Self-Consistent Current Sheets and Filaments in Relativistic Collisionless Plasma with Arbitrary Energy Distribution of Particles

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A new class of self-consistent planar current sheets and cylindrical current filaments with a functional freedom for the resultant spatial profiles is found analytically for collisionless plasma. Invariants of particle motion are employed to obtain exact stationary solutions of Vlasov-Maxwell equations for arbitrary energy distribution of particles. This method automatically takes into account complicated particle motion in a self-consistent magnetic field, can be equally well applied to relativistic and nonrelativistic plasma, and yields a much wider class of solutions as compared to models of the Harris-Bennett type and their known generalizations. We discuss typical analytical solutions and general properties of magnetostatic neutral structures: spatial scales, magnitudes of current and magnetic field, degree of anisotropy of particle distributions, and possible equipartition of magnetic and particle energies.

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The structure and properties of self-sustained current configurations and the corresponding magnetostatic fields with scales less than the mean free path of particles are of great interest in the physics of collisionless plasma [1–4]. The self-consistent current sheets and filaments are responsible for various regular and turbulent large-scale processes. Examples include the dynamics of current sheets in the Earth magnetosphere and solar corona, the formation of collisionless shocks, cosmic-ray acceleration, the evolution of accretion disk-jet systems in astrophysics and laser plasma jets in laboratory physics, and so on. Explanation of many phenomena observed in collisionless plasma, including synchrotron radiation, requires long-living magnetic fields with energy density of the order of particle energy density. Such fields are thought to be supported by quasistationary self-consistent current configurations.

In most cases the problem is complicated due to nonthermal anisotropic particle distribution functions (PDFs), that prevent the use of known solutions, such as planar Harris and cylindrically symmetrical Bennett ones, based on shifted Maxwellian distributions [5,6]. Until recently, tangled particle trajectories in self-consistent magnetic field did not allow us to find exact analytical solutions, especially for non-Maxwellian PDFs. For a review of theoretical achievements and subtle points of the problem, see [4,7–12] and references therein.

In this Letter, we employ the method of invariants of particle motion and find analytically a broad variety of magnetostatic 1D and cylindrically symmetrical current configurations with functional freedom for a choice of particle energy distribution. The idea it to use a quite general class of PDFs, which allows us to represent the magnetostatic self-consistency equation as an explicit nonlinear equation for vector potential and to describe all types of its solutions. This approach is as powerful as that of Bernstein, Greene, and Kruskal [13,14], related to a scalar potential and applied to nonlinear plasma oscillations. Previous attempts to apply invariants of particle motion to the magnetostatic problem in collisionless plasma have not led to representative analytical results and cannot clarify the general structure of self-consistent configurations, because they were limited by the particular choice of particle distributions without any functional freedom [4,7–11].

We start with the basic nonlinear magnetostatic problem and the method of invariants of particle motion in multicomponent collisionless plasma, then qualitatively analyze possible classes of neutral current configurations, give examples of typical exact analytical solutions, and describe their physical properties.

Nonlinear magnetostatic problem and invariants of particle motion.—We consider stationary plasma configurations in the case of perfect charge neutrality. The magnetic field is assumed to lie in the x-y plane; we describe it by vector potential $\mathbf{A} = A_z \mathbf{z}_0 = A(x, y)\mathbf{z}_0$. Translation invariance along the z axis implies that for each particle the z component of generalized momentum is strictly conserved, together with the total momentum, p, which defines the kinetic energy. We do not use an approximate invariant of the magnetic moment. So, we can consider the PDF as a function of two integrals of motion:

$$F_j(\mathbf{r}, \mathbf{p}) = \hat{F}_j(p, p_z + e_j A(x, y)/c). \qquad (1)$$

It satisfies the collisionless Vlasov equation [1,3,12]

$$\mathbf{p}\,\partial F_j/\partial \mathbf{r} + e_j c^{-1}[\mathbf{p} \times \mathbf{B}]\partial F_j/\partial \mathbf{p} = 0 \qquad (2)$$

for arbitrary $\hat{F}_j$ and arbitrary $\mathbf{B} = \nabla \times \mathbf{A}$. Then, Maxwell equations are reduced to one nontrivial equation

$$\Delta_{xy}A = -\frac{4\pi}{c}\sum_j e_j \int \hat{F}_j(p, p_z + e_jA/c)\frac{p_z}{m_j\gamma_j}d^3\mathbf{p}, \quad (3)$$

where $\Delta_{xy} = \partial^2/\partial x^2 + \partial^2/\partial y^2$, e_j, m_j, and $\gamma_j = (1 + p^2/m_j^2c^2)^{1/2}$ are the charge, mass, and Lorentz factor of the particles of species j. The sum in the right side of (3), that is the current density j_z, is a function of A, which can conveniently be represented via a "potential":

$$\Delta_{xy}A = -dU/dA. \quad (4)$$

The latter is an equation of the Grad-Shafranov type [15], widely used in MHD. The shape of the potential $U(A)$ is related to the structure of PDFs $\hat{F}_j$. Gauge transformation $A \to A + \text{const}$ would lead to a trivial parallel shift in $U(A)$, so we will assume some fixed gauge. We will not discuss in detail the required condition of charge neutrality (especially important if the sheet scale is less than the Debye screening length) and a possible way to satisfy it. The simplest general way is to consider current structures where particles of, say, positive charge, are still or have inverted momentum distribution as compared to that of particles of negative charge.

An analytical expression of potential $U(A)$ can be obtained, for example, for a combination of polynomial and exponential functions of $p_z + e_jA/c$ for each p ($d \geq 0$),

$$\hat{F}_j = \sum_{i=0}^{d}\hat{F}_{ji}(p)\left(\frac{p_z + e_jA/c}{m_jc}\right)^i \exp\left(\zeta_j\frac{p_z + e_jA/c}{m_jc}\right), \quad (5)$$

where the functions $\hat{F}_{ji}(p)$ can be arbitrary, provided that $\hat{F}_j$ remains non-negative for every p and $|p_z| < p$, and ζ_j are constants. Straightforward integration yields

$$\Delta_{xy}A = -8\pi^2\sum_j e_jm_jce^{\zeta_ja_j}\sum_{k=0}^{d}a_j^k\sum_{i=k}^{d}\frac{(-\zeta_j)^{k-i-2}n!}{k!(i-k)!} \times \int \hat{F}_{ji}[\Gamma(i-k+2, b_j) - \Gamma(i-k+2, -b_j)]\frac{pdp}{\gamma_j}, \quad (6)$$

where $a_j = e_jA/(m_jc^2)$, $b_j = \zeta_jp/(m_jc)$; Γ is the incomplete gamma function. The right side of (6) as a function of A is a sum of polynomials multiplied by exponential functions (cf. [4,7,8]). The same is true for the potential $U(A)$, which can be obtained by integration of the right-hand side of (6). In the purely polynomial case, $\zeta_j = 0$, we have

$$U = \sum_j\sum_{i=0}^{d}\int\frac{8\pi^2\hat{F}_{ji}(p)p[g_{ji}(p) - g_{ji}(-p)]dp}{m_j\gamma_j(i+1)(i+2)(i+3)} + \text{const}, \quad (7)$$

where $g_{ji}(p) = (e_jA/c - p)^{i+2}[(i+2)p + e_jA/c]$. Potential (7) is a polynomial in A of order d.

Explicit integration of the right-hand side of (3) can be carried out for negative or even noninteger i in the decomposition (5), giving rise to other profiles $U(A)$ and, hence, different classes of current sheets and filaments, which will be described in a separate paper. In these cases the argument $p_z + e_jA/c$ of the PDF (5) must be positive, implying a lower bound on the domain of the value of vector potential where $U(A)$ is defined: $|A| > |A|_{\min} = p_{\max}c/|e_j|$. This bound is determined by the maximal momentum of particles in the self-consistent structure.

Qualitative description of solutions.—In the 1D case ($\partial A/\partial y \equiv 0$) Eq. (4) is equivalent to an equation of classical nonlinear oscillator (with the x coordinate playing the role of time). If $\zeta_j = 0$ and positive powers i are used (including noninteger), the movement of this oscillator must be finite, otherwise the particle density would go to infinity together with A. This leaves three possibilities.

(i) Periodic dependence $A(x)$, i.e., periodic (but not necessarily harmonic) magnetic field $\mathbf{B}(x)$, which is directed along y axis and has zero mean value.

(ii) Monotonic dependence $A(x)$, where both values $A(x \to -\infty)$ and $A(x \to +\infty)$ correspond to maxima of potential $U(A)$ of the same height. The magnetic field in this case does not change its direction; it is localized between two oppositely directed current sheets and vanishes at $x \to \pm\infty$, so the net current is zero and the integral of magnetic energy density over coordinate x is finite.

(iii) Symmetric dependence $A(x) = A(2x_0 - x)$, where the value $A(x \to -\infty) = A(x \to +\infty)$ corresponds to a local maximum of $U(A)$, and x_0 is the turning point with $U(A(x_0)) = U(A(x \to \pm\infty))$. The magnetic field is antisymmetric and changes its direction exactly once, at $x = x_0$, corresponding to a symmetric current sheet centered around that point. In such a sheet the central part is surrounded by at least two countersheets, so there is no net current and magnetic field $B \to 0$ at $x \to \pm\infty$, as in the case (ii) (exponentially, except for special cases).

If we allow negative powers i (integer or not) in (5) or retain the exponential factors, $\zeta_j \neq 0$, the solution $A(x)$ can be unbounded and current sheets with nonzero total current and finite magnetic field at infinity can be described. A current sheet of this class is symmetric, and its magnetic field reverses exactly once, at the plane of symmetry. In this case, the current density j_z can have the same sign throughout the sheet and the particle densities of all species can vanish at infinity. In particular, sheets with zero magnetic field at $x \to \pm\infty$ could correspond to potential profiles $U(A)$ of case (iii) with a finite local maximum at $A \to +\infty$ or $A \to -\infty$.

We have found analogous classes of cylindrically symmetric solutions (neutral current filaments) with self-consistent anisotropic PDFs and azimuthal magnetic field both dependent on cylindrical coordinate ρ.

It should be noted that in localized solutions, where current density vanishes as x (or ρ) goes to infinity, PDFs do not, in general, become isotropic there. If at infinity the value of A approaches a constant, then the degree of PDF anisotropy for each j also approaches a constant. If the value of A goes to infinity, then the anisotropy can either vanish [if only negative powers in (5) are present and $\zeta_j = 0$] or stay constant [if $\zeta_j \neq 0$, i.e., the exponential factors in (5) are retained]. Changing the isotropic plasma component ($i = 0$, $\zeta_j = 0$) in (5) does not affect the properties of the described magnetostatic structures.

Typical solutions.—Let us show some self-consistent current structures which are described universally within a whole class of PDFs with arbitrary energy profiles.

We begin with planar structures, where the spatial dependence is on coordinate x only. Let $d = 4$, $\zeta_j = 0$, $\hat{F}_{j1} \equiv 0$, $\hat{F}_{j3} \equiv 0$; then the Grad-Shafranov potential

$$U = U_0(A^2/A_0^2 - A^4/A_0^4) \tag{8}$$

is defined by two positive parameters A_0^2 and U_0:

$$A_0^2 = S\left(-5\sum_j \int \frac{\hat{F}_{j4}(p)}{m_j^5 c^8} e_j^4 p^4 \frac{dp}{\gamma_j}\right)^{-1}, \qquad U_0 = \frac{16\pi^2}{15c} A_0^2 S; \tag{9}$$

$$S = \sum_j \int \frac{[5\hat{F}_{j2}(p)m_j^2c^2 + 6\hat{F}_{j4}(p)p^2]}{m_j^5 c^5} e_j^2 p^4 \frac{dp}{\gamma_j}. \tag{10}$$

In this case 1D-localized solution of Eq. (4) is

$$A(x) = (A_0/\sqrt{2})\tanh[(\sqrt{U_0}/A_0)x]. \tag{11}$$

It is a pair of oppositely directed current sheets with magnetic field localized between them [see Fig. 1(a)]. The field exponentially vanishes away from the sheets, while particle density approaches constant. So, far away from the structure the plasma is uniform and unmagnetized, although anisotropy does not vanish, leaving a room for anisotropy-driven Weibel instability. The stability for perturbations with $\mathbf{k}\perp\mathbf{z}$, $\mathbf{E} \parallel \mathbf{z}$ can be checked based on a condition similar to one obtained in [12], which gives the inequality $A_0^2 > 0$, already assumed. The corresponding condition for orthogonal perturbations with $\mathbf{k} \parallel \mathbf{z}$, $\mathbf{E}\perp\mathbf{z}$ is

$$\sum_j \int \frac{[5\hat{F}_{j2}(p) + 8\hat{F}_{j4}(p)p^2]}{m_j c} e_j^2 p^4 \frac{dp}{\gamma_j} < 0. \tag{12}$$

Taking the Grad-Shafranov potential (8) with $U_0 < 0$, we come to another 1D-localized solution of Eq. (4)

$$A = (A_0/\sqrt{2})\sqrt{1 - \tanh^2[(\sqrt{-U_0}/A_0)x]}, \tag{13}$$

which describes a current sheet surrounded by two sheets of oppositely directed current, so the net current is zero [see Fig. 1(b)]. It can be shown that perturbations with $\mathbf{k} \parallel \mathbf{z}$, $\mathbf{E}\perp\mathbf{z}$ are Weibel stable far away from the sheet if

$$\sum_j \int \frac{[5\hat{F}_{j2}(p) + 2\hat{F}_{j4}(p)p^2]}{m_j c} e_j^2 p^4 \frac{dp}{\gamma_j} > 0, \tag{14}$$

and the orthogonal perturbations with $\mathbf{k}\perp\mathbf{z}$, $\mathbf{E} \parallel \mathbf{z}$ do not exhibit instability, since their stability condition coincides with the inequality $A_0^2 > 0$. The above mentioned stability conditions for both solutions (11) and (13) can be satisfied for a specific range of functions $\hat{F}_{j2}$ and $\hat{F}_{j4}$. As for the comprehensive stability analysis of these and other self-consistent structures, it could hardly be done analytically, and is beyond the scope of this Letter.

Keeping in (5) only the exponential factor, i.e., for $d = 0$, $\zeta_j \neq 0$, we come to generalized Harris current structures with arbitrary energy distribution functions

$$\hat{F}_j(p, p_z + e_jA/c) = \hat{F}_{j0}(p)\exp[(cp_z/e_j + A)/A_{0j}]. \tag{15}$$

Here $A_{0j} = m_jc^2/e_j\zeta_j$; below they all are assumed equal, $A_{0j} = A_0 > 0$. Anisotropy of this distribution is the same throughout the structure, unlike the cases with $d \neq 0$, and the Grad-Shafranov equation takes the form

$$\Delta_{xy}A = -\alpha\exp(A/A_0), \tag{16}$$

$$\alpha = \sum_j \frac{8\pi^2 e_j^3 A_0^2}{m_j c^3} \int\left[\left(\frac{cp}{e_jA_0} - 1\right)\exp\left(\frac{cp}{e_jA_0}\right) + \left(\frac{cp}{e_jA_0} + 1\right)\exp\left(-\frac{cp}{e_jA_0}\right)\right]\frac{p}{\gamma_j}\hat{F}_j(p)dp. \tag{17}$$

It has the following planar solution of the Harris type:

$$A = -2A_0\ln\cosh\kappa x, \qquad B_y = 2A_0\kappa\tanh\kappa x, \qquad N_j/N_{j\,\mathrm{max}} = \cosh^{-2}\kappa x, \qquad \kappa = \sqrt{\alpha/2A_0}; \tag{18}$$

see Fig. 1(c). To obtain the classical (Maxwellian) Harris sheet from (15)–(18), one needs only to consider the nonrelativistic case and set $\hat{F}_{j0}(p) \propto \exp(-p^2/\mathrm{const})$ [4,5].

Equation (16) has very simple cylindrical solution as well (generalized Bennett pinch) (cf. [6])

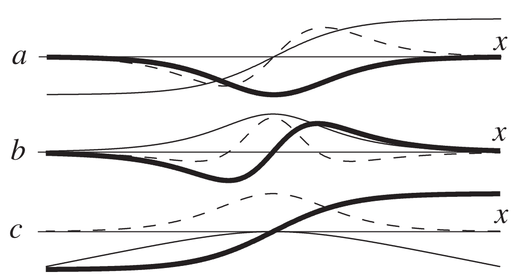

FIG. 1. Profiles of magnetic field B_y (thick solid line), current density $j_z = -(c/4\pi)dB_y/dx$ (dashed line), and vector potential A (thin solid line) for solutions (11), (13), and (18).

$$A = -2A_0 \ln[1 + (\kappa\rho/2)^2], \quad B = \frac{A_0 \kappa^2 \rho}{1 + (\kappa\rho/2)^2},$$
$$N_j/N_{j\max} = [1 + (\kappa\rho/2)^2]^{-2}, \quad \kappa = \sqrt{\alpha/2A_0}. \tag{19}$$

Here the current is mostly localized within a cylinder of radius $2\kappa^{-1}$; magnetic field reaches its peak of $B_{\max} = \kappa A_0$ at this radius, and $B \approx 4A_0/\rho$ at $\rho \gg 2\kappa^{-1}$. Particle density far away from the filament decreases as $N_j \propto \rho^{-4}$, unlike the exponential decrease in the planar case (18).

For $d = 2$, $\zeta_j = 0$ the Grad-Shafranov equation becomes linear (we assume for simplicity that $\hat{F}_{j1} \equiv 0$, and also $\hat{F}_{j0} \equiv 0$, as the latter does not affect the solution):

$$\Delta_{xy} A + k^2 A = 0; \quad k^2 = \sum_j \frac{32\pi^2 e_j^2}{3m_j^3 c^4} \int p^4 \hat{F}_{j2}(p) \frac{dp}{\gamma_j}. \tag{20}$$

Its solutions include two-dimensional structures described by sums of spatially harmonic components

$$A = \sum_l A_l \cos(kx \cos\theta_l + ky \sin\theta_l + \varphi_l), \tag{21}$$

where A_l, θ_l, and φ_l are arbitrary constants. A necessary condition for Weibel stability is that gyrofrequency of an essential fraction of the particles is greater than or of the order of the plasma frequency in every region with scale greater than k^{-1}; i.e., the magnetic field is quite strong. Indeed, analysis shows that even for one harmonic component in the solution (21) the ratio, ε_B, of average magnetic field and average particle energy densities can be up to $1/3$ (or, up to $2/3$ in the nonrelativistic case, when particle rest energy is not taken into account).

In particular, in the cylindrically symmetrical solution with radial dependence described by the Bessel function,

$$A = A_{\max} J_0(k\rho), \tag{22}$$

the above mentioned condition is satisfied only up to a certain radius ρ_f, since the magnetic field oscillations decrease as $\rho^{-1/2}$ and $\varepsilon_B \ll 1$ for $\rho \gg \rho_f$. Formally, the latter region cannot be separated from the solution (22) [without taking into account higher order terms $i > 2$ in (7)]. However, only the region $\rho \lesssim \rho_f$, where the particles are magnetized, is of interest as a specific structural element in collisionless self-magnetized plasma.

One of the simplest examples of such a structural element may be constructed in the case $d = 3$ (and $\zeta_j = 0$), when the Grad-Shafranov potential has the form

$$U = U_0(A^3/A_0^3 - A^2/A_0^2), \tag{23}$$

where A_0 and U_0 are calculated according to (7). Then, Eq. (4) gives self-consistent exponentially localized structures, qualitatively similar in 1D and cylindrically symmetrical cases. The latter is too complex for analytic presentation, so we present here only the planar solution:

$$A = A_0 \cosh^{-2}[(U_0/2A_0^2)^{1/2} x]. \tag{24}$$

As in the case of (13), the total current in this sheet (and in corresponding filament) is zero, the spatial scale can be either less than or greater than the typical gyroradius of particles, the degree of anisotropy is less than or of the order of 2, and the particle energy within the current-carrying region is greater than or of the order of the magnetic field energy.

Conclusions.—We analytically find a new broad class of neutral current configurations in collisionless multicomponent plasma, relativistic or not. It allows for a functional freedom to choose PDFs and spatial profiles of corresponding magnetic field, greatly overcovering the majority of known results (see, e.g., [4–11]). We describe their general properties and obtain a number of new planar, and cylindrical magnetostatic structures (localized and delocalized), which are self-consistent with inhomogeneous anisotropic particle distributions with essentially arbitrary energy profiles. Such solutions are valuable for analysis of physical properties of current sheets and filaments, including their synchrotron radiation, the current localization and value, the degree of anisotropy of particle distribution, and possible equipartition of particle and magnetic energy densities. In particular, the results may be applied to the problem of long-living magnetic field in collisionless plasma for various dynamical structures, e.g., shocks, winds, jets, and accretion disks.

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