Dependency Calculus
Reasoning in a General Point Relation Algebra

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Abstract

The point algebra is a fundamental formal calculus for spatial and temporal reasoning. We present a new generalization that meets all requirements to describe dependencies on networks. Applications range from traffic networks to medical diagnostics. We investigate satisfaction problems, tractable subclassses, embeddings into other relation algebras, and the associated interval algebra.

1 The Dependency Calculus

Reasoning about complex dependencies between events is a crucial task. However, qualitative reasoning has so far concentrated on spatial and temporal issues. In contrast, we present a calculus [Ragni and Scivos, 2005], a proper generalization of the nonlinear relation algebra, created for specific questions of reasoning about consequences.

This algebra, called dependency calculus (*DC*), meets all requirements to describe dependencies in networks. There are two aspects: dependencies of points are described by the point algebra PA_{dc}, and of intervals by the associated interval algebra IA_{dc}. For these we analyze questions concerning the satisfaction problems, and show the correspondence to other relation algebras. For this, we use an isomorphism preserving the tractability of subclasses. This method promises to structure the field of relation algebras and to transfer algebraic aspects and complexity results from one algebra to another.

If we observe pollution in an ecosystem of flowing water, we can draw conclusions about pollution at other points (cf. Fig. 1). If pollution is found at D, F is polluted as well. It

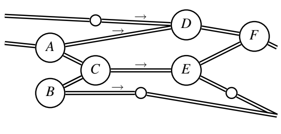

Figure 1: A pipe network. Flow occurs along the "pipes" from left to right. Is there a difference between the pairs (A,B) and (D,E)?

might be caused from a source at A, but not B, C, or E. If C is polluted, D probably is also polluted. Does the same hold true for B and D? No: they have no common point upstream. DC directly represents such differences. For pairs like (C,D) or (D,E), we have the new "fork" relation ($\curlywedge$). The relations "fork", "before", "equal", and "after" are called "dependent". The other case, like (A,B), is called "independent" ($\asymp$). A medical problem is given in Fig. 2.

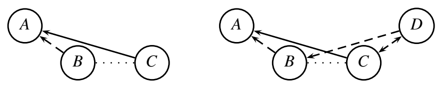

Figure 2: A virus transmittance scheme. Arrows indicate direction of assured, dashed of unassured donorship, dotted lines mean that both persons carry the same virus. No lines means that we do not have prior knowledge. The situation on the left must be incomplete. The existence of a fourth person D accounts for this. PA_{dc} concludes that indeed there was indirect transmittance from D to A.

If x, y are points in a partial order $\langle T, \leq \rangle$, then we define these relations in terms of the partial order as follows:

$x \prec y$	iff	$x \leq y$ and not $y \leq x$.
$x = y$	iff	$x \leq y$ and $y \leq x$.
$x \succ y$	iff	$y \leq x$ and not $x \leq y$.
$x \curlywedge y$	iff	$\exists z\, z \leq y \wedge z \leq x$ and neither $x \leq y$ nor $y \leq x$.
$x \asymp y$	iff	neither $\exists z\, z \leq y \wedge z \leq x$ nor $x \leq y$ nor $y \leq x$.

All relations between nodes in Fig. 1 can be described by these five basic relations.

2 Computational Complexity

Assume that a set of constraints between some points is given. One question is whether this set is consistent. Is it possible to construct a network in which all constraints are satisfied? This problem is called PA_{dc}-SAT. What is the computational effort to construct such a network?

Definition 1. *For two relation algebras Γ, Γ' a homomorphism is a function γ from Γ to Γ' such that γ preserves all operations of the boolean algebra and for all relations R, S:*

- *(converse):* $\gamma(R^{-1}) = \gamma(R)^{-1}$
- *(composition):* $\gamma(R \circ S) = \gamma(R) \circ \gamma(S)$

Definition 2. *For two relation algebras Γ, Γ' a tractability-preserving-homomorphism (tph) is a homomorphism γ from Γ to Γ' such that each subset $\beta \subseteq \Gamma$ is tractable iff $\gamma(\beta) \subseteq \Gamma'$ is tractable. An isomorphic tph is called tpi.*

Lemma 1. *A tpi γ from PA_{dc} to RCC-5 (Fig. 3) is given by:*

$$\gamma: \quad \prec \;\mapsto\; \mathsf{PP} \qquad = \;\mapsto\; \mathsf{EQ} \qquad \succ \;\mapsto\; \mathsf{PP}^{-1}$$
$$\curlywedge \;\mapsto\; \mathsf{PO} \qquad \asymp \;\mapsto\; \mathsf{DR}$$

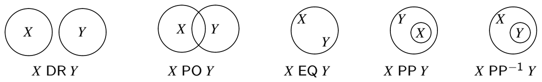

Figure 3: The RCC-5 Relations

Table 1: The tractable subclasses of PA_{dc}.

	τ_{28}	τ_{20}	τ_{17}	τ_{14}
$\perp$	•	•	•	•
$\{\asymp\}$	•	•		
$\{\curlywedge\}$	•	•		
$\{\asymp,\curlywedge\}$	•	•		
$\{\prec\}$	•			•
$\{\asymp,\prec\}$	•	•		
$\{\curlywedge,\prec\}$	•			
$\{\asymp,\curlywedge,\prec\}$	•	•		
$\{\succ\}$	•			•
$\{\asymp,\succ\}$	•	•		
$\{\curlywedge,\succ\}$	•			
$\{\asymp,\curlywedge,\succ\}$	•	•		
$\{\prec,\succ\}$				•
$\{\asymp,\prec,\succ\}$		•		•
$\{\curlywedge,\prec,\succ\}$	•			•
$\{\asymp,\curlywedge,\prec,\succ\}$	•	•		•

The relations including $\{=\}$ are contained in $\tau_{28}, \tau_{20}, \tau_{14}$ iff $R \setminus \{=\}$ is in τ_{28} (τ_{20},τ_{14}). All relations including $\{=\}$ are in τ_{17}.

With [Renz and Nebel, 1999] and Lemma 1, we get:

Theorem 1. *PA_{dc}-SAT is NP-complete.*

Theorem 2. *The four classes $\tau_{28}, \tau_{20}, \tau_{17}, \tau_{14}$ (cf. Tab. 1) are the only maximal subclasses of PA_{dc}.*

3 The Associated Interval Algebra

There are applications in which it is not sufficient to compare single points in a network. For instance, pollution in a pipe network is not restricted to single points but extends to whole sections, and automated planning and project management deal with tasks that span over time intervals.

Definition 3. *An* interval *$I = [s_I, e_I]$ is a pair of points satisfying $s_I \prec e_I$. The interval algebra IA_{dc} is the relation algebra generated by quadruples of relations as basic relations*

$$\mathcal{B} = \left\{ \begin{pmatrix} R_{ss} & R_{se} \\ R_{es} & R_{ee} \end{pmatrix} | R_{ss}, R_{se}, R_{es}, R_{ee} \in \{\prec, =, \succ, \asymp, \curlywedge\} \right\}$$

closed under $\cap, \cup, ^-, \circ, ^{-1}$. For $I = [s,e]$ and $I' = [s', e']$, being in relation $I\, R\, I'$ means $s R_{ss} s'$, $s R_{se} e'$, $e R_{es} s'$, and $e R_{ee} e'$.

IA_{po} is analogically defined, based on $\{\prec, =, \succ, \{\asymp, \curlywedge\}\}$.

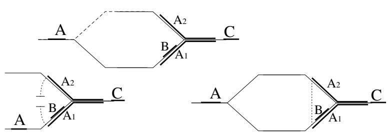

Figure 4: Reasoning with uncertainty. The dashed line indicates that it is unknown if A is before A_2. In contrast to IA_{po}, IA_{dc} concludes such knowledge: Depending on the IA_{dc} relation between A_1 and A_2, the obstruction B can or cannot be bypassed using A_2.

If $R_{ss} = R_{se} = R_{es} = R_{ee} =: R'$, we write (R'). Fig. 4 and Fig. 5 show advantages of the new, finer interval calculus: More situations can be distinguished and more conclusions are possible. The situation given in Fig. 4 is described by:

$$A\,(\prec)\,A_1,\ A_1\,(\prec)\,C,\ A_2\,(\prec)\,C,\ B\,(\prec)\,C,$$

$$B \begin{pmatrix} \{\succ,\prec\} & \prec \\ \{\prec,\prec\} & \prec \end{pmatrix} A_1, \quad B \begin{pmatrix} \{\asymp,\curlywedge\} & \prec \\ \{\asymp,\curlywedge\} & \prec \end{pmatrix} A_2, \quad A_1 \begin{pmatrix} \{\asymp,\curlywedge\} & \prec \\ \prec & = \end{pmatrix} A_2$$

By specifying the latter relation, e.g. to $A_1 \begin{pmatrix} \asymp & \prec \\ \prec & = \end{pmatrix} A_2$, new conclusions can be drawn, in this case, $A\,(\prec)\,A_2$ becomes impossible. This means, if B is an obstacle on the path from A to C via A_1, then there is no alternative route via A_2.

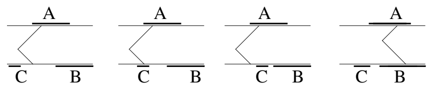

Figure 5: In contrast to IA_{po}, IA_{dc} discerns these cases which are specifications of the composition $A(\{\asymp,\curlywedge\})B \circ B(\succ)C$.

For a set of IA_{dc} constraints, we define IA_{dc}-SAT analogically to PA_{dc}-SAT. How hard is it to decide the satisfiability?

Theorem 3. *IA_{dc}-SAT is NP-hard.*

Definition 4. *For a subset $\mathcal{S} \subseteq PA_{dc}$, a relation R is called $\mathcal{S}$-pointizable if it belongs to the class*

$$\mathcal{P}_{\mathcal{S}} = \left\{ \begin{pmatrix} R_{ss} & R_{se} \\ R_{es} & R_{ee} \end{pmatrix} | R_{es}, R_{se}, R_{es}, R_{ee} \in \mathcal{S} \right\}$$

A gadget for an IA_{dc} relation R is a set of points $p_1, \ldots, p_m$ $(m \geq 4)$ with PA_{dc} relations that are satisfiable iff $[p_1, p_2] R [p_3, p_4]$ is satisfiable. For $\mathcal{S} \subseteq PA_{dc}$, an IA_{dc} relation R is called $\mathcal{S}$-gadgetable ($R \in \mathcal{G}_{\mathcal{S}}$) if all PA_{dc} relations between the endpoints extended with additional points are relations of $\mathcal{S}$. $\mathcal{G}$ is an abbreviation for $\mathcal{G}_{PA_{dc}}$ and $\mathcal{P}$ for $\mathcal{P}_{PA_{dc}}$.

Theorem 4. *$\mathcal{G}, \mathcal{P}$ are intractable subclasses of IA_{dc}.*

Not all gadgetable relations are pointizable. For instance, $(\prec) \cup (\succ)$ is gadgetable. Hence, $\mathcal{B} \subsetneq \mathcal{P} \subsetneq \mathcal{G} \subsetneq IA_{dc}$. If $\mathcal{S}$ is tractable (in PA_{dc}), then $\mathcal{P}_{\mathcal{S}}$ is tractable in IA_{dc}. What about the larger class $\mathcal{G}_{\mathcal{S}}$? Is it still tractable?

Theorem 5. *If $\mathcal{R}$ is a class of $\mathcal{S}$-gadgetable IA_{dc} relations and the satisfiability problem over $\mathcal{S}$ is tractable, then the satisfiability problem over $\mathcal{R}$ is tractable.*

Corollary 1. *$\mathcal{B}, \mathcal{G}_{\tau_{28}}, \mathcal{G}_{\tau_{20}}, \mathcal{G}_{\tau_{17}}, \mathcal{G}_{\tau_{14}} \subsetneq IA_{dc}$ are tractable.*

Corollary 2. *IA_{dc}-SAT is NP-complete.*

4 Conclusion

We presented and investigated a new algebra for reasoning about causal relations. Both PA_{dc} and IA_{dc} are NP-complete, and we identified tractable subclasses. Promising ideas for future work are introducing probabilities to this calculus to model reasoning in Bayesian networks and the temporalization of *DC* for modeling dependencies that vary over time.

References

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