A Logical Model of Nash Bargaining Solution

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Abstract

This paper presents a logical extension of Nash's Cooperative Bargaining Theory. We introduce a concept of entrenchment measurement, which maps propositions to real numbers, as a vehicle to represent agent's belief states and attitudes towards bargaining situations. We show that Nash's bargaining solution can be restated in terms of bargainers belief states. Negotiable items, bargaining outcomes and conflicting arguments can then be explicitly expressed in propositional logic meanwhile Nash's numerical solution to bargaining problem is still applicable.

1 Introduction

Negotiation or bargaining is a process to settle disputes and reach mutually beneficial agreements. Typical situations of negotiation are characterized by two or more agents who have common interests in cooperating but have conflicting interests in the way of doing so. The outcomes of negotiation depend on agents' attitudes towards their bargaining items and their expectations from the negotiation. The representation of bargainer's attitudes in game theory is implicit via a pair of von Neumann-Morgenstern utility functions [Nash 1950]. The conflicting of interests between bargainers is then implied by certain mathematical conditions of the utility functions. However, agents' attitudes can be more explicitly described in terms of logic. Negotiable items, conflicting claims, arguments of disputation and negotiation protocols are all expressible by logical statements [Sycara 1990][Sierra *et al.* 1997][Kraus *et al.* 1998][Wooldridge and Parsons 2000][Meyer *et al.* 2004a][Zhang *et al.* 2004]. One difficulty of logical frameworks of negotiation is that quantitative criteria are harder to be applied to bargaining processes, which seem necessary in the analysis of negotiation situations. This paper attempts to bridge the gap between the quantitative analysis of game-theory and qualitative description and reasoning of logic.

One of our basic assumption to negotiation is that outcomes of negotiation are determined by bargainers' belief states. Representation of agent's belief states is normally by a set of beliefs and a revision operator over the belief set in terms of the theory of belief revision [Gärdenfors 1988]. Alternatively, belief states can also be specified by an ordering over propositions, referred to as *epistemic entrenchment* [Gärdenfors 1988]. It is well-known that such two modelings of belief states is equivalent. The main finding of the paper is that *the ordering of epistemic entrenchment can be extended into a numerical measurement over propositions so that a von Neumann-Morgenstern utility function is induced*. With this quantitative representation of bargainers' belief states, Nash's numerical requirements on bargaining solutions can be restated in terms of logical properties. Bargaining items, negotiation outcomes and conflicting arguments can be also explicitly expressed via the extended concept of belief states.

The structure of the paper is the following. After a short motivation in this section, we extend the AGM's epistemic entrenchment to a map from propositions to real numbers. Bargaining problem is then defined as a pair of bargainers' belief states and outcomes of bargaining is specified by a pair of concessions made by two bargainers. In order to induce a von Neumann-Morgenstern utility function over negotiation outcomes, the standard game-theoretical process of randomization over possible outcomes is applied. When all these done, Nash's bargaining solution is ready to be restated in terms of bargainers belief states. Finally we argue that our solution to bargaining problem can not be replaced by the belief revision based solution even though they share many common logical properties.

We will work on a propositional languages $\mathcal{L}$ with finite many propositional variables. A set K of sentences in $\mathcal{L}$ is said to be *logically closed* or to be a *belief set* if $K = Cn(K)$, where $Cn(K) = \{\varphi \in \mathcal{L} : K \vdash \varphi\}$. If F_1, F_2 are two sets of sentences, $F_1 + F_2$ denotes $Cn(F_1 \cup F_2)$. We shall use $\top$ and $\bot$ to denote the propositional constants **true** and **false**, respectively.

2 Entrenchment Measurement

In the theory of belief revision, the belief state of an agent consists of a belief set - *the beliefs held by the agent* - and a revision operator - *which takes a piece of new information as input and outputs a new belief set*. We know that the belief state is uniquely determined by the agent's epistemic entrenchment - *an ordering on beliefs* [Gärdenfors 1988]. However, we shall argue that such an ordering is not enough to specify agent's belief states in bargaining. We will extend the concept of epistemic entrenchment into a more general concept so that a quantitative measurement can be used to capture bargainers' attitudes towards their bargaining items.

Definition 1 *An entrenchment measure is a pair of* (ρ, e) *where* e *is a real number and* ρ *is a function from* $\mathcal{L}$ *to the real number set* $\Re$ *which satisfies the following condition:*

(LC) *If* $\varphi_1, \cdots, \varphi_n \vdash \psi$, $\min\{\rho(\varphi_1), \cdots, \rho(\varphi_n)\} \leq \rho(\psi)$.

Let $K = \{\varphi \in \mathcal{L} : \rho(\varphi) \geq e\}$, called the *derived belief set of the entrenchment measure*. It is easy to see that K is logically closed, *i.e.,* $K = Cn(K)$, so it is a belief set. Obviously, K is consistent iff $\rho(\perp) < e$. We shall call ρ the *belief state* of an agent and e is the *bottom line* of the belief state.

The following proposition shows that an entrenchment measure uniquely determines an epistemic entrenchment, thus also determines a unique belief revision function over the derived belief set.

Proposition 1 *Let* (ρ, e) *be an entrenchment measure and* K *be its derived belief set. Define an ordering* $\leq$ *over* $\mathcal{L}$ *as follows: for any* $\varphi, \psi \in \mathcal{L}$,

1. *if* $\vdash \varphi$, $\varphi \leq \psi$ *iff* $\vdash \psi$;[1]
2. *if* $\not\vdash \varphi$ *and* $\varphi \in K$, $\varphi \leq \psi$ *iff* $\rho(\varphi) \leq \rho(\psi)$;
3. *if* $\varphi \notin K$, $\varphi \leq \psi$ *for any* ψ.

Then $\leq$ *satisfies all postulates (EE1)-(EE5) for epistemic entrenchment (See [Gärdenfors 1988]).*

Based on the result, we can define a belief revision operator $*$ that satisfies all AGM postulates: $\psi \in K * \varphi$ if and only if $\psi \in K + \{\varphi\}$ and either $\rho(\neg\varphi) < \rho(\neg\varphi \vee \psi)$ or $\vdash \neg\varphi \vee \psi$. We shall call $*$ the *revision function derived from* (ρ, e).

Since our language is finite, it is easy to define a multiple revision operator $\otimes$ by the singleton revision function:

$$K \otimes F = K * (\wedge F)$$

where $(\wedge F)$ is the conjunction of all the sentences in F. This will facilitate the comparison of our results with [Meyer *et al.* 2004b] and [Zhang *et al.* 2004].

We remark that any epistemic entrenchment ordering can be extended to an entrenchment measure while keeping the ordering on sentences. Clearly such an entrenchment measure is not unique. An entrenchment measure contains richer structure than an epistemic entrenchment ordering. Interestingly, the extra structure of entrenchment measures cannot be captured by belief revision operations.

Theorem 1 (Independence of Monotone Transformations) *Let* τ *be a strictly increasing monotonic transformation over* $\Re$, *i.e., for any* $x, y \in \Re$, $x < y$ *if and only if* $\tau(x) < \tau(y)$. *Let* $*$ *and* $*'$ *be the derived revision functions from* (ρ, e) *and* $(\tau \circ \rho, \tau(e))$, *respectively*[2]. *Then for any* φ, $K * \varphi = K *' \varphi$, *where* K *is the belief set of* (ρ, e).

3 Bargaining Problem

We shall restrict us to the bargaining situations where only two parties are involved. A bargaining situation is a situation in which both bargainers bring their negotiable items to the negotiation table expecting to reach a common agreement. Whenever a conflict presents, concessions form one or both agents are required in order to reach a compromise. Formally, a *bargaining situation* or a *bargaining game* is a pair, $((\rho_1, e_1), (\rho_2, e_2))$, of entrenchment measures over $\mathcal{L}$ such that each belief set derived by the entrenchment measures is logically consistent, *i.e.* $K_i = \{\varphi \in \mathcal{L} : \rho_i(\varphi) \geq e_i\} \not\vdash \perp$.

[1] Thanks to the anonymous reviewers for pointing out an error in the original version.

[2] $\circ$ is the composition operator on real number functions.

Example 1 *Consider the following negotiation scenario: a buyer (agent 1) negotiates with a seller (agent 2) about the price of a product. The buyer's reserve price is less than \$16 and the seller's reserve price is no less than \$10. Let's discretize the problem as follows:*

$p_1 =\{$ *the price is less than \$10* $\}$.
$p_2 =\{$ *the price is less than \$12* $\}$.
$p_3 =\{$ *the price is less than \$14* $\}$.
$p_4 =\{$ *the price is less than \$16* $\}$.

Assume that the entrenchment measure of each agent is respectively the following[3]*:*

$\rho_1(\top) = \rho_1(p_1 \to p_2) = \rho_1(p_2 \to p_3) = \rho_1(p_3 \to p_4) = \rho_1(p_4) = 4$, $\rho_1(p_3) = 3$, $\rho_1(p_2) = 2$, $\rho_1(p_1) = 1$, $\rho_1(\neg p_j) = 0$ *for* $j = 1, \cdots, 4$.

$\rho_2(\top) = \rho_2(p_1 \to p_2) = \rho_2(p_2 \to p_3) = \rho_2(p_3 \to p_4) = \rho_2(\neg p_1) = 4$, $\rho_2(\neg p_2) = 3$, $\rho_2(\neg p_3) = 2$, $\rho_2(\neg p_4) = 1$, $\rho_2(p_j) = 0$ *for* $j = 1, \cdots, 4$.

Suppose that both sides set their bottom lines to be 1. Then $K_1 = Cn(\{p_1, p_2, p_3, p_4\})$ *and* $K_2 = Cn(\{\neg p_1, \neg p_2, \neg p_3, \neg p_4\})$. ¶

Note that the values of entrenchment measures reflect the degree of entrenchment on negotiable items of each agent rather than the agent's preference. For instance, the buyer in above example entrenches p_4 more firmly than p_1 even though p_1 is more profitable because p_4 is much easier to keep than p_1[4].

Definition 2 *A bargaining game* $B = ((\rho_1, e_1), (\rho_2, e_2))$ *is said to be a subgame of another bargaining game* $B' = ((\rho'_1, e'_1), (\rho'_2, e'_2))$, *denoted by* $B \sqsubseteq B'$, *if, for each* $i = 1, 2$,

1. $\rho_i(\varphi) = \rho'_i(\varphi)$ *for any* $\varphi \in K_i$;
2. $e_i \geq e'_i$.

Note that $B \sqsubseteq B'$ implies $K_i \subseteq K'_i$ $(i = 1, 2)$. Therefore B can be viewed as a kind of *conservative* concession to each other in the sense that each player gives up part of their initial demands by raising their negotiable bottom lines while keeping their entrenchment measurements on negotiable items unchanged.

4 Possible Outcomes of Bargaining

We now consider the possible outcomes of a bargaining game. Apparently the outcome of a bargaining process is the agreements that are reached in the negotiation. In most situations when the demands of two agents conflict, concessions from players are required. The finial negotiated outcome then consists of the combination of those demands that each agent chooses to retain. In this section we shall first consider all possible compromises a bargaining process could reach and then combine these outcomes in a probabilistic fashion.

[3] To make the presentation simple, we do not give the entrenchment measurements for the whole language. The readers are invited to complete the measurements with reasonable ranking for the rest of sentences.

[4] This exactly follows the original explanation of epistemic entrenchment. For instance, φ is no less entrenched than $\varphi \wedge \psi$ because φ is easier to retain than $\varphi \wedge \psi$, so keeping φ is less costly than $\varphi \wedge \psi$ from the information economics point of view.

4.1 Pure deals of bargaining

Following [Meyer *et al.* 2004a] and [Zhang *et al.* 2004], we shall define an outcome of bargaining as a pair of subsets of two agents' bargaining item sets, interpreted as the concessions made by both agents. Considering the real-life bargaining, bargainers normally intend to keep their highly entrenched negotiable items and to give up those less entrenched items if necessary. This idea leads to the following definition of possible bargaining outcomes.

Definition 3 *Let $B = ((\rho_1, e_1), (\rho_2, e_2))$ be a bargaining game. K_i ($i = 1, 2$) is the belief set derived by (ρ_i, e_i). A pure deal of B is a pair (D_1, D_2) satisfying:*

1. $D_i \subseteq K_i$
2. $\top \in D_i$
3. $D_1 \cup D_2 \not\vdash \bot$
4. *if* $\varphi \in K_i$ *and* $\exists \psi \in D_i(\rho_i(\varphi) \geq \rho_i(\psi))$, *then* $\varphi \in D_i$

where $i = 1, 2$. The set of all pure deals of B is denoted by $\Omega(B)$.

There is little need to comment on the first three conditions. They are just the statements on the type of compromises desired. The last one expresses the idea that the procedure of concession by both agents is in the order from lower entrenched items to higher entrenched items: *higher entrenched items should be retained before any lower entrenched items are considered to be given up*. Combining this condition with the second implies the *individual rationality* – if an agent entrench an item as firmly as a tautology, this item will never be given up in any bargaining situation. Such an item is called *reserved item*.

We are now ready to define the utility of pure deals for each player. We will evaluate an agent's gain from a deal by *measuring the length of the remained negotiable items of the agent*.

Definition 4 *For any $D \in \Omega(B)$, define the utility of the deal, denoted by $u(D) = (u_1(D), u_2(D))$, as follows:*
$u_i(D) = \rho_i(\top) - \min\{\rho_i(\varphi) : \varphi \in D_i\}$

Example 2 *Consider Example 1 again. All possible pure deals of the bargaining game are:*
$D^1 = (Cn(\{p_2, p_3, p_4\}), Cn(\{\neg p_1\}))$
$D^2 = (Cn(\{p_3, p_4\}), Cn(\{\neg p_1, \neg p_2\}))$
$D^3 = (Cn(\{p_4\}), Cn(\{\neg p_1, \neg p_2, \neg p_3\}))$
The utility values of the deals are:
$u(D^1) = (2, 0)$, $u(D^2) = (1, 1)$, $u(D^3) = (1, 0)$.
Note that $(Cn(\{p_1, p_2, p_3, p_4\}), Cn(\emptyset))$ and $(Cn(\emptyset), Cn(\{\neg p_1, \neg p_2, \neg p_3, \neg p_4\}))$ are not pure deals because p_4 and $\neg p_1$ are reserved items of agent 1 and 2, respectively. ¶

Given a bargaining game B, if $\Omega(B)$ is nonempty, it must contain two extreme cases of pure deals, called *all-or-nothing deals*:

$$\overleftarrow{D} = (K_1, R_2) \text{ and } \overrightarrow{D} = (R_1, K_2)$$

where $R_i = \{\varphi \in K_i : \rho_i(\varphi) = \rho_i(\top)\}$, i.e., the reserved items of agent i.

The utilities of the all-or-nothing deals give the up-bound and low-bound of utility values for each agent:
$u(\overleftarrow{D}) = (\rho_1(\top) - e_1, 0)$ and $u(\overrightarrow{D}) = (0, \rho_2(\top) - e_2)$.

We say that a pure deal D *dominates* another pure deal D', denoted by $D \succ D'$, if either $D_1 \supset D'_1 \wedge D_2 \supseteq D'_2$ or $D_1 \supseteq D'_1 \wedge D_2 \supset D'_2$. D *weekly dominates* D', denoted by $D \succeq D'$, if $D_1 \supseteq D'_1 \wedge D_2 \supseteq D'_2$.

A pure deal D of a bargaining game B is said to be *Pareto optimal* over $\Omega(B)$ if there does not exist another pure deal in $\Omega(B)$ which dominates D.

The following theorem shows that if a negotiation function takes a Pareto optimal pure deal as its outcome, then it satisfies all logical requirements put on negotiation functions by [Meyer *et al.* 2004a] and all the postulates except *Iteration* proposed in [Zhang *et al.* 2004].

Theorem 2 *For any $(D_1, D_2) \in \Omega(B)$, if it is Pareto optimal, then it satisfies all the postulates introduced by [Meyer et al. 2004a] for so-called concession-permissible deals[5]:*

(C1) $D_i = Cn(D_i)$ *for* $i = 1, 2$.
(C2) $D_i \subseteq K_i$ *for* $i = 1, 2$.
(C3) *If* $K_1 \cup K_2$ *is consistent,* $D_i = K_i$ *for* $i = 1, 2$.
(C4) $D_1 \cup D_2$ *is consistent.*
(C5) *If either* $K_1 \cup D_2$ *or* $K_2 \cup D_1$ *is consistent, then* $K_1 \cap K_2 \subseteq Cn(D_1 \cup D_2)$.

Moreover, if $\otimes_1$ and $\otimes_2$ are the belief revision operators derived by the two entrenchment measures of B, respectively, then D satisfies the following conditions introduced by [Zhang et al. 2004]:

1. $Cn(D_1 \cup D_2) = (K_1 \otimes_1 D_2) \cap (K_2 \otimes_2 D_1)$.
2. $K_1 \cap (K_2 \otimes_2 D_1) \subseteq K_1 \otimes_1 D_2$ *and* $K_2 \cap (K_1 \otimes_1 D_2) \subseteq K_2 \otimes_2 D_1$.

The following lemma shows that the utility value of a pure deal uniquely determines the deal. The fact will be used in the next section.

Lemma 1 *For $D, D' \in \Omega(B)$,*

$$u(D) = u(D') \text{ iff } D = D'.$$

4.2 Mixed deals

In many cases, a bargaining game could end in a tie that no single agreement reaches. A standard method to deal with the problem in game theory is to play a lottery to determine the outcome of a bargaining game. Following Nash's utility theory, we call $rD' + (1-r)D''$ a *mixed deal* if D' and D'' are two pure deals or mixed deals, meaning the lottery which has the two possible outcomes, D' and D'', with probabilities r and $1-r$, respectively. The set of all mixed deals is denoted by $\overline{\Omega}(B)$[6].

For any mixed deal $D = rD' + (1-r)D''$, the utility of the deal can be defined as

$$u(D) = ru(D') + (1-r)u(D'').$$

The concept of domination then can be extended to mixed deals: *for any mixed deals $D, D' \in \overline{\Omega}(B)$,*

$$D \succ (\succeq)\ D' \text{ iff } u(D) > (\geq)\ u(D').^7$$

[5] In [Meyer *et al.* 2004a], six postulates were introduced to specify the rationality of concessionary. However the sixth, $(C6)$, holds trivially provided K_1 and K_2 are logically closed.

[6] More precisely, $\overline{\Omega}(B)$ is a conservative extension of $\Omega(B)$ by allowing lottery deals in the form $rD' + (1-r)D''$.

[7] For any $x, y \in \Re^2$, $x > y$ denotes $x_1 > y_1 \wedge x_2 \geq y_2$ or $x_1 \geq y_1 \wedge x_2 > y_2$. $x \geq y$ denotes that $x_i \geq y_i$.

Since the language we consider is finite, the set of pure deals of a bargaining game is also finite. The following lemma is easy to prove according to the construction of mixed deals.

Lemma 2 *Let $D^1, \cdots, D^m \in \Omega(B)$ be all pure deals of a bargaining game B. Any mixed deal $D \in \overline{\Omega}(B)$ can be expressed as a linear combination of $D^1, \cdots, D^m$, i.e., there exist real numbers, $\alpha_1, \cdots, \alpha_m$, satisfying*
1. *$\alpha_j \geq 0, j = 1, \cdots, m$;*
2. *$\sum \alpha_j = 1$;*
3. *$D = \sum \alpha_j D^j$.*

Moreover, $u(D) = \sum \alpha_j u(D^j)$.

In other words, $\overline{\Omega}(B)$ is a convex hull of $\Omega(B)$ if we identify a deal with its utility pair. Obviously, the utility function over mixed deals defined above is a pair of von Neumann-Morgenstern utility functions.

We have seen from Lemma 1 that a pure deal can be uniquely determined by its utility values. However, a deal can be duplicately represented in the form of mixed deals. For instance, any pure deal D can be represented as $rD + (1-r)D$. To solve the problem we can either use Lemma 2 to obtain a unique representation of a deal by orthogonalizing representation of pure deals (or utility pairs) or apply equivalence classification by defining an equivalent relation: $D \sim D'$ iff $u(D) = u(D')$. To avoid too much complexity, we omit the details of the mathematical treatment but simply assume that for any $D, D' \in \overline{\Omega}(B)$, $D \equiv D'$ iff $u(D) = u(D')$.

5 Bargaining solution

The target of bargaining theory is to find theoretical predictions of what agreement, if any, will be reached by the bargainers. John Nash in his path-breaking papers introduced an axiomatic method which permitted a unique feasible outcome to be selected as the solution of a given bargaining problem[Nash 1950][Nash 1953]. He formulated a list of properties, or axioms, that he thought solution should satisfy, and established the existence of a unique solution satisfying all the axioms. In this section we shall give a brief summary of Nash's theory and then extend Nash's result in terms of bargainer's belief states.

5.1 The Nash bargaining solution

A bargaining game in Nash's terminology is a pair (Ω, d), where $\Omega \subseteq \Re^2$ is a set of possible utility pairs and $d \in \Re^2$ the disagreement point. It was assumed that Ω is convex and compact. All bargaining games satisfying the conditions are collected in the set $\mathcal{B}$. A bargaining solution is then a function $f : \mathcal{B} \rightarrow \Re^2$ such that $f(\Omega, d) \in \Omega$. Nash proposed that a bargaining solution should possess the following four properties [Nash 1953][8]:

PO (Pareto-Optimality) There does not exist $(u_1, u_2) \in \Omega$ such that $(u_1, u_2) > f(\Omega, d)$.

SYM (*Symmetry*) If (Ω, d) is symmetric, then $f_1(\Omega, d) = f_2(\Omega, d)$.

IEUR (*Independence of Equivalent Utility Representation*) Let (Ω', d') be obtained from (Ω, d) by a strictly increasing affine transformation $\tau(x) = (\tau_1(x_1), \tau_2(x_2))$[9]. Then $f(\Omega', d') = \tau(f(\Omega, d))$

IIA (Independence of Irrelevant Alternatives) If $\Omega' \subseteq \Omega$ and $f(\Omega, d) \in \Omega'$, then $f(\Omega', d) = f(\Omega, d)$.

Theorem 3 (Nash's Theorem) *A bargaining solution f satisfies all the four properties if and only if $f = F$, where F is defined by $F(\Omega, d) = u^*$ where u^* solves the maximization problem* $\max\{u_1 u_2 : u = (u_1, u_2) \in \Omega, u \geq d\}$.

5.2 Bargaining solution with belief states

We might have noticed that a solution to bargaining problem is to provide a general model of the bargaining process rather than a selection function that picks up a point as outcome from all feasible solutions for a particular bargaining situation. A surprise with Nash's theorem is that the axioms shown above seem so natural that any negotiation process should follow whereas the solution is so specific that it uniquely determines an outcome in any bargaining situation. The question now is that whether we can restate these plausible requirements in terms of bargainers' belief states instead of utility functions.

Given a finite propositional language, let $\mathcal{B}$ denote all bargaining games in the same propositional language *with nonempty set of deals*. A *bargaining solution* is defined as a function f which maps a bargaining game $B \in \mathcal{B}$ to a mixed deal in $\overline{\Omega}(B)$. Following Nash's approach, we propose the following axioms:

PO *(Pareto Optimality)* There does not exist a deal $D \in \overline{\Omega}(B)$ such that $D \succ f(B)$.

MD (*Midpoint Domination*) $f(B) \succeq \frac{1}{2}\overleftarrow{D} + \frac{1}{2}\overrightarrow{D}$.

IEEM (*Independence of Equivalent Entrenchment Measurement*) Given a bargaining game $B = ((\rho_1, e_1), (\rho_2, e_2))$, for any strictly increasing affine transformations τ,
$$f(\tau(B)) \equiv f(B)$$
where $\tau(B) = ((\tau_1 \circ \rho_1, \tau_1(e_1)), (\tau_2 \circ \rho_2, \tau_2(e_2)))$.

IIA (*Independence of Irrelevant Alternatives*) If $B' \sqsubseteq B$ and $f(B) \in \overline{\Omega}(B')$, then $f(B') \equiv f(B)$.

We can easily see the similarities between the above axioms and Nash's. The differences are also observable. Firstly, the above axioms are presented in terms of bargainers' belief states, *i.e.*, the entrenchment measurement of two agents. This is significant not only because bargainers' belief states contains the logical implication of negotiation demands but also because the entrenchment measurements are more obtainable than the utility functions over all possible outcomes(combinatorial explosion can happen with possible outcomes). Secondly, our axioms are more intuitive and easier to verify. For instance, the MD, which is a replacement of Nash's SYM, expresses the idea that a minimal amount of cooperation among the players should allow them to do at least as well as the average of their preferred outcomes. When no easy agreement on a deterministic outcome is obtained, the tossing of a coin is always an option to determine a winner

[8] See more modern treatments and detailed explanation from [Roth 1979][Owen 1995][Houba and Bolt 2002][Napel 2002].

[9] A strictly increasing affine transformation is a pair of linear real functions with positive slope, i.e., $\tau_i(x) = a_i x + b_i$ where $a_i > 0$.

of the negotiation (see a counterpart of MD in game theory in [Thomson 1994]). Additionally, our IIA is much weaker than Nash's IIA because not every convex subset of $\Re^2$ can be a set of utility pairs of mixed deals.

Exactly like the Nash's theorem, there exists a unique bargaining solution possessing the above properties.

Theorem 4 *There is a unique solution f which satisfies PO, IEEM, MD and IIA. Moreover, the solution is the function $F(B) \equiv D^*$ such that $D^* \in \overline{\Omega}(B)$ and $u_1(D^*)u_2(D^*) = \max_{D \in \overline{\Omega}(B)} (u_1(D)u_2(D))$.*

Proof: The whole proof of the theorem is quite lengthy. To save space, we shall only present a sketch of the second half of the proof. We shall prove that if f is a bargaining solution possessing PO,IEEM, MD and IIA, then $f = F$.

Given a bargaining game $B = ((\rho_1, e_1), (\rho_2, e_2))$, let K_i be the derived belief sets of B. In the case that $K_1 \cup K_2$ is consistent, it is easy to show that $f(B) \equiv F(B) \equiv (K_1, K_2)$. So we assume that $K_1 \cup K_2$ is inconsistent.

Let $D^* \equiv F(B)$. Suppose that B' is the bargaining game derived from B by changing its entrenchment measurement with a strictly increasing affine transformation τ so that $u'(D^*) = (\frac{1}{2}, \frac{1}{2})$. By IEEM, $f(B) \equiv D^*$ iff $f(B') \equiv D^*$. Since F satisfies IEEM, D^* also maximizes $u'_1(D)u'_2(D)$ over $\overline{\Omega}(B')$. Therefore we are left to prove $f(B') \equiv D^*$.

Let $B' = ((\rho'_1, e'_1), (\rho'_2, e'_2))$. Following Nash's argument, it is not hard to verify that for any $D \in \overline{\Omega}(B')$, $(u'_1(D), u'_2(D)) \in \{(x, y) : x \geq 0, y \geq 0 \text{ and } x + y \leq 1\}$ because $\overline{\Omega}(B')$ is convex. Particularly, $\rho'_i(\top) - e'_i \leq 1$.

Since $K_1 \cup K_2$ is inconsistent and $\Omega(B)$ is nonempty, we can pick up two Pareto optimal pure deals from $\Omega(B')$ in the form (K_1, U_2) and (U_1, K_2). Construct a game $B'' = ((\rho''_1, e''_1), (\rho''_2, e''_2))$ as following: for each $i = 1, 2$,

1. for any $\varphi \in K_i$, $\rho''_i(\varphi) = \rho'_i(\varphi)$;
2. for any $\varphi \in Cn(K_i \cup \{\neg(\wedge U_i)\}) \backslash K_i$, $\rho''_i(\varphi) = \rho'_i(\top) - 1$;
3. $e''_i = \rho'_i(\top) - 1$.

It is easy to prove that B' is a subgame of B'' and $K''_i = K_i \cup \{\neg(\wedge U_i)\}$ $(i = 1, 2)$. According to MD, $u''(f(B'')) \geq u''(\frac{1}{2}\overleftarrow{D}'' + \frac{1}{2}\overrightarrow{D}'') = (\frac{1}{2}, \frac{1}{2})$. On the other hand, for any mixed deal $D \in \overline{\Omega}(B'')$, by Lemma 2, $D \equiv \sum \alpha_j D^j$, where $D^j \in \Omega(B'')$. For each j, if both $D^j_i \subseteq K_i$ $(i = 1, 2)$, then $u''(D^j) = u'(D^j)$; otherwise either $D^j = \overleftarrow{D}''$ or $D^j = \overrightarrow{D}''$, which implies $u''_1(D^j) + u''_2(D^j) = 1$. Thus $u''_1(D) + u''_2(D) \leq 1$. This implies $u''_1(f(B'')) + u''_2(f(B'')) \leq 1$. Therefore $u''(f(B'')) = (\frac{1}{2}, \frac{1}{2})$. It follows that $f(B'')$ is expressible in B', or $f(B'') \in \overline{\Omega}(B')$. According to IIA, $f(B'') \equiv f(B')$. We conclude that $f(B) \equiv f(B') \equiv D^* \equiv F(B)$. ¶

With the theorem we can easily see that the solution to the bargaining situation described in Example 1 is the equivalent class of the pure deal $D^2 = (Cn(\{p_3, p_4\}), Cn(\{\neg p_1, \neg p_2\}))$, under which both agents agree that the price of the product is less than \$14 but no less than \$12. Note that a more refined discretization of the problem might give a mixed deal as the solution of the bargaining game.

The uniqueness of bargaining solution depends on the assumption that both bargainers agree to randomize between outcomes, *i.e.*, whenever the solution of a bargaining game gives a mixed deal, a lottery will be played to decide which pure deal will be the agreement. However, the assumption is not generally applicable to most of real-life bargaining. In this case, we could use a pure deal that maximizes the product of utility pairs as an approximate prediction of bargaining outcomes even though it is not necessarily unique (Note that such a solution must be Pareto optimal).

6 Discussion

We have observed that a bargaining game consists of two independent entrenchment measurements on negotiable items of two individuals. The actual values of entrenchment measures can be "freely" scaled without changing the solution of the bargaining game. How could these "random" assignments of an entrenchment measure determine a unique solution? Before we answer the question, let's explore another example.

Example 3 [Owen 1995] *Two men are offered \$100 if they can decide how to divide the money. The first man is assumed to be very rich, while the second man has \$50 capital in all. As the first player is very rich, it is assumed that the utility of \$x for the first player, with $0 \leq x \leq 100$, is proportional to x, i.e. $u_1(x) = x$. To the second player, it is assumed that the utility of a sum of money is proportional to its (natural) logarithm, i.e., $u_2(x) = \ln(150 - x) - \ln 50$.*

The game-theoretical solution to the problem is simply maximizing the product of two agents' utilities : $u_1(x)u_2(x) = x \ln(\frac{150-x}{50})$, which gives approximately $x = 57.3$, meaning that the first player receives \$57.3 and the other takes \$43.7.

The logical solution to the problem needs a process of discretization. Let $P(x)$ denote the proposition that the first player receives no less than $\$x$, where x is a *natural number* $(0 \leq x \leq 100)$ [10].

The negotiable item set of each agent is then:

$K_1 = Cn(\{P(x) : 0 \leq x \leq 100\})$ and $K_2 = Cn(\{\neg P(x) : 0 \leq x \leq 100\})$.

Let $C = \{P(x) \rightarrow P(x - 1) : 0 < x \leq 100\}$, which is the common knowledge of both players. The associated entrenchment measure of the first player can be defined as:

$\rho_1(\varphi) = 100$ for each $\varphi \in \{\top\} \cup C$;
$\rho_1(P(x)) = 100 - x$ for any $0 \leq x \leq 100$;
$\rho_1(\varphi) = -1$ for any $\varphi \notin K_1$.

The entrenchment measure of the second player is:

$\rho_2(\varphi) = 100$ for each $\varphi \in \{\top\} \cup C$;
$\rho_2(\neg P(x)) = 100 - \ln(\frac{150-x}{50})$ for any $0 \leq x \leq 100$;
$\rho_1(\varphi) = -1$ for any $\varphi \notin K_2$.

It is easy to verify that the bargaining solution of the game is $(Cn(\{P(57)\} \cup C), Cn(\{\neg P(58)\} \cup C))$, which gives the similar result as the game-theoretical solution.

The result seems strange: the poor man receives less money than the rich man. This is because the poor man is so eager to get money that he highly entrench each single dollars $(98 \leq \rho_2(\neg P(x)) \leq 100$ for any $0 \leq x \leq 100))$ [11]. If the second agent linearly entrenched its gain as the first agent, the

[10] Note that we treat $P(x)$ as a proposition rather than a predicate.

[11] Game-theory explains such a phenomenon as the interpersonal differences of attitudes towards threats or negotiation power of dif-

negotiation would end with 50 to 50. This means that the non-linearity of the second agent's entrenchment measurement results the non-balanced allocation. In other words, the distribution of entrenchment measurement reflects the bargaining power of players. Note that the logarithm function is strictly monotone. Therefore the non-linearity of entrenchment measurement cannot be captured by any belief change operations (see Theorem 1). The AGM's entrenchment ordering is not enough to measure players' bargaining power. In fact, the entrenchment measurement plays two rules in bargaining: *its ordering determines a unique belief revision operation for minimizing the loss of bargainer's negotiable items* and *the distribution of entrenchment values decides the negotiating power of bargainers*.

7 Conclusion and Related Work

We have presented a logical framework for Nash's bargaining solution. To do so, we introduced a concept of entrenchment measure, which maps a proposition to a real number, and used it as a vehicle to convey agents' belief states in different bargaining situations. We have shown that Nash bargaining solution can be restated in terms of the extended concept of belief states. Negotiable items, bargaining outcomes and conflicting arguments can then be expressed in propositional logic meanwhile Nash's numerical solution to bargaining problem is still applicable. This offers a combinative approach of qualitative and quantitative analysis to bargaining situations. As a direct application of the result, a logic-based solution to automated negotiation can be proposed. Given a two-agent bargaining situation, we invite both agents to describe their negotiable items in terms of propositional logic and to provide their own measurements of entrenchment on their negotiable items. All the information comes to an arbitrator (or a server) who will announce the outcome of the negotiation after conducting a process of belief revision and a calculation of Nash's solution. This procedure should be considered to be fair because the result of negotiation is in effect determined by the participants' belief states rather by the arbitrator.

This paper is closely related to the work on the belief revision model of negotiation [Booth 2001][Meyer *et al.* 2004a][Meyer *et al.* 2004b][Zhang *et al.* 2004]. We have proved that if the bargaining solution takes a pure deal as the outcome, it will satisfy almost all assumptions that are put on the negotiation function constructed by belief revision operators(see Theorem 2). However, as we have discussed in the last section, belief revision operation cannot measure bargainers' negotiation power. Therefore a negotiation function defined purely by belief revision operation is unable to determine negotiation outcomes.

The model we presented in the paper is a logical extension of Nash's *cooperative model* of bargaining, which deals with the bargaining situations described by a set of abstract rules. *Non-cooperative models* of bargaining, in contrast, analyze interaction which is based on explicit rules of bargaining [Binmore *et al.* 1992]. A series of work have been done that attempts to represent negotiation procedures in terms of logic [Kraus *et al.* 1998][Sycara 1990][Sierra *et al.* 1997]. It is interesting to see whether the classical results about the relationship between cooperative solutions and noncooperative solutions of bargaining problem can be extended to the logical models of bargaining.

Nash's axiomatic approach to bargaining has reached a high sophistication through the development of last five decades [Roth 1979][Thomson 1994][Napel 2002]. A variety of alternative assumptions have been proposed to derive a given solution concept. It could be a promising research topic to investigate logical properties of these alternative solutions. On the other hand, the logical implication of the distribution of entrenchment measurements is also worthwhile to be further explored.

ferent negotiators [Owen 1995][Houba and Bolt 2002][Fatima *et al.* 2005].

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