Positive Subsumption in Fuzzy $\mathcal{EL}$ with General t-Norms*

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Abstract

The Description Logic $\mathcal{EL}$ is used to formulate several large biomedical ontologies. Fuzzy extensions of $\mathcal{EL}$ can express the vagueness inherent in many biomedical concepts. We study the reasoning problem of deciding positive subsumption in fuzzy $\mathcal{EL}$ with semantics based on general t-norms. We show that the complexity of this problem depends on the specific t-norm chosen. More precisely, if the t-norm has zero divisors, then the problem is co-NP-hard; otherwise, it can be decided in polynomial time. We also show that the best subsumption degree cannot be computed in polynomial time if the t-norm contains the Łukasiewicz t-norm.

1 Introduction

Description Logics [Baader *et al.*, 2007] (DLs) are a family of knowledge representation formalisms that are specially suited for the representation of the conceptual knowledge of an application domain. In these logics, *concepts* represent sets of individuals in the domain, and *roles* state binary relations between domain elements. From a formal point of view, concepts and roles correspond to unary and binary predicates from first-order logic, respectively. Different DLs are motivated by a trade-off between expressivity and complexity.

$\mathcal{EL}$ is a light-weight description logic capable of expressing conjunctions and existential restrictions, but no negations. In this logic, domain knowledge is expressed through a TBox: a finite set of so-called *general concept inclusion axioms* (GCIs) that express causal relations between concepts. The relevant reasoning task is then to decide *subsumption* between concepts, i.e. whether one concept is always a subclass of another. Computing all the subsumption relations between basic concepts is called *classification*. One of the main features of $\mathcal{EL}$ is that TBoxes can be classified in polynomial time [Baader, 2003; Brandt, 2004]. Its low complexity has been a driving force for the development of very large TBoxes, such as SNOMED CT[1] and the Gene Ontology,[2] for representing knowledge from the biomedical domain. Its success as a knowledge representation language is witnessed by it being the basis for the OWL 2 EL profile of the standard ontology language for the Semantic Web,[3] and the implementation of highly optimized classification tools, such as jcel[4] and ELK.[5]

In their classical form, DLs cannot deal with the imprecision that is endemic to biomedical knowledge. For example, the current version of SNOMED CT defines the disorder "Perinatal Cyanotic Attack" as a cardiovascular disorder occurring in the perinatal period and manifested through cyanosis. This definition depends on two vague notions, namely the *perinatal period*—the period of time around birth—and *cyanosis*—a bluish discoloration of the skin. While it is possible to say that one year after birth is not perinatal, and a few hours from birth is, there is no precise threshold on the end of the perinatal period. However, it makes sense to say that every child is *less* in its perinatal period as time goes by. A similar consideration can be made for skin turning from red to blue in cases of cyanosis. The use of several *degrees of truth* has been proposed for dealing with these gradual changes, as well as other kinds of imprecisions.

Mathematical Fuzzy Logic [Hájek, 2001] generalizes classical logic by allowing all real numbers from the interval $[0, 1]$ to act as truth degrees. It is then possible to express, e.g. that a newborn child is in the perinatal period with degree 1, but a three-week-old belongs to this period only with degree 0.3. In Fuzzy Logic, the interpretation of the logical constructors, such as conjunction, disjunction, and implication, is determined by the choice of a binary *triangular norm* (or t-norm for short). Fuzzy Description Logics combine DLs with Mathematical Fuzzy Logic as a means of formally representing and reasoning with vague conceptual knowledge [Tresp and Molitor, 1998; Straccia, 2001]. So far, research on fuzzy DLs was mainly focused on the expressive side of the spectrum, considering fuzzy extensions of propositionally closed DLs. Unfortunately, in fuzzy DLs with a negation constructor, it is often undecidable whether a set of GCIs is consistent, i.e. non-contradictory [Borgwardt and Peñaloza, 2012b; Cerami and Straccia, 2013].

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[1] http://www.ihtsdo.org/snomed-ct/

[2] http://www.geneontology.org

[3] http://www.w3.org/TR/owl2-overview/

[4] http://jcel.sourceforge.net/

[5] http://www.cs.ox.ac.uk/isg/tools/ELK/

To the best of our knowledge, the only fuzzy extension of $\mathcal{EL}$ that has been studied so far is based on the Gödel t-norm [Mailis *et al.*, 2012].[6] In that paper, the authors describe a polynomial-time algorithm for deciding fuzzy subsumption between concepts. Beyond this tractable case, very little is known about the complexity of subsumption with general t-norms. If we restrict the set of membership degrees to be finite, then subsumption can be decided in exponential time [Borgwardt and Peñaloza, 2013; Bobillo and Straccia, 2013], but for the interval $[0, 1]$ nothing is known, even for more expressive fuzzy DLs in which consistency is decidable [Borgwardt *et al.*, 2012b].

We consider fuzzy extensions of $\mathcal{EL}$ with general t-norm semantics and identify for which cases reasoning remains polynomial. As for the classical case, we are interested in deciding subsumption between concepts. However, the different membership degrees must also be taken into account. For that reason, we consider the *positive subsumption* problem: deciding whether the (fuzzy) implication between two concepts is always greater than 0. Intuitively, a positive subsumption between two fuzzy concepts expresses that they are causally related *to some degree*. We show that the complexity of this problem depends on the properties of the t-norm chosen: if the t-norm has zero divisors, then positive subsumption is co-NP-hard; otherwise, the problem is reducible in linear time to classical subsumption. We also consider the computation problem of finding the best lower bound for the subsumption degree and show that the corresponding decision problem is co-NP-hard if the t-norm contains the Łukasiewicz t-norm.

2 Fuzzy $\mathcal{EL}$

In this section we introduce the fuzzy Description Logic $\otimes$-$\mathcal{EL}$ and its reasoning tasks, along with some of the properties that will be used throughout the paper. The semantics of $\otimes$-$\mathcal{EL}$ depend on the choice of a t-norm $\otimes$.

A *t-norm* is an associative, commutative, and monotone binary operator $\otimes\colon [0, 1] \times [0, 1] \to [0, 1]$ that has unit 1 [Klement *et al.*, 2000]. We consider only *continuous* t-norms, i.e. those that are continuous as a function. Every continuous t-norm defines a unique *residuum* $\Rightarrow\colon [0, 1] \times [0, 1] \to [0, 1]$ where $x \Rightarrow y := \sup\{z \mid x \otimes z \leq y\}$. From this it follows that (i) $x \Rightarrow y = 1$ iff $x \leq y$, and (ii) $1 \Rightarrow y = y$ hold for all $x, y \in [0, 1]$. The *residual negation* $\ominus$ is defined as $\ominus x := x \Rightarrow 0$. Table 1 lists three important continuous t-norms and their residua. It is well known that all other continuous t-norms can be described as the ordinal sums of copies of these three t-norms, as described next.

Let $((a_i, b_i))_{i\in I}$ be a (possibly infinite) family of nonempty, disjoint open subintervals of $[0, 1]$ and $(\otimes_i)_{i\in I}$ be a family of continuous t-norms over the same index set I. The *ordinal sum* of $(((a_i, b_i), \otimes_i))_{i\in I}$ is the t-norm $\otimes$, where

$$x \otimes y := a_i + (b_i - a_i)\left(\tfrac{x-a_i}{b_i-a_i} \otimes_i \tfrac{y-a_i}{b_i-a_i}\right)$$

if $x, y \in [a_i, b_i]$ for some $i \in I$, and $x \otimes y := \min\{x, y\}$ otherwise. This yields a continuous t-norm, whose residuum $x \Rightarrow y$ is given by

$$\begin{cases} 1 & \text{if } x \leq y, \\ a_i + (b_i - a_i)\left(\tfrac{x-a_i}{b_i-a_i} \Rightarrow_i \tfrac{y-a_i}{b_i-a_i}\right) & \text{if } a_i \leq y < x \leq b_i, \\ y & \text{otherwise,} \end{cases}$$

where $\Rightarrow_i$ is the residuum of $\otimes_i$, for each $i \in I$. Intuitively, this means that the t-norm $\otimes$ and its residuum "behave like" $\otimes_i$ and its residuum in each of the intervals $[a_i, b_i]$, and like the Gödel t-norm and residuum everywhere else.

Table 1: The three fundamental continuous t-norms.

Name	t-norm ($x \otimes y$)	residuum ($x \Rightarrow y$)
Gödel	$\min\{x, y\}$	$\begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
Product	$x \cdot y$	$\begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$
Łukasiewicz	$\max\{x + y - 1, 0\}$	$\min\{1 - x + y, 1\}$

Theorem 1 ([Mostert and Shields, 1957]). *Every continuous t-norm is isomorphic to the ordinal sum of copies of the Łukasiewicz and product t-norms.*

Let $\otimes$ be a continuous t-norm and $(((a_i, b_i), \otimes_i))_{i\in I}$ be its representation as ordinal sum given by Theorem 1.[7] Note that the only elements $x \in [0, 1]$ that are *idempotent* w.r.t. $\otimes$, i.e. that satisfy $x \otimes x = x$, are those that are not in (a_i, b_i) for any $i \in I$. Thus, every continuous t-norm except the Gödel t-norm has infinitely many non-idempotent elements. We call $(((a_i, b_i), \otimes_i))_{i\in I}$ the *components* of $\otimes$. We further say that $\otimes$ *contains* a t-norm $\otimes'$ if it has a component of the form $((a_i, b_i), \otimes')$. It *starts with Łukasiewicz* if it has a component of the form $((0, b), \otimes_Ł)$, where $\otimes_Ł$ is the Łukasiewicz t-norm; and is *product-free* if it does not contain the product t-norm.

A value $x \in (0, 1]$ is called a *zero divisor* for a t-norm $\otimes$ if there is a $y \in (0, 1]$ such that $x \otimes y = 0$. It can be shown [Klement *et al.*, 2000] that for every t-norm without zero divisors, the residual negation corresponds to the Gödel negation. More precisely, if $\otimes$ has no zero divisors, then

$$\ominus x = \begin{cases} 0 & \text{if } x > 0, \\ 1 & \text{otherwise.} \end{cases}$$

Of the three continuous t-norms from Table 1, only the Łukasiewicz t-norm has zero divisors: every value $x \in (0, 1)$ is a zero divisor for this t-norm since $1 - x > 0$ and $x \otimes (1 - x) = 0$. In fact, a continuous t-norm can only have zero divisors if it starts with the Łukasiewicz t-norm.

Lemma 2 ([Klement *et al.*, 2000]). *A continuous t-norm has zero divisors iff it starts with the Łukasiewicz t-norm.*

Every continuous t-norm $\otimes$ defines a fuzzy DL $\otimes$-$\mathcal{EL}$. The syntax of $\otimes$-$\mathcal{EL}$ is identical to the one of the classical DL $\mathcal{EL}$, which allows only for the top concept, conjunctions, and existential restrictions. Formally, from two disjoint sets $\mathsf{N_C}$

[6] Mailis *et al.* consider an extension of $\mathcal{EL}$ called $\mathcal{EL}^{++}$.

[7] For ease of presentation, we treat the isomorphism as equality.

and $\mathsf{N_R}$ of *concept names* and *role names*, respectively, $\otimes$-$\mathcal{EL}$-*concepts* are built through the syntactic rule

$$C ::= A \mid \top \mid C_1 \sqcap C_2 \mid \exists r.C$$

where $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$. We use the abbreviation C^n for the n-ary conjunction of a $\otimes$-$\mathcal{EL}$-concept C with itself, i.e.

$$C^n := \bigsqcap_{i=1}^{n} C.$$

A $\otimes$-$\mathcal{EL}$-*TBox* is a finite set of *general concept inclusion axioms* (GCIs) of the form $\langle C \sqsubseteq D \geq q\rangle$, where C, D are $\otimes$-$\mathcal{EL}$-concepts and $q \in [0,1]$. A $\otimes$-$\mathcal{EL}$-TBox is called *crisp* if it contains only GCIs of the form $\langle C \sqsubseteq D \geq 1\rangle$. In the following we will often drop the prefix $\otimes$-$\mathcal{EL}$ and speak simply of, e.g. concepts and TBoxes.

The semantics of this logic extends the classical DL semantics by interpreting concepts and roles as fuzzy sets and fuzzy binary relations, respectively, over some interpretation domain. Given a non-empty domain Δ, a *fuzzy set* is a function $F\colon \Delta \to [0,1]$. The intuition of this function is that an element $\delta \in \Delta$ belongs to the fuzzy set F *with degree* $F(\delta)$.

Formally, an *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty *domain*, and the interpretation function $\cdot^{\mathcal{I}}$ maps each concept name A to a function $A^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \to [0,1]$ and each role name r to a function $r^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$. The interpretation function is extended to $\otimes$-$\mathcal{EL}$-concepts by setting, for every $\delta \in \Delta$,

$$\begin{aligned}\top^{\mathcal{I}}(\delta) &:= 1,\\ (C_1 \sqcap C_2)^{\mathcal{I}}(\delta) &:= C_1^{\mathcal{I}}(\delta) \otimes C_2^{\mathcal{I}}(\delta),\\ (\exists r.C)^{\mathcal{I}}(\delta) &:= \sup_{\gamma \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(\delta,\gamma) \otimes C^{\mathcal{I}}(\gamma).\end{aligned}$$

Such an interpretation $\mathcal{I}$ *satisfies* the GCI $\langle C \sqsubseteq D \geq q\rangle$ iff $\inf_{\delta \in \Delta^{\mathcal{I}}}(C^{\mathcal{I}}(\delta) \Rightarrow D^{\mathcal{I}}(\delta)) \geq q$. It is a *model* of the TBox $\mathcal{T}$ if it satisfies all the GCIs in $\mathcal{T}$. An interpretation $\mathcal{I}$ is called *crisp* if $A^{\mathcal{I}}(\delta) \in \{0,1\}$ and $r^{\mathcal{I}}(\delta,\gamma) \in \{0,1\}$ hold for every concept name A, role name r, and $\delta, \gamma \in \Delta^{\mathcal{I}}$.

Example 3. The concept of perinatal cyanotic attacks (PCA) can be described using the GCI

$$\begin{aligned}\langle \mathsf{PCA} \sqsubseteq\ & \mathsf{CardiovascularDisorder} \sqcap\\ & \exists \mathsf{occurrence.PerinatalPeriod} \sqcap\\ & \exists \mathsf{manifestation.Cyanosis} \geq 1\rangle,\end{aligned}$$

which is in fact very close to the definition found in SNOMED CT. Under the Łukasiewicz t-norm, an individual that belongs to each of the three concepts on the right-hand side with degree 0.7 will belong to PCA with degree at most $0.7 + 0.7 + 0.7 - 2 = 0.1$. While this makes sense from a diagnostic point of view—lesser symptomatic manifestations should yield a weaker diagnosis—, SNOMED CT is meant to *describe* clinical terms, rather than diagnose them. It thus makes sense to divide the previous GCI into the three axioms

$$\begin{aligned}&\langle \mathsf{PCA} \sqsubseteq \mathsf{CardiovascularDisorder} \geq 1\rangle,\\ &\langle \mathsf{PCA} \sqsubseteq \exists \mathsf{occurrence.PerinatalPeriod} \geq 1\rangle, \text{ and}\\ &\langle \mathsf{PCA} \sqsubseteq \exists \mathsf{manifestation.Cyanosis} \geq 1\rangle.\end{aligned}$$

In fuzzy description logics, it is customary to restrict reasoning to so-called *witnessed* interpretations $\mathcal{I}$ only [Hájek, 2005]. Witnessed interpretations are those in which the supremum $(\exists r.C)^{\mathcal{I}}(\delta)$ is in fact a maximum; formally, there is a $\gamma \in \Delta^{\mathcal{I}}$ such that $(\exists r.C)^{\mathcal{I}}(\delta) = r^{\mathcal{I}}(\delta,\gamma) \otimes C^{\mathcal{I}}(\gamma)$. This assumption is often needed to simplify reasoning and was in fact introduced in [Hájek, 2005] to correct the existing algorithm for fuzzy $\mathcal{ALC}$ in [Tresp and Molitor, 1998]. In this paper we do not need this additional assumption; all our results are valid w.r.t. general *and* witnessed semantics.

As in classical $\mathcal{EL}$, every $\otimes$-$\mathcal{EL}$-TBox has the trivial model $\mathcal{I} = (\{\delta\}, \cdot^{\mathcal{I}})$ where $A^{\mathcal{I}}(\delta) = 1$ for every concept name A and $r^{\mathcal{I}}(\delta,\delta) = 1$ for every role name r. Thus, TBox *consistency* is trivial in this logic. We are therefore interested in deciding subsumption between two concepts.

Definition 4. Let $\mathcal{T}$ be a TBox, C, D be two concepts, and $p \in (0,1]$. C is p-*subsumed* by D w.r.t. $\mathcal{T}$ ($C \sqsubseteq_{\mathcal{T}}^{p} D$) if every model of $\mathcal{T}$ satisfies $\langle C \sqsubseteq D \geq p\rangle$. C is *positively subsumed* by D w.r.t. $\mathcal{T}$ ($C \sqsubseteq_{\mathcal{T}}^{>0} D$) if every model $\mathcal{I}$ of $\mathcal{T}$ and every $\delta \in \Delta^{\mathcal{I}}$ satisfies $C^{\mathcal{I}}(\delta) \Rightarrow D^{\mathcal{I}}(\delta) > 0$. The *best subsumption degree* of $C \sqsubseteq D$ w.r.t. $\mathcal{T}$ is

$$\mathsf{bsd}_{\mathcal{T}}(C \sqsubseteq D) := \sup\{p \mid C \sqsubseteq_{\mathcal{T}}^{p} D\}.$$

Clearly, if $\mathsf{bsd}_{\mathcal{T}}(C \sqsubseteq D) > 0$, then $C \sqsubseteq_{\mathcal{T}}^{>0} D$. However, the converse does not necessarily hold (see Example 15).

3 Positive Subsumption

We first analyze the complexity of deciding positive subsumption in $\otimes$-$\mathcal{EL}$, which depends on the existence of zero divisors for the t-norm $\otimes$. In Section 4, we will consider the problem of computing the best subsumption degree.

3.1 T-norms with Zero Divisors

For t-norms with zero divisors, positive subsumption is co-NP-hard. We show this by reducing the NP-hard vertex cover problem [Karp, 1972] to the complement of our problem.

Definition 5. Let $V = \{v_1, \ldots, v_m\}$ be a finite set, and $\mathcal{E}$ a set of subsets of V of cardinality 2. A *vertex cover* is a set $S \subseteq V$ such that $S \cap E \neq \emptyset$ holds for all $E \in \mathcal{E}$. The *vertex cover problem* consists in deciding, given a natural number $k \leq m$, whether there is a vertex cover of cardinality $\leq k$.

Observe that every superset of a vertex cover is also a vertex cover, and thus one can equivalently ask for a vertex cover of size exactly k. Let $\otimes$ be a t-norm with zero divisors, i.e. it starts with the Łukasiewicz t-norm in an interval $[0,b]$ with $0 < b \leq 1$ (see Lemma 2). Given an instance $\mathcal{V} := (V, \mathcal{E}, k)$ of the vertex cover problem, we construct a $\otimes$-$\mathcal{EL}$-TBox $\mathcal{T}_{\mathcal{V}}$ such that $\top$ is *not* positively subsumed by the concept name A w.r.t. $\mathcal{T}_{\mathcal{V}}$ iff there is a vertex cover of size k.

Let $V_i, 0 \leq i \leq m$, be concept names, where $m = |V|$, i.e. we have a concept name V_i for every element $v_i \in V$, and an additional concept name V_0. For each $i, 1 \leq i \leq m$, we set

$$\mathcal{T}_i := \{\langle V_i^{m-k} \sqsubseteq V_i^{m-k+1} \geq 1\rangle,\ \langle \top \sqsubseteq V_i \geq b \cdot \tfrac{m-k-1}{m-k}\rangle\}$$

and $\mathcal{T}_0 := \{\langle \top \sqsubseteq V_0 \geq b \cdot \frac{m-k-1}{m-k}\rangle\}$. Every model $\mathcal{I}$ of $\bigcup_{i=0}^{m} \mathcal{T}_i$ and $\delta \in \Delta^{\mathcal{I}}$ satisfies that $V_0^{\mathcal{I}}(\delta) \geq b \cdot \frac{m-k-1}{m-k}$ and

$V_i^{\mathcal{I}}(\delta) \in \{b \cdot \frac{m-k-1}{m-k}\} \cup [b, 1]$ for $1 \leq i \leq n$. We now define

$$\mathcal{T}_{\mathcal{V}} := \bigcup_{i=0}^{m} \mathcal{T}_i \,\cup\, \{\langle V_1 \sqcap \ldots \sqcap V_m \sqsubseteq A \geq 1\rangle\} \cup \{\langle V_0 \sqsubseteq V_{j_1} \sqcap V_{j_2} \geq 1\rangle \mid \{v_{j_1}, v_{j_2}\} \in \mathcal{E}\}. \quad (1)$$

Theorem 6. *There is a vertex cover of $V, \mathcal{E}$ of size k iff $\top$ is not positively subsumed by A w.r.t. $\mathcal{T}_{\mathcal{V}}$.*

Proof. Let $S = \{v_{i1}, \ldots, v_{ik}\}$ be a vertex cover of size k. Build the interpretation $\mathcal{I}_S := (\{\delta\}, \cdot^{\mathcal{I}_S})$ with $A^{\mathcal{I}_S}(\delta) := 0$, $V_0^{\mathcal{I}_S}(\delta) := b \cdot \frac{m-k-1}{m-k}$, and for $i, 1 \leq i \leq m$,

$$V_i^{\mathcal{I}_S}(\delta) := \begin{cases} 1 & \text{if } v_i \in S \\ b \cdot \frac{m-k-1}{m-k} & \text{otherwise.} \end{cases}$$

It is easy to verify that $\mathcal{I}_S$ is a model of $\mathcal{T}_{\mathcal{V}}$ and we have $\top^{\mathcal{I}_S}(\delta) \Rightarrow A^{\mathcal{I}_S}(\delta) = 0$.

For the converse, let $\mathcal{I}$ be a model of $\mathcal{T}_{\mathcal{V}}$ and $\delta \in \Delta^{\mathcal{I}}$ be such that $A^{\mathcal{I}}(\delta) = \top^{\mathcal{I}}(\delta) \Rightarrow A^{\mathcal{I}}(\delta) = 0$. We define

$$S_{\mathcal{I}} := \{v_i \mid V_i^{\mathcal{I}}(\delta) \geq b, 1 \leq i \leq m\}.$$

Since $V_1^{\mathcal{I}}(\delta) \otimes \ldots \otimes V_m^{\mathcal{I}}(\delta) = 0$, there must be at least $m - k$ concept names V_j such that $V_j^{\mathcal{I}}(\delta) = b \cdot \frac{m-k-1}{m-k}$, and hence $S_{\mathcal{I}}$ has at most k elements. Moreover, since $\mathcal{I}$ satisfies the axioms in (1), for every $\{v_{j_1}, v_{j_2}\} \in \mathcal{E}$, at least one of $V_{j_1}^{\mathcal{I}}(\delta), V_{j_2}^{\mathcal{I}}(\delta)$ is $\geq b$. Thus, $S_{\mathcal{I}}$ is a vertex cover. □

Corollary 7. *If $\otimes$ has zero divisors, then positive subsumption in $\otimes$-$\mathcal{EL}$ is co-*NP*-hard.*

If we consider only the sublogic $\otimes$-$\mathcal{L}$ of $\otimes$-$\mathcal{EL}$ in which existential restrictions are not allowed, we can use complexity results for propositional fuzzy logics [Hájek, 2006] to show that for certain *strongly r-admissible* t-norms this complexity bound is tight. Strongly r-admissible t-norms satisfy several restrictions that limit reasoning to the *rational* numbers in $[0, 1]$ (see [Hájek, 2006] for details). Additionally, $\otimes$ must be a product-free t-norm with finitely many components.

We map every concept name A to a unique propositional variable p_A, each conjunction C of concept names to the propositional conjunction φ_C of the corresponding variables, and a GCI $\alpha = \langle C \sqsubseteq D \geq q\rangle$ to $\varphi_\alpha := \overline{q} \to (\varphi_C \to \varphi_D)$, where $\overline{q}$ is a constant that is interpreted as q. Finally, we express a TBox $\mathcal{T}$ by the conjunction of all φ_α for $\alpha \in \mathcal{T}$. Let now C_0, D_0 be concepts and $\mathcal{T}$ be a TBox containing only rational numbers in its GCIs. It follows that C_0 is not positively subsumed by D_0 w.r.t. $\mathcal{T}$ iff the conjunction of $\varphi_{\mathcal{T}}$ and $(\varphi_{C_0} \to \varphi_{D_0}) \to \overline{0}$ is satisfiable in the fuzzy propositional logic $\mathcal{RL}(\otimes)$. Since the latter problem is NP-complete [Hájek, 2006], the former is in co-NP.

Proposition 8. *If $\otimes$ is strongly r-admissible, product-free, and has only finitely many components, then positive subsumption in $\otimes$-$\mathcal{L}$ is in co-*NP.

3.2 T-norms without Zero Divisors

If the underlying t-norm $\otimes$ has no zero divisors, i.e. it does not start with the Łukasiewicz t-norm, then positive subsumption turns out to be decidable in polynomial time, as in the crisp case [Brandt, 2004; Baader *et al.*, 2005]. Under Gödel semantics, positive subsumption is equivalent to deciding whether the best subsumption degree is greater than zero. Thus, a consequence of the polynomial time algorithm for computing best subsumption degree from [Mailis *et al.*, 2012] is that positive subsumption is polynomial for the Gödel t-norm. We generalize this result to all t-norms without zero divisors. To show this, we provide a reduction similar to the one from [Borgwardt *et al.*, 2012b], where consistency in expressive fuzzy DLs is reduced to the corresponding crisp DLs. Our reduction transforms in linear time a $\otimes$-$\mathcal{EL}$-TBox into a crisp TBox that describes all positive subsumption relations. Given a TBox $\mathcal{T}$, we define

$$\mathcal{T}^{>0} := \{\langle C \sqsubseteq D \geq 1\rangle \mid \langle C \sqsubseteq D \geq q\rangle \in \mathcal{T}, q > 0\}.$$

Notice that every model of $\mathcal{T}^{>0}$ is also a model of $\mathcal{T}$, since the axioms where $q = 0$ are satisfied by all interpretations. We thus have the following theorem.

Theorem 9. *Let $\mathcal{T}$ be a TBox and C_0, D_0 two concepts. Then C_0 is positively subsumed by D_0 w.r.t. $\mathcal{T}$ iff for every crisp model $\mathcal{J}$ of $\mathcal{T}^{>0}$ and $\delta \in \Delta^{\mathcal{J}}$ it holds that $C_0^{\mathcal{J}}(\delta) \leq D_0^{\mathcal{J}}(\delta)$.*

Proof. First, assume that there is a crisp model $\mathcal{J}$ of $\mathcal{T}^{>0}$ and a $\delta_0 \in \Delta^{\mathcal{J}}$ with $C_0^{\mathcal{J}}(\delta_0) = 1$ and $D_0^{\mathcal{J}}(\delta_0) = 0$, and thus $C_0^{\mathcal{J}}(\delta_0) \Rightarrow D_0^{\mathcal{J}}(\delta_0) = 0$. Since $\mathcal{J}$ is also a model of $\mathcal{T}$, we know that C_0 is not positively subsumed by D_0 w.r.t. $\mathcal{T}$.

For the converse direction, let $\mathcal{I}$ be a model of $\mathcal{T}$ and $\delta_0 \in \Delta^{\mathcal{I}}$ such that $C_0^{\mathcal{I}}(\delta_0) \Rightarrow D_0^{\mathcal{I}}(\delta_0) = 0$. We construct the crisp interpretation $\mathcal{J}$ over the domain $\Delta^{\mathcal{J}} := \Delta^{\mathcal{I}}$ as follows. Let $\mathbb{1}\colon [0, 1] \to \{0, 1\}$ be the function defined by $\mathbb{1}(0) := 0$ and $\mathbb{1}(q) := 1$ for all $q > 0$ (cf. [Cignoli and Torrens, 2003]). For all $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and $\delta, \gamma \in \Delta^{\mathcal{J}}$, we set $A^{\mathcal{J}}(\delta) := \mathbb{1}(A^{\mathcal{I}}(\delta))$ and $r^{\mathcal{J}}(\delta, \gamma) := \mathbb{1}(r^{\mathcal{I}}(\delta, \gamma))$.

We first show that $C^{\mathcal{J}}(\delta) = \mathbb{1}(C^{\mathcal{I}}(\delta))$ holds for all concepts C and all $\delta \in \Delta^{\mathcal{I}}$. If C is a concept name, the claim holds by definition of $\mathcal{J}$, and for $C = \top$ the claim is trivial. If $C = C_1 \sqcap C_2$, then $C^{\mathcal{I}}(\delta) = C_1^{\mathcal{I}}(\delta) \otimes C_2^{\mathcal{I}}(\delta) = 0$ iff we have $C_1^{\mathcal{I}}(\delta) = 0$ or $C_2^{\mathcal{I}}(\delta) = 0$ since $\otimes$ has no zero divisors. Thus, we have $C^{\mathcal{J}}(\delta) = \mathbb{1}(C_1^{\mathcal{I}}(\delta)) \otimes \mathbb{1}(C_2^{\mathcal{I}}(\delta)) = \mathbb{1}(C^{\mathcal{I}}(\delta))$. Finally, if $C = \exists r.C_1$, then

$$C^{\mathcal{J}}(\delta) = \sup_{\gamma \in \Delta^{\mathcal{I}}} \mathbb{1}(r^{\mathcal{I}}(\delta, \gamma)) \otimes \mathbb{1}(C_1^{\mathcal{I}}(\gamma))$$
$$= \mathbb{1}(\sup_{\gamma \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(\delta, \gamma) \otimes C_1^{\mathcal{I}}(\gamma)) = \mathbb{1}(C^{\mathcal{I}}(\delta))$$

by similar arguments as above and the fact that the supremum over a set of values is 0 iff all of these values are 0.

We now show that $\mathcal{J}$ is a model of $\mathcal{T}^{>0}$. Consider a GCI $\langle C \sqsubseteq D \geq q\rangle \in \mathcal{T}$. For all $\delta \in \Delta^{\mathcal{I}}$, we have $C^{\mathcal{I}}(\delta) \Rightarrow D^{\mathcal{I}}(\delta) \geq q$ since $\mathcal{I}$ is a model of $\mathcal{T}$. If $q = 0$, then $C^{\mathcal{J}}(\delta) \Rightarrow D^{\mathcal{J}}(\delta) \geq 0 = q$. If $q > 0$, then $C^{\mathcal{I}}(\delta) > 0$ implies that $D^{\mathcal{I}}(\delta) > 0$. Indeed, $C^{\mathcal{I}}(\delta) > 0$ and $D^{\mathcal{I}}(\delta) = 0$ would yield that $C^{\mathcal{I}}(\delta) \Rightarrow D^{\mathcal{I}}(\delta) = 0 < q$, contradicting the assumption.[8] Thus, $\langle C \sqsubseteq D \geq 1\rangle$ is satisfied by $\mathcal{J}$.

[8] Recall that the residual negation is the Gödel negation.

Finally, since $D_0^{\mathcal{I}}(\delta_0) \leq C_0^{\mathcal{I}}(\delta_0) \Rightarrow D_0^{\mathcal{I}}(\delta_0) = 0$, we have $C_0^{\mathcal{I}}(\delta_0) > 0$, and thus $C_0^{\mathcal{J}}(\delta_0) = 1$ and $D_0^{\mathcal{J}}(\delta_0) = 0$. □

The latter condition in this theorem is equivalent to subsumption between C_0 and D_0 in classical $\mathcal{EL}$, which can be decided in polynomial time [Brandt, 2004].

Corollary 10. *If $\otimes$ has no zero divisors, then positive subsumption in $\otimes$-$\mathcal{EL}$ is decidable in polynomial time.*

We have so far focused on deciding positive subsumption between concepts. A related problem of interest in the context of fuzzy DLs is the computation of the best subsumption degree between concepts. In the following section, we show that the picture of the best subsumption degree is more elaborate than that of positive subsumption.

4 The Best Subsumption Degree

We consider the problem of computing the best subsumption degree of two concepts C, D w.r.t. a TBox $\mathcal{T}$, and the corresponding decision problem of whether $C \sqsubseteq_{\mathcal{T}}^{p} D$ holds for a given $p \in (0, 1]$. We again make a distinction on the structure of the underlying t-norm. We show that for any t-norm containing the Łukasiewicz t-norm, the problem is co-NP-hard. We then argue why we believe this problem to be hard also for all other t-norms, except for the Gödel t-norm.

4.1 T-norms Containing Łukasiewicz

For t-norms with zero divisors, deciding p-subsumption is also co-NP-hard. Consider the reduction presented in the proof of Theorem 6 to show co-NP-hardness of positive subsumption. Since none of the concept names $V_i, 1 \leq i \leq m$, can be interpreted with any degree between $b \cdot \frac{m-k-1}{m-k}$ and b, if the conjunction of these concept names is smaller than b, then it must be of the form $b \cdot \frac{n}{m-k}$ for some natural number n. It thus follows that $\top$ is positively subsumed by A, and hence there is no vertex cover of size k, if and only if $\top$ is $\frac{b}{m-k}$-subsumed by A.

Proposition 11. *If $\otimes$ has zero divisors, then p-subsumption in $\otimes$-$\mathcal{EL}$ is co-*NP*-hard.*

Again, this bound is tight if we restrict to $\otimes$-$\mathcal{L}$, where $\otimes$ is a t-norm as in Proposition 8. Indeed, C_0 is p-subsumed by D_0 w.r.t. $\mathcal{T}$ iff the propositional formula $\overline{p} \rightarrow (\varphi_{C_0} \rightarrow \varphi_{D_0})$ is a semantic consequence of $\varphi_{\mathcal{T}}$ in $\mathcal{RL}(\otimes)$. The latter problem is co-NP-complete [Hájek, 2006].

Proposition 12. *If $\otimes$ is strongly r-admissible, product-free, and has only finitely many components, then p-subsumption in $\otimes$-$\mathcal{L}$ is in co-*NP.

Contrary to positive subsumption, p-subsumption is also co-NP-hard for some t-norms without zero divisors. Indeed, hardness arises as soon as $\otimes$ *contains* the Łukasiewicz t-norm. This is a consequence of the following result.

Theorem 13. *Let $\otimes_1, \otimes_2$ be continuous t-norms, $b \in (0, 1)$, and $\otimes$ be the ordinal sum of $((0, b), \otimes_1)$, $((b, 1), \otimes_2)$. Then p-subsumption in $\otimes$-$\mathcal{EL}$ is at least as hard as p-subsumption in $\otimes_2$-$\mathcal{EL}$.*

Proof. Let $h : [0, 1] \rightarrow [b, 1]$ be the bijective function where $h(x) = b + (1 - b)x$, $\mathcal{T}$ be a $\otimes_2$-$\mathcal{EL}$-TBox, and $\Rightarrow, \Rightarrow_2$ be the residua of $\otimes, \otimes_2$, respectively. We construct the TBox

$$\mathcal{T}_{\otimes} := \{\langle C \sqsubseteq D \geq h(q)\rangle \mid \langle C \sqsubseteq D \geq q\rangle \in \mathcal{T}\}.$$

Given two concepts C_0, D_0 and $p \in (0, 1]$, we show that $C_0 \sqsubseteq_{\mathcal{T}}^{p} D_0$ over $\otimes_2$ iff $C_0 \sqsubseteq_{\mathcal{T}_{\otimes}}^{h(p)} D_0$ over $\otimes$.

Let $\mathcal{I}$ be a model of $\mathcal{T}$ with $C_0^{\mathcal{I}}(\delta_0) \Rightarrow_2 D_0^{\mathcal{I}}(\delta_0) < p$ for a $\delta_0 \in \Delta^{\mathcal{I}}$. We construct $\mathcal{J} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{J}})$, where, for $\delta, \gamma \in \Delta^{\mathcal{I}}$,

$$A^{\mathcal{J}}(\delta) := h(A^{\mathcal{I}}(\delta)), \qquad r^{\mathcal{J}}(\delta, \gamma) := h(r^{\mathcal{I}}(\delta, \gamma)).$$

Using an induction argument similar to the one of Theorem 9, we can show $C^{\mathcal{J}}(\delta) = h(C^{\mathcal{I}}(\delta))$ for every concept C and $\delta \in \Delta^{\mathcal{I}}$, and in particular $\mathcal{J}$ is a model of $\mathcal{T}_{\otimes}$ with $C_0^{\mathcal{J}}(\delta_0) \Rightarrow D_0^{\mathcal{J}}(\delta_0) < h(p)$ since h is strictly increasing (recall the definition of ordinal sums from Section 2).

Conversely, let $\mathcal{J}$ be a model of $\mathcal{T}_{\otimes}$ and $\delta_0 \in \Delta^{\mathcal{J}}$ such that $C_0^{\mathcal{J}}(\delta_0) \Rightarrow D_0^{\mathcal{J}}(\delta_0) < h(p)$. A similar argument shows that the interpretation $\mathcal{I} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{I}})$ where, for every $\delta, \gamma \in \Delta^{\mathcal{I}}$,

$$A^{\mathcal{I}}(\delta) = \begin{cases} h^{-1}(A^{\mathcal{J}}(\delta)) & \text{if } A^{\mathcal{J}}(\delta) \geq b, \\ 0 & \text{otherwise} \end{cases}$$

$$r^{\mathcal{I}}(\delta, \gamma) = \begin{cases} h^{-1}(r^{\mathcal{J}}(\delta, \gamma)) & \text{if } r^{\mathcal{J}}(\delta, \gamma) \geq b, \\ 0 & \text{otherwise} \end{cases}$$

is a model of $\mathcal{T}$ such that

$$C_0^{\mathcal{I}}(\delta_0) \Rightarrow_2 D_0^{\mathcal{I}}(\delta_0) < h^{-1}(h(p)) = p.$$

Since $|\mathcal{T}_{\otimes}|$ is linear in $|\mathcal{T}|$, this yields the result. □

Every t-norm that contains the Łukasiewicz t-norm can be expressed as the ordinal sum of two components $((0, b), \otimes_1)$, $((b, 1), \otimes_2)$, where $\otimes_2$ starts with Łukasiewicz. Thus, Proposition 11 and Theorem 13 yield the following.

Corollary 14. *If $\otimes$ contains the Łukasiewicz t-norm, then p-subsumption in $\otimes$-$\mathcal{EL}$ is co-*NP*-hard.*

In particular, this shows that the best subsumption degree in $\otimes$-$\mathcal{EL}$ cannot be computed in polynomial time if $\otimes$ contains the Łukasiewicz t-norm (unless P = NP).

4.2 T-norms without Łukasiewicz

From Theorem 1 it follows that every t-norm that does not contain Łukasiewicz must be expressible as the ordinal sum of copies of the product t-norm. In particular, it either is the Gödel t-norm, or has at least one component using the product t-norm. For the Gödel t-norm, it is known that the best subsumption degree can be computed in polynomial time using a variant of the completion algorithm for classical $\mathcal{EL}$ [Mailis *et al.*, 2012]. The only remaining cases are those t-norms that contain the product t-norm.

Recall that all t-norms different from the Gödel t-norm have infinitely many elements that are not idempotent. For those cases, the approach used in [Mailis *et al.*, 2012] cannot be applied directly. We now provide some arguments that suggest that p-subsumption is in fact hard for all t-norms containing the product t-norm. We consider first the basic case of the product t-norm itself. If p-subsumption is indeed hard for this t-norm, then similar arguments should be applicable to t-norms *starting with* the product t-norm, and by Theorem 13 to all other elements of this family.

The following example shows that under product t-norm semantics $C \sqsubseteq_{\mathcal{T}}^{>0} D$ does not imply that $\mathsf{bsd}_{\mathcal{T}}(C, D) > 0$. In other words, although positive subsumption can be decided in polynomial time, this result cannot be used to decide whether the best subsumption degree is greater than zero.

Example 15. Consider the product t-norm and $A \in \mathsf{N_C}$. For every interpretation $\mathcal{I}$ and $\delta \in \Delta^{\mathcal{I}}$, it holds that if $A^{\mathcal{I}}(\delta) > 0$, then $A^{\mathcal{I}}(\delta) \Rightarrow (A^2)^{\mathcal{I}}(\delta) = A^{\mathcal{I}}(\delta) > 0$.[9] Thus A is positively subsumed by A^2. However, for every $p > 0$ we can build an interpretation $\mathcal{I} = (\{\delta\}, \cdot^{\mathcal{I}})$ with $A^{\mathcal{I}}(\delta) = p/2$. Then, $A^{\mathcal{I}}(\delta) \Rightarrow (A^2)^{\mathcal{I}}(\delta) = A^{\mathcal{I}}(\delta) = p/2 < p$. As this holds for every $p > 0$, it follows that $\mathsf{bsd}(A \sqsubseteq A^2) = 0$.

This example also shows that a *direct* crispification approach, akin to the one presented in Section 3.2 cannot be used to decide whether the best subsumption degree is zero or not. Indeed, no TBox was used in the example, and over crisp interpretations A is always subsumed by A^2 (with degree 1). Thus, if p-subsumption is decidable in polynomial time, one would need to find an algorithm that can deal with the different degrees appearing in the axioms, without using more than a polynomial number of combinations of them.

An obvious approach is to generalize the completion algorithm for classical $\mathcal{EL}$ from [Baader *et al.*, 2005] in the style of Mailis *et al.* to allow for product operations. The algorithms from [Baader *et al.*, 2005; Mailis *et al.*, 2012] first transform the TBox into an equivalent one in normal form. A TBox $\mathcal{T}$ is in *normal form* if all the GCIs in $\mathcal{T}$ are of the form

$$\langle A_1 \sqcap A_2 \sqsubseteq B \geq q \rangle,\ \langle A \sqsubseteq \exists r.B \geq q \rangle,\ \text{or}\ \langle \exists r.A \sqsubseteq B \geq q \rangle,$$

with $A, A_1, A_2, B \in \mathsf{N_C} \cup \{\top\}$ and $r \in \mathsf{N_R}$. It is well known that in classical $\mathcal{EL}$ and $\otimes$-$\mathcal{EL}$ using the Gödel t-norm any TBox $\mathcal{T}$ can be transformed to an equivalent one in normal form of size linear in the size of $\mathcal{T}$ [Brandt, 2004; Baader *et al.*, 2005; Mailis *et al.*, 2012]. We show that this is not true for $\otimes$-$\mathcal{EL}$ in general with the help of the following proposition, which holds for any t-norm $\otimes$. The proof is by a simple case analysis on the shape of the axioms in $\mathcal{T}$.

Proposition 16. *Let $\mathcal{T}$ be a TBox in normal form, $p \in [0, 1]$, $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ an interpretation and $\mathcal{I}_p = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}_p})$ the interpretation where for every $\delta, \gamma \in \Delta^{\mathcal{I}}$, $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$*

$$A^{\mathcal{I}_p}(\delta) = \max\{A^{\mathcal{I}}(\delta), p\},\ r^{\mathcal{I}_p}(\delta, \gamma) = \max\{r^{\mathcal{I}}(\delta, \gamma), p\}.$$

If $\mathcal{I}$ is a model of $\mathcal{T}$, then $\mathcal{I}_p$ is also a model of $\mathcal{T}$.

We now prove that for any t-norm $\otimes$ except the Gödel t-norm it is impossible to construct a $\otimes$-$\mathcal{EL}$ TBox in normal form that is equivalent to the GCI $\langle A \sqsubseteq B \sqcap C \geq 1 \rangle$. Suppose that such a TBox $\mathcal{T}$ exists. The interpretation $\mathcal{I} = (\{\delta\}, \cdot^{\mathcal{I}})$ with $A^{\mathcal{I}}(\delta) = B^{\mathcal{I}}(\delta) = C^{\mathcal{I}}(\delta) = 0$ must then be a model of $\mathcal{T}$. Since $\otimes$ has non-idempotent elements, there must be a value $p \in (0, 1)$ with $p \otimes p < p$. By Proposition 16, the interpretation $\mathcal{I}_p$ is also a model of $\mathcal{T}$. However,

$$A^{\mathcal{I}_p}(\delta) \Rightarrow (B \sqcap C)^{\mathcal{I}_p}(\delta) = p \Rightarrow p \otimes p < 1,$$

which violates the axiom $\langle A \sqsubseteq B \sqcap C \geq 1 \rangle$. Thus $\mathcal{I}_p$ cannot be a model of $\mathcal{T}$, yielding a contradiction.

Even if the input TBox $\mathcal{T}$ is already in normal form, the completion rules from [Mailis *et al.*, 2012] cannot be directly transformed to handle the product t-norm. For instance, the correctness of the rule that handles conjunctions on the left-hand side (rule **CR2** in [Baader *et al.*, 2005, p. 366] and [Mailis *et al.*, 2012, p. 417]) is based on the intuition that if $A \sqsubseteq_{\mathcal{T}}^1 B$ and $A \sqsubseteq_{\mathcal{T}}^1 C$ hold, then also $A \sqsubseteq_{\mathcal{T}}^1 B \sqcap C$. While this is true for classical semantics and the Gödel t-norm, it fails for the product t-norm, as depicted in Example 15. The only deduction one can make from the two premises is that $A^2 \sqsubseteq_{\mathcal{T}}^1 B \sqcap C$ holds. Applying this idea, it is not hard to find a TBox $\mathcal{T}$ of size n such that $A^{2^n} \sqsubseteq_{\mathcal{T}}^p B$ holds for some $p \in (0, 1]$, but $A^k \not\sqsubseteq_{\mathcal{T}}^1 B$ for every $k, 1 \leq k < 2^n$.

Any algorithm that can decide p-subsumption would need to keep track of the subsumers of concepts of the form A^n, since, e.g. $A^n \sqsubseteq_{\mathcal{T}}^{q_1} B$ and $\top \sqsubseteq_{\mathcal{T}}^{q_2} B$ together imply $A \sqsubseteq_{\mathcal{T}}^p B$, where $p := \sqrt[n]{q_2^{n-1} \cdot q_1}$. This suggests that no deterministic algorithm that decides p-subsumption can avoid the application of exponentially many steps. Although we have not been able to prove that this problem is indeed hard, we have strong reasons to suspect it.

5 Conclusions

We have analyzed subsumption problems in fuzzy extensions of $\mathcal{EL}$ with semantics based on general t-norms. For the complexity of positive subsumption, we have shown a dichotomy between polynomial for t-norms without zero divisors, and co-NP-hard (and therefore probably not polynomial) for all t-norms with zero divisors. For the former case, positive subsumption is linearly reducible to subsumption in the classical DL $\mathcal{EL}$. This dichotomy goes well in hand with the complexity of deciding TBox consistency in more expressive fuzzy DLs: for t-norms without zero divisors, the problem is linearly reducible to classical reasoning [Borgwardt *et al.*, 2012a; 2012b], and in particular decidable, but becomes undecidable for all other t-norms [Cerami and Straccia, 2013; Borgwardt and Peñaloza, 2012a; 2012b].

The problem of deciding p-subsumption exhibits a different complexity pattern. We showed that there exist t-norms without zero divisors for which this problem is also co-NP-hard. In fact, this lower bounds holds for any t-norm containing the Łukasiewicz t-norm. So far, we have not been able to obtain complexity results for other t-norms, beyond the previously known case of the Gödel t-norm. However, we presented some arguments that suggest that p-subsumption is probably intractable for these t-norms as well. As future work, we plan to prove this claim and find matching upper bounds for all our hardness results.

Although our hardness results cast a shadow on the possibility of reasoning in large fuzzy ontologies, we believe that for well-structured ontologies, such as SNOMED CT, which contains no cyclic relations between concepts and where most axioms can be normalized without affecting their intended semantics, tractability can be regained. A deeper analysis of this situation is part of our plans for future work.

[9] Recall that A^2 stands for $A \sqcap A$.

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