Iris (or $\ell = 9$ for Breast Cancer), a hidden layer with 4 nodes ($u = 4$), and an output layer with $d = 3$ for Iris (or $d = 2$ for Breast Cancer). Other parameters for the protocol are chosen the same as our previous benchmarks.

We converted the real numbers in both dataset and model parameters into integers by scaling each value by $S = 2^{30}$.To enable evaluating f_h and f_o using our protocol, we scale model parameters and define corresponding matrices

$$M_h = \left(S \cdot \mathbf{W}_h, S^2 \cdot \beta_h\right), M_o = \left(S \cdot \mathbf{W}_o, S^5 \cdot \beta_o\right).$$

Then, after performing a linear operation, the output is scaled as follows (rounding is performed after scaling but ignored in the description below).

1. To evaluate the hidden layer f_h, compute the scaled output as follows.

$$(M_h \otimes M_h) \cdot ((S \cdot \mathbf{a} \| 1) \otimes (S \cdot \mathbf{a} \| 1)) = \left((S^2 \cdot \mathbf{W}_{h,i} \cdot \mathbf{a} + S^2 \cdot \beta_{h,i})^2\right)_{1 \leq i \leq u} = S^4 \cdot f_h(\mathbf{a}).$$

2. To evaluate the output layer f_o, compute the scaled output as follows.

$$(M_o \otimes M_o) \cdot \left((S^4 \cdot f_h(\mathbf{a}) \| 1) \otimes (S^4 \cdot f_h(\mathbf{a}) \| 1)\right) = \left((S^5 \cdot \mathbf{W}_{o,j} \cdot f_h(\mathbf{a}) + S^5 \cdot \beta_{o,j})^2\right)_{1 \leq j \leq d} = S^{10} \cdot f_o(f_h(\mathbf{a})).$$

We rescale the result by dividing it by $S^{10} = 2^{300}$ to obtain the desired output $f_o(f_h(\mathbf{a}))$ as real numbers. This scaling factor allowed us to achieve inference results in our protocol that were nearly identical to the results obtained in plain computation, with an error of less than 10^{-7}.

The benchmark results, including the time taken for setup, key generation, decryption, and encryption, are presented in Table 5.

	Key Distributor		Evaluator	Client
	Setup	KeyGen	Dec	Enc
Iris	1.47s	1.05h	1.11h	0.93s
Breast	4.92s	3.61h	3.32h	1.13s

Table 5. Benchmarks for secure inference per one input for the 128-bit security level. Note that 1) we precompute 3072-bit prime in advance so the time to select 3072-bit prime is not included during the Setup. 2) Iris data and Breast data use parameters $(\ell, u, d) = (4, 4, 3)$ and $(9, 4, 2)$, respectively.

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A Functional Encryption for Linear/Quadratic Polynomials due to Musciagna

We overview FE schemes for linear/quadratic polynomials, proposed by Musciagna [29]. They generalize the technique for building the FE for quadratic polynomials assuming k-LIN due to Wee [43].

Both constructions in this section achieve the semi-adaptive simulation based security (SA-SIM), where it is more stronger security, under several assumptions in Section 2.3. The semi-adaptive game introduced by [42], where the adversary is restricted to perform functional key queries to the KeyGen oracle only after it sees the challenge ciphertext.

Notation. Suppose that 2-linear map $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$ is given. For appropriate sizes of matrices $[\mathbf{M}_i]_i \in \mathbb{G}_i$, $e([\mathbf{M}_1]_1, [\mathbf{M}_2]_2)$ is defined by $[[\mathbf{M}_1]_1 \cdot [\mathbf{M}_2]_2]_T$ by exploiting e.

Let $\mathsf{GGen}(1^\lambda)$ be a PPT algorithm which takes as input the security parameter λ, and returns $(\mathbb{G}, g, p)$. Here, $\mathbb{G}$ is a group of composite order $p \gg 2^\lambda$ and g is its generator, which has λ-bit hardness of the discrete log problem. We also use a bracket notation $[x]$ to denote g^x. We note that while the order p corresponds to a composite number $\mathit{\Delta} = N \cdot \phi(N)$, an order of hybrid protocol in the main body, we use the notation p to follow the same description of the scheme [29].

A.1 Functional Encryption for Linear Polynomials

In this section, we provide a brief overview of the simplified IPFE proposed by Musciagna in [29], without security proofs. For more detailed information, we refer readers to the original paper [29].

We begin by describing Mus.IPFE, a FE scheme for inner products (IPFE) proposed by Musciagna [29], in Fig. 11. The security of Mus.IPFE is based on the hardness of the χ-MDDH assumption (Definition 10).

- $\mathsf{Mus.Setup}_{\mathsf{IPFE}}(1^\lambda, n, e))$:
  1. Sample $\mathbf{a} \leftarrow \mathbb{Z}_p^2$, $\mathbf{U} \leftarrow \mathbb{Z}_p^{n\times 2}$.
  2. Set $\mathsf{mpk} = \{[\mathbf{a}]_1, [\mathbf{U}\cdot\mathbf{a}]_1\}$ and $\mathsf{msk} = \{\mathbf{a}, \mathbf{U}\}$ and return $\mathsf{mpk}, \mathsf{msk}$.
- $\mathsf{Mus.KeyGen}_{\mathsf{IPFE}}(\mathsf{msk}, \mathsf{mpk}, [\mathbf{y}]_2 \in \mathbb{G}_2^n)$:
  1. Return $\mathsf{sk}_{\mathbf{y}} = \left[\begin{pmatrix} -\mathbf{U}^T\cdot\mathbf{y} \\ \mathbf{y} \end{pmatrix}\right]_2 \in \mathbb{G}_2^{n+2}$.
- $\mathsf{Mus.Enc}_{\mathsf{IPFE}}(\mathsf{mpk}, [\mathbf{x}]_1 \in \mathbb{G}_1^n)$:
  1. Sample $r \leftarrow \mathbb{Z}_p$.
  2. Return $\mathsf{ct} = \left[\begin{pmatrix} r\cdot\mathbf{a} \\ \mathbf{x} + r\cdot\mathbf{U}\cdot\mathbf{a} \end{pmatrix}\right]_1 \in \mathbb{G}_1^{n+2}$.
- $\mathsf{Mus.Dec}_{\mathsf{IPFE}}(\mathsf{sk}_f, \mathsf{ct}, \mathsf{mpk})$:
  1. Return $\left[\mathsf{ct}^T \cdot \mathsf{sk}_{\mathbf{y}}\right]_T$.

Fig. 11. Mus.IPFE.

Correctness (of Fig. 11). From the decryption procedure, we observe that

$$\begin{aligned}[\mathsf{ct}^T\cdot\mathsf{sk}_{\mathbf{y}}]_T &= \left[\begin{pmatrix} r\cdot\mathbf{a} \\ \mathbf{x}+r\cdot\mathbf{U}\cdot\mathbf{a}\end{pmatrix}^T \cdot \begin{pmatrix} -\mathbf{U}^T\cdot\mathbf{y} \\ \mathbf{y}\end{pmatrix}\right]_T \\ &= [(r\cdot\mathbf{a})^T(-\mathbf{U}\cdot\mathbf{y}) + \mathbf{x}^T\mathbf{y} + (r\cdot\mathbf{U}\cdot\mathbf{a})^T\cdot\mathbf{y}]_T \\ &= [\mathbf{x}^T\cdot\mathbf{y}]_T\end{aligned}$$

We omit the security proof of Mus.IPFE. For more details, we refer the original paper [29, Section 4].

Theorem 4 (Security of Mus.IPFE [29]). *This scheme provides the semi-adaptive simulation based security if the $\mathcal{D}_1$-MDDH assumption holds on $\mathbb{G}_1$.*

A.2 FE for Quadratic Polynomials

We will now describe FE scheme for quadratic polynomials (QFE) in Fig. 12. The goal of QFE is to compute a vector of the form $(\mathbf{x}\otimes\mathbf{y})^T\cdot\mathbf{f}$ for some $\mathbf{x}\in\mathbb{Z}_p^n, \mathbf{y}\in\mathbb{Z}_p^n$, and $\mathbf{f}\in\mathbb{Z}_p^{n^2}$.

Correctness (of Fig. 12). We first observe the term $[(\mathsf{ct}_{\mathbf{x}}\otimes\mathsf{ct}_{\mathbf{y}})\cdot\mathbf{f}]_T$:

$$\begin{aligned}&[(\mathsf{ct}_{\mathbf{x}}\otimes\mathsf{ct}_{\mathbf{y}})\cdot\mathbf{f}]_T \\ &= [(\mathbf{x}\otimes\mathbf{y} + \mathbf{x}\otimes(\mathbf{V}\cdot\mathbf{s}) + (\mathbf{U}\cdot\mathbf{r})\otimes\mathbf{y} + (\mathbf{U}\cdot\mathbf{r})\otimes(\mathbf{V}\cdot\mathbf{s}))^T\cdot\mathbf{f}]_T \\ &= [(\mathbf{x}\otimes\mathbf{y} + (\mathbf{U}\cdot\mathbf{r})\otimes\mathbf{y} + \mathsf{ct}_{\mathbf{x}}\otimes(\mathbf{V}\cdot\mathbf{s}))^T\cdot\mathbf{f}]_T \\ &= [(\mathbf{x}\otimes\mathbf{y} + (\mathbf{U}\otimes\mathbf{I}_n)(\mathbf{r}\otimes\mathbf{y}) + (\mathbf{I}_n\otimes\mathbf{V})(\mathsf{ct}_{\mathbf{x}}\otimes\mathbf{s}))^T\cdot\mathbf{f}]_T \\ &= [(\mathbf{x}\otimes\mathbf{y})^T\cdot\mathbf{f}]_T + \left[\underbrace{\begin{pmatrix}\mathbf{r}\otimes\mathbf{y} \\ \mathsf{ct}_{\mathbf{x}}\otimes\mathbf{s}\end{pmatrix}^T}_{\mathbf{h}^T}\cdot\underbrace{\begin{pmatrix}\mathbf{U}^T\otimes\mathbf{I}_n \\ \mathbf{I}_n\otimes\mathbf{V}^T\end{pmatrix}\cdot\mathbf{f}}_{\mathbf{M}\cdot\mathbf{f}}\right]_T.\end{aligned}$$

On the other hand, by the correctness of $\mathsf{Mus.Dec}_{\mathsf{IPFE}}$, we obtain that

$$\mathsf{Mus.Dec}_{\mathsf{IPFE}}(\mathsf{mpk}_{\mathsf{IPFE}}, \mathsf{ct}_{\mathsf{IPFE}}, \mathsf{sk}_{\mathsf{IPFE}}) = [\mathbf{h}^T\cdot\mathbf{M}\cdot\mathbf{f}]_T.$$

Hence, the correctness must hold since we obtain $[(\mathbf{x}\otimes\mathbf{y})^T\cdot\mathbf{f}]_T$.

As the above, we also omit the security proof of Mus.QFE. For more details, we refer an original paper [29, Section 5].

- $\mathsf{Mus.Setup}_{\mathsf{QFE}}(1^\lambda, e, n)$:
  1. Sample $\mathbf{U} \leftarrow \mathbb{Z}_p^{n\times 2}$, $\mathbf{V} \leftarrow \mathbb{Z}_p^{n\times 2}$ and compute
  $$\mathbf{M} = \begin{pmatrix} \mathbf{U}^T \otimes \mathbf{I}_n \\ \mathbf{I}_n \otimes \mathbf{V}^T \end{pmatrix}.$$
  2. Sample
  $$\{\mathsf{msk}_{\mathsf{IPFE}}, \mathsf{mpk}_{\mathsf{IPFE}}\} \leftarrow \mathsf{Mus.Setup}_{\mathsf{IPFE}}(1^\lambda, 4n).$$
  3. Set $\mathsf{mpk}, \mathsf{msk}$ as follows:
  $$\begin{aligned} \mathsf{mpk} &= \{[\mathbf{U}]_a, [\mathbf{V}]_b, \mathsf{mpk}_{\mathsf{IPFE}}\} \\ \mathsf{msk} &= \{\mathbf{U}, \mathbf{V}, \mathbf{M}, \mathsf{msk}_{\mathsf{IPFE}}\} \end{aligned}$$
  4. Return $\{\mathsf{mpk}, \mathsf{msk}\}$.
- $\mathsf{Mus.KeyGen}_{\mathsf{QFE}}(\mathsf{msk}, \mathsf{mpk}, \mathbf{f} \in \mathbb{Z}_p^{n^2})$:
  1. Sample $\mathsf{sk}_{\mathsf{IPFE}} \leftarrow \mathsf{KeyGen}_{\mathsf{IPFE}}(\mathsf{msk}_{\mathsf{IPFE}}, [\mathbf{M} \cdot \mathbf{f}]_2)$.
  2. Return $\mathsf{sk}_{\mathbf{f}} = \{\mathsf{sk}_{\mathsf{IPFE}}, \mathbf{f}\}$.
- $\mathsf{Mus.Enc}_{\mathsf{QFE}}(\mathsf{mpk}, \mathbf{x} \in \mathbb{Z}_p^n, \mathbf{y} \in \mathbb{Z}_p^n)$:
  1. Sample $\mathbf{r} \leftarrow \mathbb{Z}_p^2$, $\mathbf{s} \leftarrow \mathbb{Z}_p^2$ and compute the following:
  $$\begin{aligned} [\mathsf{ct}_{\mathbf{x}}]_1 &= [\mathbf{x} + \mathbf{U} \cdot \mathbf{r}]_1, \\ [\mathsf{ct}_{\mathbf{y}}]_2 &= [\mathbf{y} + \mathbf{V} \cdot \mathbf{s}]_2, \\ [\mathbf{h}]_1 &= \left[\begin{pmatrix} \mathbf{r} \otimes \mathbf{y} \\ \mathsf{ct}_{\mathbf{x}} \otimes \mathbf{s} \end{pmatrix}\right]_1, \\ \mathsf{ct}_{\mathsf{IPFE}} &\leftarrow \mathsf{Mus.Enc}_{\mathsf{IPFE}}(\mathsf{msk}_{\mathsf{IPFE}}, [\mathbf{h}]_1) \end{aligned}$$
  2. Return $\mathsf{ct} = \{[\mathsf{ct}_{\mathbf{x}}]_1, [\mathsf{ct}_{\mathbf{y}}]_2, \mathsf{ct}_{\mathsf{IPFE}}\}$.
- $\mathsf{Mus.Dec}_{\mathsf{QFE}}(\mathsf{sk}_{\mathbf{f}}, \mathsf{ct}, \mathsf{mpk})$:
  1. Compute
  $$\mathbf{t} = \mathsf{Mus.Dec}_{\mathsf{IPFE}}(\mathsf{mpk}_{\mathsf{IPFE}}, \mathsf{ct}_{\mathsf{IPFE}}, \mathsf{sk}_{\mathsf{IPFE}}).$$
  2. Return $\log_{g_T}([(\mathsf{ct}_{\mathbf{x}} \otimes \mathsf{ct}_{\mathbf{y}}) \cdot \mathbf{f}]_T / \mathbf{t})$.

Fig. 12. Mus.QFE.

Theorem 5 (Security of Mus.QFE). *Mus.QFE is semi-adaptive simulation based secure if the bilateral* 2*-LIN assumptions holds in* $(\mathbb{G}_1, \mathbb{G}_2)$ *and the Mus.IPFE is semi-adaptive simulation based secure.*

B Deferred Security Proof

The purpose of this section is to present the security proof of Theorem 2. We will mainly focus on the security proof of Mus.QFE, as the technique is identical to the proof for Mus.IPFE throughout this section.

Strategy. To prove the theorem, we convert from the Mus.QFE scheme [29] into our composable QFE scheme in Section 4 under semi-adaptive security game. That is, we will show that

$$\mathsf{Adv}_{\mathsf{cQFE}} \leq \mathsf{Adv}_{\mathsf{Mus.QFE}}$$

which finishes proof. This is because that [29] proved that the scheme achieves the semi-adaptive security under the cryptographic hard assumptions.

We make use of an assumption that there exists a secure bilinear map oracle $\mathcal{M}$ capable of computing

$$\mathcal{M}(g_1^x, g_2^y) = g_T^{xy}$$

for some groups $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$ with generators $g_1, g_2, g_T \in \mathbb{Z}_{N^2}$. The only required thing for proof is that the whole group $\mathbb{G}_i$ is an order p group. Thus we skip the group description. It is important to note that efficiency, in terms of storage cost, is not a requirement for the bilinear maps under consideration. Rather, the main concern is security, which makes lookup tables a viable option. The bilinear map oracle will be used for decryption during the security game.

We then show that the composable QFE scheme is semi-adaptive secure when only Q-ciphertext queries are given.

Lemma 4. cQFE *achieves* Q*-ciphertext bounded semi-adaptive simulation based security under the assumption that the bilateral* 2*-LIN assumptions holds in* $(\mathbb{G}_1, \mathbb{G}_2)$ *and the above IPFE is semi-adaptive simulation based secure.*

Proof of Lemma 4. Our goal is to demonstrate that if an adversary $\mathcal{A}_{\mathsf{cQFE}}$ can break the semi-adaptive simulation-based security of our scheme, then Mus.QFE can also be broken. To accomplish this, we construct an adversary of Mus.QFE using $\mathcal{A}_{\mathsf{QFE}}$ that breaks the cQFE.

Now, we introduce an algorithm, called matrix-vector splitting algorithm (Alg. 2) to bridge between ciphertexts of Mus.QFE and cQFE. The algorithm is a core technique of the reduction for the proof.

For ease of presentations, we omit the underlying group, $\mathbb{G}$, and the dimension of given vectors, L[**], in the input of each algorithm.

Algorithm 2 Matrix-vector splitting

```
Input: Vectors $\{\mathbf{v}_i\}_{i\in[1,Q]} \subset \mathbb{G}^L$
Output: $[\mathbf{D}] \in \mathbb{G}^{L\times L+Q}$ and $\{\mathbf{d}_i\}_{i\in[1,Q]} \subset \mathbb{Z}_{N\cdot\phi(N)}^{L+Q}$ such that $[\mathbf{D}]^{\mathbf{d}_i} = \mathbf{v}_i$
Set $\mathbf{A} \leftarrow \mathcal{D}_{\mathbb{Z}^{L+Q\times L+Q}}$ until $\mathbf{A}$ is invertible.
Sample $\widetilde{\mathbf{B}} \leftarrow \mathbb{Z}_{N\cdot\phi(N)}^{L\times L}$ and set
    $$[\mathbf{D}] = (\mathbf{v}_1 \| \cdots \| \mathbf{v}_Q \| [\widetilde{\mathbf{B}}])^{\mathbf{A}^{-1}} \in \mathbb{G}^{L\times(L+Q)}$$
Set $\mathbf{d}_i = \mathbf{A} \cdot \mathbf{e}_i$ for $i \in [1, Q]$.
return $[\mathbf{D}]$ and $\{\mathbf{d}_i\}$.
```

Alg. 2 is used to convert ciphertexts of Mus.QFE (resp. Mus.IPFE) into cQFE (resp. c-IPFE). The detailed algorithms for transforming Mus.IPFE and Mus.QFE into cIPFE and cQFE are given by Fig. 13 and Fig. 14, respectively.

[**] This notation is independent to the L in the main body.

We would like to note that the set of output vectors $\{\mathbf{d}_{\mathbf{x}_i}, \mathbf{d}_{\mathbf{y}_i}, \mathbf{d}_{\mathsf{IPFE},i}\}_i$ obtained from Fig. 14 has the same distribution of $\mathsf{Enc}_{\mathsf{QFE}}$ in Section 4 when $\mathbf{x}_i = \mathbf{y}_i$. Additionally, we note that the terms $\widetilde{\mathsf{KeyGen}}_{\mathsf{QFE}}(\mathbf{f}, \{[\mathbf{D}_0]_1, [\mathbf{D}_1]_2, [\mathbf{D}_{\mathsf{IPFE}}]_2\}, \mathsf{KeyGen}_{\mathsf{QFE}}(\mathbf{f}))$ can serve as functional keys for cQFE because it has the same distribution of $\mathsf{KeyGen}_{\mathsf{QFE}}$ as well. As a result, it is able to obtain a set of ciphertexts and a family of functions that serve as legitimate inputs.

Let $\mathsf{Mus.QFE}^{\mathcal{S}}$ (resp. $\mathsf{Mus.IPFE}^{\mathcal{S}}$) be a simulator of $\mathsf{Mus.QFE}$ (resp. $\mathsf{Mus.IPFE}$). Since these schemes are semi adaptive simulation based secure scheme, such a simulator exists. In the same vein, Alg. 2 converts the $\mathsf{Mus.QFE}^{\mathcal{S}}$ into a simulator of cQFE. We denote it as $\mathsf{cQFE}^{\mathcal{S}}$.

The probability of distinguishing between cQFE and $\mathsf{cQFE}^{\mathcal{S}}$ should be non-negligible if not $\mathsf{Mus.QFE}$ and its simulator should be distinguished with non-negligible probability. This completes the proof.

```
$\widetilde{\mathsf{Enc}}_{\mathsf{IPFE}}(\mathsf{Mus{:}ctxt}_{\mathsf{IPFE}}(\mathbf{x}_i))$ :
    1. Sample $[\mathbf{D}]_2, \mathbf{d}_i \leftarrow$ Alg. 2($\mathsf{Mus{:}ctxt}_{\mathsf{IPFE}}(\mathbf{x}_i)$).
    2. Return $\mathbf{d}_i$ for every $i$.
$\widetilde{\mathsf{KeyGen}}_{\mathsf{IPFE}}(\mathbf{f}, [\mathbf{D}]_2, \mathsf{Mus.KeyGen}_{\mathsf{IPFE}}(\mathbf{f}))$ :
    1. Denote $\mathsf{Mus.KeyGen}_{\mathsf{IPFE}}(\mathbf{f})$ by $[\mathsf{sk}_f]_1$.
    2. Return $e([\mathsf{sk}_f^T]_1, [\mathbf{D}]_2) = [\mathsf{sk}_f^T \cdot \mathbf{D}]_T$.
```

Fig. 13. Construction of $\widetilde{\mathsf{Enc}}_{\mathsf{IPFE}}$ and $\widetilde{\mathsf{KeyGen}}_{\mathsf{IPFE}}$ given $\{\mathsf{Mus{:}ctxt}_{\mathsf{IPFE}}(\mathbf{x}_i)\}$ and $\mathbf{f}$.

```
$\widetilde{\mathsf{Enc}}_{\mathsf{QFE}}(\{\mathsf{Mus{:}ctxt}_{\mathsf{QFE}}(\mathbf{x}_i, \mathbf{y}_i)\}_{i \le Q})$ :
    1. Sample $[\mathbf{D}_0]_1, \{\mathbf{d}_{\mathbf{x}_i}\}_{i \le Q} \leftarrow$ Alg. 2($\{\mathsf{ct}_{\mathbf{x}_i}\}_{i \le Q}$).
    2. Sample $[\mathbf{D}_1]_2, \{\mathbf{d}_{\mathbf{y}_i}\}_{i \le Q} \leftarrow$ Alg. 2($\{\mathsf{ct}_{\mathbf{y}_i}\}_{i \le Q}$).
    3. Sample
           $[\mathbf{D}_{\mathsf{IPFE}}]_2, \{\mathbf{d}_{\mathsf{IPFE},i}\}_{i \le Q} \leftarrow \widetilde{\mathsf{Enc}}_{\mathsf{IPFE}}(\{\mathsf{ct}_{\mathsf{IPFE},i}\}_{i \le Q})$.
    4. Return $(\{\mathbf{d}_{\mathbf{x}_i}, \mathbf{d}_{\mathbf{y}_i}, \mathbf{d}_{\mathsf{IPFE},i}\}_{i \le Q})$.
$\widetilde{\mathsf{KeyGen}}_{\mathsf{QFE}}(\mathbf{f}, \{[\mathbf{D}_0]_1, [\mathbf{D}_1]_2, [\mathbf{D}_{\mathsf{IPFE}}]_2\}, \mathsf{KeyGen}_{\mathsf{QFE}}(\mathbf{f}))$ :
    1. Parse $\mathsf{KeyGen}_{\mathsf{QFE}}(\mathbf{f})$ by $(\mathsf{sk}_{\mathsf{IPFE},\mathbf{f}}, [\mathbf{f}]_c)$.
    2. Compute $\mathsf{fk}_1 := [\mathbf{f}^T \cdot (\mathbf{D}_0 \otimes \mathbf{D}_1)]_T$ via bilinear map $e$ and $\mathbf{f}^T$, $[\mathbf{D}_0]_1$, and $[\mathbf{D}_1]_2$.
    3. Sample
           $\mathsf{fk}_{\mathsf{IPFE}} \leftarrow \widetilde{\mathsf{KeyGen}}_{\mathsf{IPFE}}(\mathsf{sk}_{\mathsf{IPFE},f}, [\mathbf{D}_{\mathsf{IPFE}}]_2)$.
    4. Return $\mathsf{fk}_1$ and $\mathsf{fk}_{\mathsf{IPFE}}$.
```

Fig. 14. Construction of $\widetilde{\mathsf{Enc}}_{\mathsf{QFE}}$ and $\widetilde{\mathsf{KeyGen}}_{\mathsf{QFE}}$ given $\{\mathsf{Mus{:}ctxt}_{\mathsf{QFE}}(\mathbf{x}_i, \mathbf{y}_i) = \{\mathsf{ct}_{\mathbf{x}_i}, \mathsf{ct}_{\mathbf{y}_i}, \mathsf{ct}_{\mathsf{IPFE},i}\}\}_{i \le Q}$ and $\mathbf{f}$.

□

B.1 Detailed Proof in Theorem 3

In this section, we provide a detailed proof of $\mathcal{P}_{2,i} \sim \mathcal{P}_{2,i+1}$. For this purpose, we prove the following lemma.

Lemma 5. *For any $1 \le i < \mathcal{E}$, $\mathcal{P}_{2,i}$ and $\mathcal{P}_{2,i+1}$ are computationally indistinguishable under the assumption that* cQFE *is $(2\ell+1)$-bounded semi-adaptively secure. The advantage of distinguishing* *between $\mathcal{D}_{2,i}$ and $\mathcal{D}_{2,i+1}$ is smaller than $\mathsf{Adv}_{\mathsf{cQFE}}$. Furthermore, no one computationally obtains any information about $\bigcirc_{i=1}^{\mathcal{E}} f_{0,i}(\mathbf{x}_0)$ from $\mathsf{E}_{i+1}(h_{0,i,0}(\bigcirc_{t=1}^{i} f_{0,t}(\mathbf{x}_0)))$.*

Proof of Lemma 5. Suppose that there exists a PPT adversary $\mathcal{B}_i$ that can distinguish between $\mathcal{P}_{2,i}$ and $\mathcal{P}_{2,i+1}$. Then, we construct an adversary $\mathcal{A}_{\mathsf{cQFE}}$ to break the security game of composable-QFE. Assume that $\mathcal{C}$ is a challenger of the security game of composable-QFE.

1. $\mathcal{C}$ samples $(\mathsf{msk}_{\mathsf{QFE},i+1}, \mathsf{pp}_{\mathsf{QFE},i+1}) \leftarrow \mathsf{Setup}_{\mathsf{QFE}}(\lambda, \ell, 2\ell+1)$ and sends $\mathsf{pp}_{\mathsf{QFE},i+1}$ to $\mathcal{A}_{\mathsf{cQFE}}$.
2. $\mathcal{A}_{\mathsf{cQFE}}$ proceeds up to steps as follows:
   (a) Sample
   $$\{\mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \mathsf{pk}^{\mathcal{S}}_{\mathsf{pkIPFE}}\} \leftarrow \mathsf{Setup}^{\mathcal{S}}_{\mathsf{pkIPFE}}(\lambda, 2\ell+1, B_X, B_F),$$
   $$\{\mathsf{msk}_{\mathsf{QFE},t}, \mathsf{pp}_{\mathsf{QFE},t}\} \leftarrow \mathsf{Setup}_{\mathsf{QFE}}(\lambda, \ell, 2\ell+1)$$
   for each $t \in [1, \mathcal{E}] \setminus \{i+1\}$.
   (b) Set pk as in $\mathcal{P}$ and send it to $\mathcal{B}_i$.
   (c) Sample a pair $(\mathbf{H}_{0,t,0}, \mathbf{H}'_{0,t,0})$ for $t \in [0, \mathcal{E}] \setminus \{i\}$.
   (d) Sample a matrix $\overline{\overline{\mathbf{H}}}_{0,t,0}$ satisfying $\mathbf{H}'_{0,t,0} \cdot \overline{\overline{\mathbf{H}}}_{0,t,0} = \mathbf{H}'_{0,t,0} \cdot \mathbf{H}_{0,t,0} \bmod \mathit{\Delta}$ for $0 \leq t < i$.
   (e) Sample a pair $(\mathbf{\Gamma}_{l,0}, \mathbf{\Gamma}^{-1}_{l,0})$ and a simulated ciphertext $\mathsf{Enc}^{\mathcal{S}}_1$ as follows.
   $$\mathsf{Enc}^{\mathcal{S}}_1 \leftarrow \mathsf{Enc}^{\mathcal{S}}_{\mathsf{pkIPFE}}(\mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \mathsf{pk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \{h_{0,0,0} \circ \gamma^{-1}_{l,0}, h_{0,0,0}(\mathbf{x})\})$$
3. $\mathcal{B}_i$ queries a set of models $\{F_l\}$ to the $\mathcal{A}_{\mathsf{cQFE}}$.
4. $\mathcal{A}_{\mathsf{cQFE}}$ computes a functional key of $\widetilde{G}_{0,0,0} = G_{0,0,0} = h_{0,0,0} \circ \gamma^{-1}_{l,0}$ as follows.
   $$\mathsf{fk}_{\mathsf{E}_1 \circ G^{\mathcal{S}}_{0,0,0}} \leftarrow \mathsf{cKeyGen}^{\mathcal{S}}_{\mathsf{pkIPFE},0}(\mathsf{msk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \mathsf{pk}^{\mathcal{S}}_{\mathsf{pkIPFE}}, \mathsf{msk}_{\mathsf{QFE},1}, \mathsf{pp}_{\mathsf{QFE},1}, G_{0,0,0}, L)$$
5. For every $t < i$, $\mathcal{A}_{\mathsf{cQFE}}$ computes functions keys $\mathsf{fk}_{\mathsf{E}_{t+1} \circ G^{\mathcal{S}}_{0,t,0}}$ as follows:
   $$\mathsf{fk}_{\mathsf{E}_{t+1} \circ G^{\mathcal{S}}_{0,t,0}} \leftarrow \mathsf{cKeyGen}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE},t}, \mathsf{pp}_{\mathsf{QFE},t}, \mathsf{msk}_{\mathsf{QFE},t+1}, \mathsf{pp}_{\mathsf{QFE},t+1}, G^{\mathcal{S}}_{0,t,0}, L),$$
   where $G^{\mathcal{S}}_{0,t,0} = \bar{h}_{0,t,0} \circ f_{l,t} \circ h'_{0,t-1,0}$.
6. For every $t > i, i+1$, $\mathcal{A}_{\mathsf{cQFE}}$ computes functions keys $\mathsf{fk}_{\mathsf{E}_{t+1} \circ G_{0,t,0}}$ as follows:
   $$\mathsf{fk}_{\mathsf{E}_{t+1} \circ G_{0,t,0}} \leftarrow \mathsf{cKeyGen}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE},t}, \mathsf{pp}_{\mathsf{QFE},t}, \mathsf{msk}_{\mathsf{QFE},t+1}, \mathsf{pp}_{\mathsf{QFE},t+1}, G_{0,t,0}, L)$$
   where $G_{0,t,0} = h_{0,t,0} \circ f_{l,t} \circ h'_{0,t-1,0}$.
7. To compute $\mathsf{fk}_{\mathsf{E}_{i+1} \circ G^{\mathcal{S}}_{0,i,0}}, \mathsf{fk}_{\mathsf{E}_{i+2} \circ G_{0,i+1,0}}$, $\mathcal{A}_{\mathsf{cQFE}}$ sends $(\mathbf{H}_{0,i,0}, \bar{\mathbf{H}}_{0,i,0})$ to $\mathcal{C}$.
8. $\mathcal{C}$ randomly chooses $h_{0,i,0} \in \{\mathbf{H}_{0,i,0}, \overline{\overline{\mathbf{H}}}_{0,i,0}\}$ and returns $\mathsf{ct} \leftarrow \mathsf{E}_{i+1}(h_{0,i,0})$.
9. $\mathcal{A}_{\mathsf{cQFE}}$ computes
   $$\mathsf{fk}_{\mathsf{E}_{i+1} \circ G^{\mathcal{S}}_{0,i,0}} \leftarrow \mathsf{KeyGen}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE},i}, \mathsf{pp}_{\mathsf{QFE},i}, \mathsf{ct} \cdot \mathbf{M}_{f_{0,i}} \cdot (\mathbf{H}'_{0,i-1,0} \otimes \mathbf{H}'_{0,i-1,0}), L).$$
10. $\mathcal{A}_{\mathsf{cQFE}}$ requests a function query for $G_{0,i+1,0} = h_{0,i+1,0} \circ f_{0,i} \circ h'_{0,i,0}$.
11. $\mathcal{C}$ computes a functional key $\mathsf{fk}_{\mathsf{E}_{i+2} \circ G_{0,i+1,0}}$ as follows
    $$\mathsf{fk}_{\mathsf{E}_{i+2} \circ G_{0,i+1,0}} \leftarrow \mathsf{cKeyGen}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE},i+1}, \mathsf{pp}_{\mathsf{QFE},i+1}, \mathsf{msk}_{\mathsf{QFE},i+2}, \mathsf{pp}_{\mathsf{QFE},i+2}, G_{0,i+1,0}, L)$$
    and sends it to $\mathcal{A}_{\mathsf{cQFE}}$. Last, $\mathcal{A}_{\mathsf{cQFE}}$ transmits it to $\mathcal{B}_i$.
12. $\mathcal{B}_i$ returns $\beta \in \{i, i+1\}$ to $\mathcal{A}_{\mathsf{cQFE}}$.
13. If $\beta = i+1$, $\mathcal{A}_{\mathsf{Enc}_{2,i+1}}$ outputs $\overline{\overline{\mathbf{H}}}_{0,i,0}$. Otherwise, returns $\mathbf{H}_{0,i,0}$.

By construction, $\mathsf{fk}_{\mathsf{E}_{i+1} \circ G^{\mathcal{S}}_{0,i,0}}$ is identical to an output of

$$\mathsf{cKeyGen}_{\mathsf{QFE}}(\mathsf{msk}_{\mathsf{QFE},i}, \mathsf{pp}_{\mathsf{QFE},i}, \mathsf{msk}_{\mathsf{QFE},i+1}, \mathsf{pp}_{\mathsf{QFE},i+1}, h_{0,i,0} \circ f_{0,i} \circ h_{0,i-1,0}, L).$$

Thus, this game directly implies that

$$\mathsf{Adv}_{i,i+1} \leq \mathsf{Adv}_{\mathsf{cQFE}} \text{ for any } i,$$

where $\mathsf{Adv}_{i,i+1}$ stands for denoting the advantage to distinguish two protocols $\mathcal{P}_{2,i}$ and $\mathcal{P}_{2,i+1}$.

Analogs to the case of $\overline{\overline{\mathbf{H}}}_{0,0,0}$, we may assume that the adversary $\mathcal{A}$ knows $\mathbf{H}'_{0,i,0}$ and $\overline{\overline{\mathbf{H}}}_{0,i,0} \cdot \bigcirc_{t=1}^{i} f_{l,t}(\mathbf{x})$ exactly. However the possible number of $\overline{\overline{\mathbf{H}}}_{0,i,0}$ is larger than 2^{λ}. Hence, no one computationally recovers the $\bigcirc_{t=1}^{i} f_{l,t}(\mathbf{x}_0)$ for each i. □