Semi-Universal Portfolios with Transaction Costs*

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Abstract

Online portfolio selection (PS) has been extensively studied in artificial intelligence and machine learning communities in recent years. An important practical issue of online PS is transaction cost, which is unavoidable and nontrivial in real financial trading markets. Most existing strategies, such as *universal portfolio* (UP) based strategies, often rebalance their target portfolio vectors at every investment period, and thus the total transaction cost increases rapidly and the final cumulative wealth degrades severely. To overcome the limitation, in this paper we investigate new investment strategies that rebalances its portfolio only at some selected instants. Specifically, we design a novel on-line PS strategy named *semi-universal portfolio* (SUP) strategy under transaction cost, which attempts to avoid rebalancing when the transaction cost outweighs the benefit of trading. We show that the proposed SUP strategy is universal and has an upper bound on the regret. We present an efficient implementation of the strategy based on non-uniform random walks and online factor graph algorithms. Empirical simulation on real historical markets show that SUP can overcome the drawback of existing UP based transaction cost aware algorithms and achieve significantly better performance. Furthermore, SUP has a polynomial complexity in the number of stocks and thus is efficient and scalable in practice.

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1 Introduction

Portfolio selection (PS) aims to optimize the allocation of wealth across a set of assets to achieve certain long-term financial goal. Some early research on PS can be dated back to the *mean variance theory* [Markowitz, 1952], which optimizes portfolios by trading off the expected return (mean) and risk (variance). The subsequent *capital growth theory* [Kelly, 1956] focuses on multiple-periods or sequential PS, aiming at maximizing portfolio's expected growth rate, or expected log return. While both theories can solve the PS problem, the latter is fitted to the *online* scenario and thus constitutes the basis of online PS, and has been extensively explored in artificial intelligence [Cover, 1991; Cover and Ordentlich, 1996; 1998] and machine learning communities [Agarwal *et al.*, 2006; Borodin *et al.*, 2004; Huang *et al.*, 2013].

Most state-of-the-art on-line PS strategies ignore transaction cost [Li and Hoi, 2014]. Transaction cost, as one central friction in financial markets, is prevalent in almost all of the financial trading. When investors face transaction costs in financial markets, their trading strategies may be much different. Therefore, how investors should trade in the presence of transaction cost remains an open yet important question.

Recently, some on-line PS studies [Blum and Kalai, 1997; Helmbold *et al.*, 1998; Kozat and Singer, 2011; Das *et al.*, 2013; 2014] have attempted to address transaction cost. Though these transaction cost aware algorithms, such as *universal portfolio* (UP) based strategies [Blum and Kalai, 1997], achieve encouraging results on many datasets, the actual suffering under transaction cost is still very high. This is because the existing UP based transaction cost aware strategies often rebalance their target portfolio vectors at every investment period, which is not always necessary, and makes the total transaction cost increase rapidly. Furthermore, these algorithms have an exponential complexity in the number of stocks [Blum and Kalai, 1997], which makes them impractical in real world scenarios.

To address the above drawbacks, in this paper, we present a new multi-period online PS strategy named *semi-universal portfolio* (SUP) with transaction cost. The basic idea is to employ Cover's UP as a moving target portfolio and rebalance the portfolio only at some selected instants, which may avoid rebalancing when the transaction cost outweighs the benefit of rebalancing. Then, we approach this strategy from a competitive algorithmic perspective, and compete against all such SUPs with arbitrary numbers of rebalancing times and at arbitrary corresponding rebalancing instants. We show that the proposed SUP strategy is universal and has an upper bound on the regret. We also develop an efficient implementation of the universal algorithm based on non-uniform random walks and learn the semi-universal portfolios based on an online factor graph algorithm.

To the best of our knowledge, the proposed SUP is the first algorithm that explicitly exploits the transaction cost for online PS by considering a Cover's moving target portfolio with occasional rebalancing. Though simple in nature, SUP suffers much less under transaction cost than existing algorithms and has been empirically validated via extensive experiments on real markets. On the one hand, the SUP strategy significantly surpasses a number of state-of-the-art transaction cost aware strategies in terms of long-term compound return. On the other hand, it has a polynomial complexity in the number of stocks and thus has better scalability in practice.

The rest of the paper is organized as follows: Section 2 formulates the on-line PS problem with transaction cost, and Section 3 reviews some related works. Section 4 presents the proposed algorithm, and Section 5 empirically evaluates its effectiveness on real markets. Section 6 concludes the paper.

2 Problem Setting

Consider a financial market with m assets for n trading periods. The asset prices on the t^{th} period are represented by a *close price vector* $\mathbf{p}_t \in \mathbb{R}^m_+$, and each element p^i_t is the close price of asset i. The changes of asset prices are denoted by a *price relative vector* $\mathbf{x}_t = (x^1_t, \ldots, x^m_t) \in \mathbb{R}^m_+$, where x^j_t is the ratio of current close price to last close price of asset j at the t^{th} period, *i.e.*, $x^j_t = p^j_t / p^j_{t-1}$. We denote $\mathbf{x}^n_1 = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ as the sequence of price relative vectors for n periods. We abbreviate $\mathbf{x}^n_1$ to $\mathbf{x}^n$ below.

The capital of the m assets at the beginning of the t^{th} period is denoted as a *portfolio vector* $\mathbf{b}_t = (b^1_t, \ldots, b^m_t) \in \mathbb{R}^m_+$, where b^j_t is the proportion of wealth invested in the j^{th} asset at the t^{th} period. Typically, we assume that the portfolio is self-financed and no margin/short is allowed, which means $\mathbf{b}_t \in \Delta_m$, where $\Delta_m = \{\mathbf{b}_t : \mathbf{b}_t \in \mathbb{R}^m_+, \sum_{j=1}^m b^j_t = 1\}$. The investment procedure is denoted as a *portfolio strategy*, *i.e.*, $\mathbf{b}_1 = \frac{1}{m}\mathbf{1}$ and following sequence of mappings $\mathbf{b}_t : (\mathbb{R}^m_+)^{t-1} \to \Delta_m, t = 1, 2, \ldots$, where $\mathbf{b}_t = \mathbf{b}_t(\mathbf{x}^{t-1})$ is the portfolio used at the t^{th} trading period, given the last market sequence $\mathbf{x}^{t-1} = (\mathbf{x}_1, \ldots, \mathbf{x}_{t-1})$. We denote by $\mathbf{B}_n = [\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n]$ the strategy for n periods.

At the t^{th} trading period, a portfolio $\mathbf{b}_t$ achieves a *portfolio period return* s_t, *i.e.*, the wealth increases by a factor of $s_t = \mathbf{b}_t^T \mathbf{x}_t = \sum_{j=1}^m b^j_t x^j_t$. When the portfolio manager sets up his new portfolio at the beginning of the $(t+1)^{th}$ period, *i.e.*, buys/sells stocks according to the actual portfolio vector $\mathbf{b}_{t+1}$, he has to pay transaction cost. We assume a *symmetric* proportional transaction cost with *cost ratio* c, where $c = c_s + c_b$ for both selling c_s and buying c_b and $0 \le c \le 1$, *i.e.*, the trade of 1 dollar worth of asset i nets only $1 - c$ dollars. Hence, the portfolio manager should spend the *transaction costs* $\mathbf{C}_t = [C^1_t, \ldots, C^m_t]^T \in \mathbb{R}^m_+$ at the beginning of the $(t+1)^{th}$ period, where $C^i_t = s_t|\overline{b}^i_t - b^i_{t+1}|c$ is the cost for the i^{th} asset, $\overline{b}^i_t$ represents the *current portfolio* of the i^{th} asset at the end of the t^{th} period and the *current portfolio vector* $\overline{\mathbf{b}}_t = [\overline{b}^1_t, \ldots, \overline{b}^m_t]^T = [\frac{b^1_t x^1_t}{\sum_{i=1}^m b^i_t x^i_t}, ..., \frac{b^m_t x^m_t}{\sum_{i=1}^m b^i_t x^i_t}]^T$. The *total costs* for all assets at the beginning of the $(t+1)^{th}$ period are denoted as $\mathrm{Cost}_t = \mathbf{C}_t^T \mathbf{e}$, where $\mathbf{e} = [1, \ldots, 1]^T \in \mathbb{R}^m_+$. We denote $s^c_t = s_t - \mathrm{Cost}_t$ as the *net wealth* at the end of the t^{th} period, which is less than s_t after the transaction. We will use the above formula to solve the SUP strategy with transaction cost in this paper.

Since we reinvest and adopt price relative, the portfolio wealth will multiplicatively grow. Thus, after n trading periods, a portfolio strategy $\mathbf{B}_{n+1}$ produces a *portfolio cumulative wealth* with transaction cost S^c_n, $S^c_n(\mathbf{x}^n) = S_0 \prod_{t=1}^n (s^c_t)$, where S_0 is the initial wealth, which is set to 1 in this paper.

Finally, we formulate the on-line PS problem as a sequential decision task. The portfolio manager aims to design a strategy $\mathbf{B}_{n+1}$ to maximize the portfolio cumulative wealth S^c_n. The portfolios are selected in a sequential fashion. At each period t, given the historical information, the manager learns to select a new portfolio vector $\mathbf{b}_t$ for the next price relative vector $\mathbf{x}_t$, where the decision criterion varies among different managers. The resulting portfolio $\mathbf{b}_t$ is scored based on the portfolio period return of s^c_t. Such procedure repeats until the end of trading periods and the portfolio strategy is finally scored by the cumulative wealth S^c_n.

3 Background Review

On-line PS has been extensively explored following the principle of Kelly investment [Kelly, 1956]. Although the need for considering transaction cost has been mentioned in [Cover, 1991; 1996; Ordentlich and Cover, 1996; Helmbold *et al.*, 1998; Borodin *et al.*, 2004; Li and Hoi, 2014], only a few works deal with online PS with transaction cost.

Blum and Kalai [1997] exploited *universal portfolios* [Cover, 1991] with proportional transaction cost, which pays transaction cost proportionally from each asset and utilizes Cover's UP formulation as a moving target portfolio, and then rebalances the portfolio at each investment period. Bean and Singer [2011; 2012], Kozat and Singer [2008; 2009] discussed *switching strategies* with transaction cost, which transfers the *benchmark* portfolio according to the market information and also rebalances at each investment period. On the other hand, some strategies focus on occasional rebalancing. Helmbold *et al.* [1998] considered *constant rebalanced portfolio* (CRP) with transaction cost, and further introduced a *semi-CRP* (SCRP), which rebalances only at some periods. Blum and Kalai [1997] showed that no strategy can guarantee the exponential growth rate of the best SCRP in hindsight, even without commission. Recently, Kozat and Singer [2011] extended this idea to a universal SCRP strategy that rebalances the target portfolio vectors in arbitrary investment periods.

Moreover, there is another category of online PS that investigates transaction cost via parameters update of stochastic optimization model. Das *et al*. [2013] proposed *online lazy updates* (OLU) and *online lazy updates with group sparsity* (OLU-GS) [2014] for transaction cost based on the *exponential gradient* algorithm [Helmbold *et al*., 1998], which rebalances the portfolio vector by *sparse* or *lazy* updates of the parameters in the optimization model.

Finally, there are some algorithms that deal with growth optimal investment with transaction cost in discrete times. Cover and Iyengar [2000] formulated the problem of horse race markets. Iyengar [2002] considered growth optimal investment with several assets. Bobryk and Stettner [1999] investigated the case of PS with consumption. Schafer [2002] considered the maximization of the long run expected growth rate. Gyorfi and Vajda [2008] investigated discrete time infinite horizon growth optimal investment with transactions cost. Ormos and Urbn [2011] presented a nonparametric model using kernel-based agents to approximate the maximum theoretical growth rate with transaction cost.

3.1 Analysis of Existing Work

The most related works to this paper are the UP-based, SCRP and OLU strategies. Here we analyze their drawbacks.

First, let us focus on the transaction costs of these existing works. In practice, at the beginning of the t^{th} period, a Kelly portfolio manager intends to rebalance the portfolio from closing price rebalanced portfolio $\overline{\mathbf{b}}_{t-1}$ to a new portfolio $\mathbf{b}_t$. After the rebalancing portfolio period, return s_{t-1} will be decomposed into two parts: the net wealth s^c_{t-1} in the new portfolio $\mathbf{b}_t$ and the transaction costs incurred during the buying and selling. If the wealth on asset i before rebalancing at the $(t-1)^{th}$ period is higher than that after reblancing, that is, $\overline{b}^i_{t-1}s_{t-1} \geq b^i_t s^c_{t-1}$, then there will be a selling rebalancing with transaction cost rates c_s. Otherwise, a buying rebalancing with transaction cost rates c_b is required. Denote $\mathbf{V}_{t-1} = [V^1_{t-1}, \cdots, V^m_{t-1}]$ as the *trading volume* at the $(t-1)^{th}$ period, where $V^i_{t-1} = \overline{b}^i_{t-1}s_{t-1} - b^i_t s^c_{t-1}$, and let x^+ be the positive part of x. Formally,

$$s_{t-1} = s^c_{t-1} + c_s \sum_{i=1}^{m} (V^i_{t-1})^+ + c_b \sum_{i=1}^{m} (-V^i_{t-1})^+.$$

Thus, after k trading periods, the portfolio manager obtains a *portfolio cumulative wealth* with transaction cost S^c_k:

$$S^c_k = \prod_{j=1}^{k} [s_j - (c_s \sum_{i=1}^{m} (V^i_j)^+ + c_b \sum_{i=1}^{m} (-V^i_j)^+)].$$

With this, we can see that the cumulative wealth and transaction cost heavily rely on the target portfolio $\mathbf{b}_t$, price relative $\mathbf{x}_t$ $(t = 1, \cdots, n)$ and the *total trading times* k $(k \leq n)$.

Now we turn to the algorithms: UP-based, SCRP and OLU, which investigate the online PS with transaction cost in the framework of competitive analysis and all belong to the *benchmarks* and *follow-the-winner* category [Li and Hoi, 2014]. The UP-based transaction cost aware strategy is extended from Cover's UP and employs $\mathbf{b}_i = \frac{\int_{\Delta m} \mathbf{b} S^c_{i-1}(\mathbf{b}) \mathrm{d}\mu(\mathbf{b})}{\int_{\Delta m} S^c_{i-1}(\mathbf{b}) \mathrm{d}\mu(\mathbf{b})}$ as a target portfolio. Although this strategy keeps a moving target portfolio that can avoid the impact of market information, it rebalances the portfolio vector at every period, which means that the trade volume $\mathbf{V}_j \neq 0$, and thus loss is $c_s \sum_{i=1}^{m} (V^i_j)^+ + c_b \sum_{i=1}^{m} (-V^i_j)^+$ at every period j, where $j = 1, \cdots, n$.

Rather than rebalancing at every investment period, SCRP considers choosing k suitable periods from the whole investment periods $(1, \cdots, n)$ to rebalance and keeps a fixed target portfolio $\mathbf{b} = [1/m, \ldots, 1/m]^T \in \mathbb{R}^m_+$, which means that the trading volume $\mathbf{V}_j > 0$ for the $t_1{}^{th}, \ldots, t_k{}^{th}$ periods, and $\mathbf{V}_j = 0$ for the other n-k periods, therefore the transaction cost is reduced severely.

OLU uses an online update setting where the target portfolio satisfies $\mathbf{b}_{t+1} = \underset{\mathbf{b} \in \Delta m}{\arg\min}(-log(\mathbf{b}^T \mathbf{x}_t)) + \alpha \left\| \mathbf{b} - \mathbf{b}_t \right\|_1 + \frac{1}{2} \left\| \mathbf{b} - \mathbf{b}_t \right\|_2^2$. The parameter $\alpha > 0$ decides how often the trading is done. A large α leads to lazy update of the portfolio with a small number of transactions, while a small α allows the portfolio to change more frequently, which means $\mathbf{V}_j$ is close or equals to zero for some investment period j, and thus the cost is reduced.

Though these strategies empirically effective on most datasets, they have potential problems. First, the UP based algorithms rebalance it portfolio vector at every investment period, thus the total transaction cost increases rapidly, which results in poor performance in practice. Moreover, UP is also computationally demanding and has unsatisfactory empirical performance [Helmbold *et al*., 1998]. Second, SCRP uses the same weight on each asset/stock in all investment periods, which does not change with the dynamic market and thus degrades the final cumulative wealth. Third, for OLU the difficulty of parameter selection substantially influences the effectiveness of the algorithm and even the final cumulative wealth. Considering these drawbacks of the existing works, we try to develop a new SUP strategy with transaction cost, which combines the idea of occasional rebalancing with a moving target portfolio.

4 Semi-Universal Portfolios

4.1 Motivation

Theoretical and empirical results [Helmbold *et al*., 1998; Kozat and Singer, 2011] for CRP strategy show that if a portfolio rebalance only on a subset of the possible trading days, it will reduce the transaction commissions remarkably and thus achieve maximum cumulative wealth on most datasets. This idea is also true for UP strategy. Let us consider a toy example to illustrate this point.

Assume a market consists of two stocks and the investment periods is set to 6 days. Let $[3, 3, 3, 0.33, 0.33, 0.33]$ and $[1/1.1, 1/1.1, 1/1.1, 1.1, 1.1, 1.1]$ be the price sequence of the two stocks. Here $c = 0.1$. We assume the initial vector of UP, SCRP, OLU, SUP are all $\mathbf{b}_1 = [1/2, 1/2]^T$. Let $S_n(UP)$ be the wealth achieved by the UP strategy:

	C_2	C_3	C_4	C_5	C_6	Cost	S_n^c
UP	0.046	0.092	0.095	0.059	0.038	0.329	1.384
SCRP	0.052	0.099	0	0	0	0.152	1.567
OLU	0.039	0.067	0.103	0.032	0.028	0.269	1.457
SUP	0.046	0.092	0	0	0	0.137	1.577

Table 1: The cost and wealth gain of SUP, UP, SCRP, OLU when $c = 0.1$.

$S_n^c(UP) = S_0 \prod_{t=1}^n (s_t^c)$. The portfolio will rebalance at every period. For SUP, if the cost of rebalancing Cost_t is larger than the factor of increasing wealth s_t in UP at the end of t^{th} period, we will not trade at the beginning of $(t+1)^{th}$ and the portfolio will run itself. Otherwise, the portfolio will rebalance to Cover's UP $\mathbf{b}_{t+1}$. This is the main point of SUP strategy. For OLU, we use updates setting in Section 3.1 where the parameters are $\eta = 20, \alpha = 0.1$. OLU will also trade at every period. For SCRP, we will not rebalance at $(t+1)^{th}$ period when $\mathrm{Cost}_t > s_t$. Otherwise, the portfolio will rebalance to the target portfolio $\mathbf{b} = [1/2, 1/2]^T$. We compute the cost, the wealth gain and rebalancing times for each strategy, which are illustrated in Table 1.

The results clearly show that the total cost of UP is the largest with 5 times rebalancing, which thus leads to the smallest wealth gain. By Contrast, the proposed methods has the least transaction commissions with 2 times rebalancing and attains the largest total wealth. Furthermore, both SUP and SCRP do not change the portfolio at $4^{th}, 5^{th}, 6^{th}$ period, but the wealth gain of SUP is larger than SCRP, which shows that the moving target portfolio used in SUP is superior to the fixed target portfolio used in SCRP. Note that although the toy example is on two assets, such advantage can be easily extended to the scenario of multiple assets. Based on the above motivation, we propose SUP strategy in the following and then solve it by developing an universal algorithm.

4.2 Two Semi-Universal Portfolio Strategies

SUP is to exploit the rebalancing of Cover's UP at some selected instants. We first give a standard SUP strategy, which rebalances the portfolio vector to UP at selected instants with all the knowledge of the past market. However, with the increase of investment periods, it might be better to consider only the most recent market. Therefore, we present an alternative version SUP-q strategy by considering a market sliding window–q, in which the rebalanced portfolio depends on the price relative vector of the last q periods.

SUP strategy

We divide the total investment periods n into an arbitrary number (say k) segments, and fit each segment to a Cover's UP. Let $\mathcal{T}_{k,n}$ be a rebalancing path with k rebalancings, which is represented by $(t_1, \cdots, t_k)$. Given n and k, there exists $\binom{n-1}{k}$ such possible choices for $\mathcal{T}_{k,n}$.

For an arbitrary sequence of price relative vectors $\mathbf{x}^n$ and a given Cover UP $\mathbf{b}_i, i = 1 \dots k$, a competing SUP with a rebalancing path $\mathcal{T}_{k,n}$ divides $\mathbf{x}^n$ into $k+1$ segments such that $\mathbf{x}^n$ is obtained by the concatenation of

$$\{\mathbf{x}_1, \dots, \mathbf{x}_{t_1 - 1}\}\{\mathbf{x}_{t_1}, \dots, \mathbf{x}_{t_2 - 1}\} \dots \{\mathbf{x}_{t_k}, \dots, \mathbf{x}_n\}.$$

The SUP with the rebalancing path $\mathcal{T}_{k,n}$ rebalances only to Cover's UP $\mathbf{b}_i$ ($i = 1, \dots, k$) on the selected times $(t_1, ..., t_k)$. For notational simplicity, we assume $t_0 = 1$ and $t_{k+1} = n + 1$. Suppose we pay transaction cost only at the start of each segment that rebalances to Cover's UP $\mathbf{b}_i$ ($i = 1, \dots, k$).

In each segment, this SUP will achieve the accumulated wealth $s_i = \mathbf{b}_i^T \bigotimes_{t=t_{i-1}}^{t_i - 1} \mathbf{x}_t$, where $\bigotimes$ denotes element-wise product, $\mathbf{b}_1 = \left(\frac{1}{m}, \dots, \frac{1}{m}\right), \mathbf{b}_i = \frac{\int_{\Delta m} \mathbf{b} S_{t_i-1}^c(\mathbf{b}) \mathrm{d}\mu(\mathbf{b})}{\int_{\Delta m} S_{t_i-1}^c(\mathbf{b}) \mathrm{d}\mu(\mathbf{b})}$, and $S_{t_i-1}^c(\mathbf{b}) = S_{t_i-1}(\mathbf{b}) - \sum_{j=1}^{t_i-1} \mathrm{Cost}_j, S_{t_i-1}(\mathbf{b}) = \prod_{j=1}^{t_i-1} \mathbf{b}^T \mathbf{x}_j$, $\mu(\mathbf{b})$ equals to the uniform distribution. Then, for a selected rebalancing path $\mathcal{T}_{k,n}$, the total accumulated wealth including commission costs $c = c_s + c_b$ on $\mathbf{x}^n$ by using a Cover's UP $\mathbf{b}$ is $S^c(\mathbf{x}^n | \mathbf{B}_k, \mathcal{T}_{k,n}) = \Pi_{i=1}^{k+1} s_i^c$, which is the combined gains of each segment's, where s_i^c is the net wealth in i^{th} segment, i.e., $s_i^c = s_i - \mathrm{Cost}_i$.

SUP-q strategy

For the SUP strategy, $S_{t_i-1}^c(\mathbf{b})$ is the net wealth in all the last $t_i - 1$ periods. While for SUP-q strategy, $S_{t_i-1}^c(\mathbf{b})$ means the net wealth in the last q periods. That is, when we rebalance to the universal portfolio, the portfolio depends only on the prices of the last q periods, *i.e.*, $S_{t_i-1}^c(\mathbf{b}) = \prod_{k=t_i-1-q}^{t_i-1} s_k^c$. The other parts are the same as in the SUP strategy.

We next investigate the proposed strategies from the framework of competing algorithms. In determining the best algorithm in the competing class, we attempt to outperform all such portfolios, including the one that has been optimized by choosing the rebalancing path $\mathcal{T}_{k,n}$ and k and UP strategy $\mathbf{b}_i$ in each segment based on observing the entire sequence $\mathbf{x}_t, t = 1, ..., n-1$ in advance, including the transaction costs. In other words, we try to seek an algorithm such that even in the worst case, it will achieve the performance of the best algorithm in the competition class, uniformly for all sequences $\mathbf{x}^n$ and all n. As such, we try to minimize the following *wealth ratio* with transaction cost:

$$R_{\mathbf{b}}^c(n) = \sup_{\mathbf{x}^n} \frac{\sup_{\mathcal{T}_{k,n}} S^c(\mathbf{x}^n | \mathbf{B}_k, \mathcal{T}_{k,n})}{S^c(\mathbf{x}^n \left| \widehat{\mathbf{b}}_{u,n}\right.)}$$

where $S^c(\mathbf{x}^n | \widehat{\mathbf{b}}_{u,n}) = \prod_{t=1}^n (\widehat{\mathbf{b}}_{u,t}^T \mathbf{x}_t - \mathrm{Cost_t})$ is the wealth achieved by this algorithm; $\widehat{\mathbf{b}}_{u,t}$ is a sequential assignment at time t, i.e., $\widehat{\mathbf{b}}_{u,t}$ may be a function of $\mathbf{x}_1, ..., \mathbf{x}_{t-1}$ but does not depend on the future; $\mathcal{T}_{k,n}$ represents any rebalancing times $(t_1, ..., t_k)$ and any k. Since the *geometric average* return over the period is $(\mathbf{S}_n^c)^{(1/n)}$, we will minimize the *log geometric average ratio* $\frac{\ln R_{\mathbf{b}}^c(n)}{n}$ alternatively. In this way, we will show that we can construct a sequential portfolio for which the logarithm of this ratio is at most $k \ln(n) + O(k+1)$ for any of

$\mathcal{T}_{k,n}, k$ or n, without knowledge of $\mathcal{T}_{k,n}, k$ and n a priori. So we have the following universal theorem.

4.3 Universal Theorem

Theorem 1 *Let $\{\mathbf{x}^t\}_{t\geq 1}$ be an arbitrary sequence of price relative vectors such that $\mathbf{x}_t \in \mathbb{R}_+^m$ for all t and where some components of $\mathbf{x}_t$ can be zero. Then, for all $\varepsilon > 0$ and given a Cover's universal portfolio $\mathbf{b}_i, i = 1 \ldots k$, $\mathbf{b}_i \in \mathbb{R}_+^m, \Sigma_{j=1}^m b_i^j = 1$, we can construct sequential portfolios $\widehat{b}_{u,t}$ with complexity linear in t per investment period, such that, when applied to $\{\mathbf{x}^t\}_{t\geq 1}$ for any $c = c_s + c_b$, and for all k, n and a parameter $\delta > 0$ for random walks, the wealth ratio $R_{\mathbf{b}}^c(n)$ satisfies*
(i) SUP strategy,

$$\frac{\ln R_{\mathbf{b}}^c(n)}{n} \leq \delta(k+1)(m-1)\frac{\ln((1+c)n+1)}{n} + (k+\varepsilon)\frac{\ln n}{n} + \frac{1}{n}(log(1+\varepsilon) + klog\frac{1}{\varepsilon})$$

(ii) SUP-q strategy,

$$\frac{\ln R_{\mathbf{b}}^c(n)}{n} \leq \delta(k+1)(m-1)\frac{\ln((1+c)q+1)}{n} + (k+\varepsilon)\frac{\ln n}{n} + \frac{1}{n}(log(1+\varepsilon) + klog\frac{1}{\varepsilon})$$

for any $\mathcal{T}_{k,n}$ representing rebalancing times $(t_1, ..., t_k)$ and any k, such that $\widehat{\mathbf{b}}_{u,t}$ does not depend on $\mathcal{T}_{k,n}$, k or n.

Theorem 1 states that given a UP strategy $\mathbf{b}_i$ $(i = 1 \ldots k)$, the logarithm of the wealth ratio of the universal sequential portfolio $\widehat{\mathbf{b}}_{u,t}$ is within $O(k\ln(n))$ for any n of the best batch SUP with any k rebalancing times (tuned to the underlying sequence), uniformly, for every sequence of price relatives $\{\mathbf{x}^t\}_{t\geq 1}$ and c.

Note that the bound in Theorem 1 is different from those given in SCRP [Kozat and Singer, 2011]. Our bound consists of two parts: one is the regret on universal portfolio $\mathbf{b}_i$, which comes from [Blum and Kalai, 1997]; another is the regret on the rebalancing path $\mathcal{T}_{k,n}$, where a Krichevsky-Trofimov (KT) weighting [Willems, 1996] of the probability of the path is used, which is the same as that in SCRP.In addition, our regret is related to the parameter δ of random walks [Kalai and Vempala, 2000] and c, which is also different from SCRP. Due to space limit, here we omit the proof of Theorem 1.

4.4 Factor Graph-Based Implementation

We now need to solve semi-universal portfolio $\widehat{\mathbf{b}}_{u,t}$. Because SUP defined in Section 4.1 is ideal, that is, if we compare the wealth and cost at every period, for given n and k, there exist $\binom{n-1}{k}$ such possible rebalancing time $\mathcal{T}_{k,n}$. Moreover, with no information of the future price, we don't know the parameter k in the real world. Therefore, there are 2^{n-1} different rebalancing path and the algorithm complexity will be very large.

Due to the above fact, we will use the factor graph-based implementation [Kschischang *et al.*, 2001] to solve the $\widehat{\mathbf{b}}_{u,t}$. In Figure 1, any directed path represents a rebalancing path where a horizontal move denotes no rebalancing, while an upward move represents a rebalancing. We

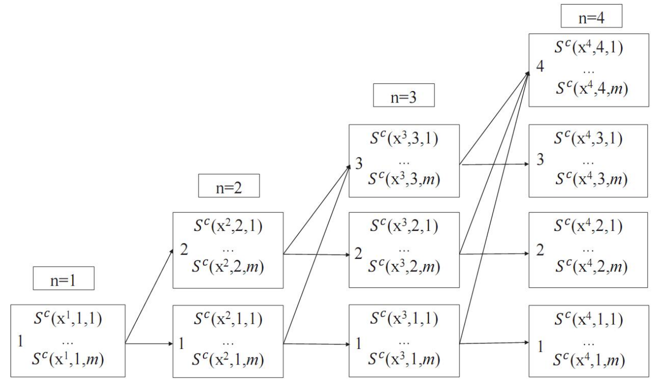

Figure 1: *The factor graph. Each box represents a state, where each number in the box is the time of the last rebalancing instant. In each box, we have accumulated wealth for each stock, $j = 1, m$.*

label the last transition times with state variables $s_t = 1, 2, ..., t$, and each class represented as a box. We then define $S_t^c(\mathbf{x}^t, s, j)$ as the net wealth of j^{th} asset/stock achieved on $\mathbf{x}^t$ by all sequential strategies that have the last rebalancing time at s_t , i.e., if $s_t = s$ then $S_t^c(\mathbf{x}^t, \mathrm{s}, j) = P_{\mathrm{kt}}(\mathrm{s}_t = \mathrm{s}|\mathrm{s}_{t-1} = \mathrm{s})S_{t-1}^c(\mathbf{x}^{t-1}, \mathrm{s}, j)x_t^j, s = 1, \cdots, t-1$, where P_{kt} is KT weighting [Willems, 1996]. If there is a rebalancing at time t, then, the net wealth will rebalance from $S_{t-1}^c(\mathbf{x}^{t-1}, s, j)$ to $S_t^c(\mathbf{x}^t, t, j)$, and $S_t^c(\mathbf{x}^t, t, j) = \sum_{\mathrm{s}=1}^{t-1}(\sum_{\mathrm{j}=1}^{m} S_{t-1}^c(\mathbf{x}^{t-1}, \mathrm{s}, j))P_{\mathrm{kt}}(\mathrm{s}_t = t|\mathrm{s}_{t-1} = \mathrm{s})b_t^j x_t^j$, b_t^j is the universal portfolio of j asset at t^{th} period. We can formulate the wealth achieved by each subsets, and then, the wealth gain can be the sum of all the subset $S_{\mathrm{u}}^c(\mathbf{x}^t) = \sum_{\mathrm{s}=1}^{t-1} S_t^c(\mathbf{x}^t, \mathrm{s}, j) + S_t^c(\mathbf{x}^t, t, j)$. Since, $S_{\mathrm{u}}^c(\mathbf{x}^{\mathrm{n}}) = \prod_{t=1}^{\mathrm{n}} \frac{S_{\mathrm{u}}^c(\mathbf{x}^t)}{S_{\mathrm{u}}^c(\mathbf{x}^{t-1})} = \prod_{t=1}^{\mathrm{n}} (\widehat{\mathbf{b}}_{u,t})^T \cdot \mathbf{x}^t$. Finally, we can obtain the recurrence formula of $\widehat{\mathbf{b}}_{u,t}$ as follows:

$$\widehat{\mathbf{b}}_{u,t} = \sum_{\mathrm{s}=1}^{t-1}\sum_{j=1}^{m} \sigma_{t-1}(\mathrm{s}, j)\{P_{\mathrm{kt}}(\mathrm{s}_t = \mathrm{s}\,|\mathrm{s}_{t-1} = \mathrm{s})\,\mathbf{e}_j + P_{\mathrm{kt}}(\mathrm{s}_t = t\,|\mathrm{s}_{t-1} = \mathrm{s})\,\mathbf{b}_t\} \quad (1)$$

Here, $\sigma_{t-1}(\mathrm{s}, j) = \frac{S_{t-1}^c(\mathbf{x}^{t-1}, \mathrm{s}, j)}{\sum_{\mathrm{s}=1}^{t-1}\sum_{\mathrm{r}=1}^{m} S_{t-1}^c(\mathbf{x}^{t-1}, \mathrm{s}, \mathrm{r})}$, $\mathbf{b}_t$ is universal portfolio at t^{th} investment period. Due to space limit, here we omit some detailed formulas and computation analysis.

Based on the analysis above, we can design the proposed algorithm with a polynomial complexity. The computation process for $\widehat{\mathbf{b}}_{u,t}$ of SUP strategy is outlined in Algorithm 1, and for $m > 2$ it is also outlined in the random walk Algorithm 2.

5 Experiments

In this section, we use the *index-wealth gain* and *turnover* to measure the performance of the SUP algorithms (SUP and SUP-q), and compare them with four of the most relevant transaction cost aware strategies (CRP [Cover, 1991],

Algorithm 1 Online factor graph implementation for SUP

Input: $\mathbf{x}^n$ price relative vector
Output: b the portfolio
Procedure:
Initialize $\mathbf{b}_1 = [1/m, \ldots, 1/m]$, $S_0(0, 0, :) = \mathbf{b}_1$
$\widehat{\mathbf{b}}_{u,t} = b_1$
for $t = 1$ **to** N **do**
  Wealth gain $\widehat{\mathbf{b}}_{u,t}^T \mathbf{x}_t$
  for $s = 1$ **to** $t-1$ **do**
    for $j = 1$ **to** m **do**
      calculate $S_t^c(\mathbf{x}^t, \mathrm{s}, j)$
    end for
  end for
  for $s = 1$ **to** $t-1$ **do**
    for $j = 1$ **to** m **do**
      calculate $S_t^c(\mathbf{x}^t, t, j)$
    end for
  end for
  calculate $\widehat{\mathbf{b}}_{u,t+1}$ from (1)
end for

Algorithm 2 Random walk implementation for $m > 2$

Input: Minimum coordinate:δ_0 Spacing of grid:δ Number of steps in random walk: S
Output: $TheSamples\Delta m$
Procedure:
for $i = 1$ **to** $Samples$ **do**
  for $s = 1$ **to** $walk$ **do**
    Choose $1 \leq j \leq n-1$ at random
    Choose $X \in \{-1, +1\}$ randomly
    if $\delta_0 \leq r^i + X\delta$ and $\delta_0 \leq r^n - X\delta$ **then**
      $x := Q_t(r^1, r^2, ..., r^n)$
      $y := Q_t(r^1, r^2, ..., r^j + X\delta, ..., r^n - X\delta)$ % [Kalai and Vempala, 2000].
      With probability $Min(1, x/y)$ $r^j := r^j + X\delta$, $r^n = r^n - X\delta$;
    end if
  end for
end for

SCRP [Kozat and Singer, 2011], OLU [Das *et al.*, 2013] and UP [Blum and Kalai, 1997]) on two datasets: NYSE(O) and SP500. The first two strategies CRP and SCRP are the "Benchmark" strategies, UP and OLU belong to "follow the winner" category. Moreover, SCRP and OLU are the two newest transaction cost aware strategies for online PS. We focus on the performance of investing into two stocks and evaluate the average wealth over 50 pairs of stocks.

5.1 Datasets

Experiments are conducted on two historical datasets[1]. The first one is the well-known NYSE(O) dataset, which consists of 36 stocks in New York Stock Exchange for a 22-year period [Cover, 1991]. The second is SP500, which is the Standard & Poor's 500 with 1276 daily prices from Jan. 2, 1998 to Jan. 31, 2003. We choose 2 days as an investment period.

[1] All datasets and their compositions can be downloaded from http://olps.stevenhoi.org/.

Table 2: Average net wealth for 50 independent trials (c=0, 0.001, 0.01, 0.02 and 0.05) on the NYSE(O) dataset.

Strategy	c=0	c=0.001	c=0.01	c=0.02	c=0.05
CRP	23.519	23.387	22.233	21.017	17.754
SCRP	23.754	23.753	23.740	23.726	23.684
OLU	23.624	23.487	23.012	22.235	20.037
UP	23.513	23.429	22.693	21.909	19.758
SUP	23.909	23.908	23.894	23.880	23.837
SUP-q	**24.092**	**24.090**	**24.078**	**24.064**	**24.024**

Table 3: Average net wealth for 50 independent trials (c=0, 0.001, 0.01, 0.02 and 0.05) on the SP500 dataset.

Strategy	c=0	c=0.001	c=0.01	c=0.02	c=0.05
CRP	1.7340	1.7339	1.7101	1.6764	1.5792
SCRP	1.7444	1.7410	1.7334	1.7329	1.7312
OLU	1.7439	1.6481	1.6316	1.6152	1.5664
UP	1.7428	1.7406	1.7210	1.6995	1.6371
SUP	1.7538	1.7538	1.7533	1.7527	1.7511
SUP-q	**1.7541**	**1.7541**	**1.7535**	**1.7530**	**1.7513**

5.2 Results

The two datasets NYSE(O) and SP500 contain 36 and 25 different stocks respectively, from each of which we randomly select 50 pairs of stocks, and invest them using the six strategies: CRP, SCRP, OLU, UP, SUP and SUP-q. Here, we set q=100 in SUP-q, η=20 and α=0.1 in OLU. Table 2 and Table 3 presents respectively the wealth achieved by various strategies on NYSE(O) and SP500 when the cost ratio c=0, 0.001, 0.01, 0.02 and 0.05. In all cases, we can see that SUP and SUP-q outperform the other strategies. Intuitively, SUP and SUP-q can reduce the transaction costs and react well to the dynamic market.

Fig. 2 shows the *turnover* results of the six strategies. *turnover* indicates the average percentage of wealth traded in each period, which is used for stability analysis of trading strategy. From Fig. 2, we can see that SUP and SUP-q achieve smaller turnover than the others. With smaller turnover and higher net wealth, the SUP and SUP-q strategies are more profitable and stable, and incur less transaction cost.

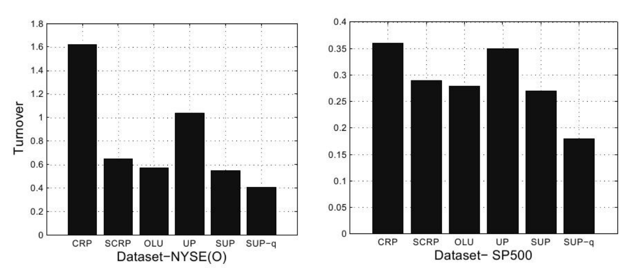

Figure 2: The turnover results with c=0.01.

6 Conclusion

In this paper, we proposed a novel on-line PS strategy named semi-universal portfolio (SUP) with transaction cost, which rebalances its portfolio only at some selected instants. The proposed approach can solve the transaction cost problem of UP strategy caused by frequently adjusting portfolio at every period. Experiments on real markets show that the proposed SUP strategy can achieve better performance than major existing strategies. Future work will study other universal portfolios and do theoretical analysis on the rebalancing problem.

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