A Simple Probabilistic Extension of Modal Mu-Calculus

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Abstract

Probabilistic systems are an important theme in AI domain. As the specification language, PCTL is the most frequently used logic for reasoning about probabilistic properties. In this paper, we present a natural and succinct probabilistic extension of μ-calculus, another prominent logic in the concurrency theory. We study the relationship with PCTL. Surprisingly, the expressiveness is highly orthogonal with PCTL. The proposed logic captures some useful properties which cannot be expressed in PCTL. We investigate the model checking and satisfiability problem, and show that the model checking problem is in **UP** $\cap$**co-UP**, and the satisfiability checking can be decided via reducing into solving parity games. This is in contrast to PCTL as well, whose satisfiability checking is still an open problem.

1 Introduction

Temporal logics are heavily used in theoretical computer science and AI-related fields. Among those, modal μ-calculus receives a lot of attraction ever since Kozen's seminal work [Kozen, 1983]. See for example, [Banieqbal and Barringer, 1987; Katoen, 1998; Walukiewicz, 2000; Berezin, 2002]. Moreover, various temporal logics including LTL [Pnueli, 1977], CTL [Emerson and Clarke, 1980], CTL* [Emerson and Halpern, 1986] are extensively studied. It is known that their expressiveness is strictly less [Dam, 1995] than μ-calculus (aka. μTL), and their model checking algorithm has been proposed: for CTL the problem can be solved in polynomial time, whereas for LTL the problem is **PSPACE**-complete [Sistla and Clarke, 1985].

Probabilistic systems, such as Markov chains and Markov decision processes, are an important theme in AI domain. To reason about properties for probabilistic systems, the logic CTL was first extended with probabilistic quantifiers in [Hansson and Jonsson, 1994] , resulting in the logic PCTL. Intuitively, $(a \mathsf{U}^{\geq 0.9} b)$ means that the probability of reaching b-states along a-states is at least 0.9. At the same time, probabilistic LTL and its extension PCTL* have all been studied. As in the classical setting, model checking problem for PCTL can be solved in polynomial time, whereas only exponential algorithms are known for LTL [Couvreur *et al.*, 2003]. There have also been several attempts to extend μTL with probabilities in the literature. As we shall discuss in the related work, the extensions are either highly non-trivial in terms of the complexity of the corresponding model checking and satisfiability problems, or hindered from the restriction of fixpoint nesting.

We propose a natural and succinct extension of μTL in this paper, and name it PμTL. The logic is acquired by equipping the next operator with probability quantifiers, and keeping other parts as standard μTL. We have for instance the formula $\nu Z.(a \wedge \mathsf{X}^{\geq 0.8} Z)$. We investigate the model checking, expressiveness, and satisfiability problems of PμTL.

In detail, we first investigate the model checking problem of PμTL upon Markov chains. It turns out to be a straightforward adaptation of the classical algorithms for μTL, and the complexity remains in **UP** $\cap$ **co-UP**. We then give a comprehensive study on the expressiveness of PμTL by comparing with PCTL, and prove that PμTL is orthogonal with PCTL in expressiveness. However, for the qualitative fragments (i.e., probabilities may appear in a formula are only 0 and 1), we show that qualitative PμTL is strictly more expressive (w.r.t. finite Markov chains). On the other side, the satisfiability checking is quite challenging: we exploit the notion of probabilistic alternating parity automata (PAPA, for short), and reduce the SATISFIABILITY problem into the EMPTINESS problem of PAPA. Further, this is reduced to solving parity games, and it is shown that both of these two problems are in 2**EXPTIME**. This is in contrast to PCTL as well, whose SATISFIABILITY checking is still an open problem (cf. [Brázdil *et al.*, ; Bertrand *et al.*, 2012]).

An illustrating example We introduce a running example to motivate our work: Suppose there is a hacker trying to attack a remote server. The hacker has a supercomputer at hand and is trying to guess the password in a brute-force manner. For simplicity, we assume the password is a sequence of l let-

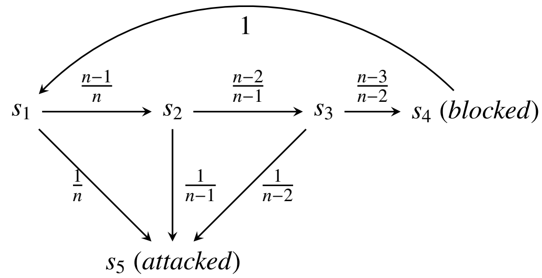

Figure 1: An illustration of the hacking process

ters, each of which is from '0'-'9', 'a'-'z', and 'A'-'Z'. Therefore, the total number of possible passwords is $n = 62^l$. The hacker let the supercomputer randomly generate a password, and see whether the decryption succeeds. If yes, the hacker wins; otherwise he tries with another one. However, if the supercomputer generates three wrong passwords in a row, it will be blocked for a certain amount of time until it can start another round of attacking — assuming that the password may be changed during the blocked moment, hence it does not make sense for the supercomputer to store all generated passwords. The whole process is illustrated in Fig. 1. Starting from s_1, we can see that the probability of eventually reaching *attacked*, i.e., the hacker decrypts successfully, equal 1, no matter how big l is (hence, the PCTL formula $\mathsf{F}^{\geq 1}$*attacked* holds), and we may conclude that the system is unsafe — this is of course against our intuition, as such system is considered to be safe if l is big enough. However, as we will show later, all PCTL formulae are not capable of expressing this property. By making use of PμTL, such property of security can be characterized easily as follows: $\nu Z.(\neg attacked \wedge \mathsf{X}^{\geq p} Z))$ with $p = {}^{n-3}/_{n-2}$, where $\neg$*attacked* denotes all other states in Fig. 1 different from s_5.

Motivation from AI perspective The presented logic has the following potential application in AI domain:

- First of all, Markov chains and Markov decision processes are the basic models in several areas of AI. As a logic with semantics defined w.r.t. such models, it could definitely be used in designating probability-relevant properties upon them. Particularly, the properties that could not be expressed by PCTL.
- Motion planing is an important topic in AI area, where standard μTL has once been adopted [Bhatia *et al.*, 2011], because of its powerful expressiveness and the decidability of its Satisfiability problem. Thus, we expect that PμTL could be used in stochastic motion planning — since, PμTL is a decidability-preserving extension of μTL.
- Fixpoints play an important role in mathematics and computer science. In AI area, it is used to designate non-terminating behaviors of intelligent systems, such as maintenance goals [Singh, 1998]. Fixpoints act as the elementary ingredients in PμTL, hence such logic can also be used in such a situation.

Related work Probabilistic extensions of μTL have been studied by many authors: e.g., μ-calculi proposed in [Morgan and McIver, 1997; Huth and Kwiatkowska, 1997; de Alfaro and Majumdar, 2001; McIver and Morgan, 2002; 2007; Mio, 2012b] interpret a formula as a function from states to real values in $[0, 1]$, whose semantics is different from PμTL. A further extension of μ-calculus was proposed in [Mio, 2012a], which is able to encode the full PCTL. However, the model checking and Satisfiability algorithms are still unknown for these calculi and are "far from trivial" [Mio, 2012b]. The other probabilistic μ-calculus was introduced in [Cleaveland *et al.*, 2005] along with a model checking algorithm for it. Moreover, it is able to encode PCTL formulae as well. However, that calculus only allows alternation-free formulae (cf. [Emerson and Lei, 1986]).

Very recently — and independently —, Castro, Kilmurray, and Piterman present another extension by adding fixpoints to full PCTL [Castro *et al.*, 2015]. The calculus they introduced is more expressive than logics PCTL and PCTL*. Moreover, it is also easy to see that it is a proper super logic of our logic PμTL as well. They show the model checking problem is in **NP** $\cap$**co-NP**. We note that some examples in our paper are similarly investigated in [Castro *et al.*, 2015]. Since the logic in [Castro *et al.*, 2015] subsumes PCTL, its Satisfiability problem is also left open. However in this paper we show Satisfiability of PμTL could be reduced to solving parity games, which makes this problem solvable in 2**EXPTIME**.

2 Preliminaries

In this paper, we fix a countable set $\mathcal{A}$ of *atomic propositions*, ranging over a, b, a_1 etc, and fix a countable set $\mathcal{Z}$ of *formula variables*, ranging over Z, Z_1 etc.

A *Markov chain* is a tuple $M = (S, \boldsymbol{T}, L)$, where S is a finite set of *states*; $\boldsymbol{T} : S \times S \to [0, 1]$ is the matrix of transition-probabilities, fulfilling $\sum_{s' \in S} \boldsymbol{T}(s, s') = 1$ for every $s \in S$; and $L : S \to 2^{\mathcal{A}}$ is the labeling function. A *pointed Markov chain* is a pair (M, s) where M is a Markov chain $(S, \boldsymbol{T}, L)$ and $s \in S$ is the *initial state*.

An (infinite) *path* π of M is an infinite sequence of states $s_0, s_1, \ldots,$ such that $s_i \in S$ and $\boldsymbol{T}(s_i, s_{i+1}) > 0$ for each i. A basic *cylinder* $\mathrm{cyl}(s_0, s_1, \ldots, s_n)$ of M is the set of infinite paths having $s_0, s_1, \ldots, s_n$ as the prefix.

According to the standard theory of Markov process, the pointed Markov chain (M, s) uniquely derives a *measure space* $(\Pi_{M,s}, \Delta_{M,s}, \mathrm{prob}_{M,s})$ where $\Pi_{M,s}$ consists of all infinite paths of M; $\Delta_{M,s}$ is the minimal Borel field containing all basic cylinder of M (i.e., $\Delta_{(M,s)}$ is closed under complementation and countable intersection); and the measuring function $\mathrm{prob}_{M,s}$ fulfills: $\mathrm{prob}_{M,s}(\mathrm{cyl}(s_0, s_1, \ldots, s_n))$ equals 0 if $s \neq s_0$, and equals $\prod_{i<n} \boldsymbol{T}(s_i, s_{i+1})$ otherwise. We say a set $P \subseteq \Pi_{M,s}$ is *measurable* if $P \in \Delta_{M,s}$. [Vardi, 1985] shows that the intersection of $\Pi_{M,s}$ and an omega-regular set must be measurable.

The syntax of PCTL formulae is described by the following abstract grammar:

$$f ::= \top \mid \bot \mid a \mid \neg a \mid \mathsf{X}^{\sim p} f \mid f \wedge f \mid f \vee f \mid f \mathsf{U}^{\sim p} f \mid f \mathsf{R}^{\sim p} f$$

where $\sim \in \{>, \geq\}$ and $p \in [0, 1]$. We also abbreviate $\top \mathsf{U}^{\sim p} f$ and $\bot \mathsf{R}^{\sim p} f$ as $\mathsf{F}^{\sim p} f$ and $\mathsf{G}^{\sim p} f$, respectively.

Semantics of a PCTL formula is given w.r.t. a Markov chain. For each PCTL formula f and a Markov chain $M = (S, \boldsymbol{T}, L)$, we will use $[\![f]\!]_M$ to denote the subset of S satisfying f, inductively defined as follows.

- $[\![\top]\!]_M = S$; $[\![\bot]\!]_M = \emptyset$.
- $[\![a]\!]_M = \{s \in S \mid a \in L(s)\}$; $[\![\neg a]\!]_M = \{s \in S \mid a \notin L(s)\}$.
- $[\![\mathsf{X}^{\sim p} f]\!]_M = \{s \in S \mid \sum_{s' \in [\![f]\!]_M} \boldsymbol{T}(s, s') \sim p\}$.
- $[\![f_1 \wedge f_2]\!]_M = [\![f_1]\!]_M \cap [\![f_2]\!]_M$; $[\![f_1 \vee f_2]\!]_M = [\![f_1]\!]_M \cup [\![f_2]\!]_M$.
- $[\![f_1 \mathsf{U}^{\sim p} f_2]\!]_M = \{s \in S \mid \mathrm{prob}_{M,s}\{\pi \in \mathrm{cyl}(s) \mid \pi \models f_1 \mathsf{U} f_2\} \sim p\}$ and $[\![f_1 \mathsf{R}^{\sim p} f_2]\!]_M = \{s \in S \mid \mathrm{prob}_{M,s}\{\pi \in \mathrm{cyl}(s) \mid \pi \models f_1 \mathsf{R} f_2\} \sim p\}$.

In addition, for an infinite path $\pi = s_0, s_1, \ldots$ of M, the notation $\pi \models f_1 \mathsf{U} f_2$ stands for that there is some $i \geq 0$ such that $s_i \in [\![f_2]\!]_M$ and $s_j \in [\![f_1]\!]_M$ for each $j < i$. Meanwhile, $\pi \models f_1 \mathsf{R} f_2$ holds if either $\pi \models f_2 \mathsf{U}(f_1 \wedge f_2)$ or $s_j \in [\![f_2]\!]_M$ for each j. To simplify notations, in what follows we denote by $M, s \models f$ whenever $s \in [\![f]\!]_M$ holds.

3 PμTL, Syntax and Semantics

In this section we present a simple probabilistic extension of modal μ-calculus, called PμTL. The syntax of PμTL formulae is depicted as follows:

$$f ::= \top \mid \bot \mid a \mid \neg a \mid Z \mid \mathsf{X}^{\sim p} f \mid f \wedge f \mid f \vee f \mid \mu Z.f \mid \nu Z.f$$

Semantics of a PμTL formula is given w.r.t. a Markov chain $M = (S, \boldsymbol{T}, L)$ and an assignment $e : \mathcal{Z} \to 2^S$. Similarly, for each PμTL formula f, we denote by $[\![f]\!]_M(e)$ the state set satisfying f under e. Inductively:

- $[\![\top]\!]_M(e) = S$ and $[\![\bot]\!]_M(e) = \emptyset$.
- $[\![a]\!]_M(e) = \{s \in S \mid a \in L(s)\}$ and $[\![\neg a]\!]_M(e) = \{s \in S \mid a \notin L(s)\}$.
- $[\![Z]\!]_M(e) = e(Z)$.
- $[\![\mathsf{X}^{\sim p} f]\!]_M(e) = \{s \in S \mid \sum_{s' \in [\![f]\!]_M(e)} \boldsymbol{T}(s, s') \sim p\}$.
- $[\![f_1 \wedge f_2]\!]_M(e) = [\![f_1]\!]_M(e) \cap [\![f_2]\!]_M(e)$ and $[\![f_1 \vee f_2]\!]_M(e) = [\![f_1]\!]_M(e) \cup [\![f_2]\!]_M(e)$.
- $[\![\mu Z.f]\!]_M(e) = \bigcap\{S' \subseteq S \mid [\![f]\!]_M(e[Z \mapsto S']) \subseteq S'\}$ and $[\![\nu Z.f]\!]_M(e) = \bigcup\{S' \subseteq S \mid [\![f]\!]_M(e[Z \mapsto S']) \supseteq S'\}$.

Indeed, $[\![\mu Z.f]\!]_M(e)$ (resp. $[\![\nu Z.f]\!]_M(e)$) could be computed as in the classical setting via the following iteration:

1. let $S_0 = \emptyset$ (resp. $S_0 = S$);
2. subsequently, let $S_{i+1} = [\![f]\!]_M(e[Z \mapsto S_i])$;
3. stops if $S_{\ell+1} = S_\ell$, and returns S_ℓ.

Note that the algorithm obtains a monotonic chain with such an iteration, and hence it must terminate within finite steps. Actually, $[\![\mu Z.f]\!]_M(e)$ (resp. $[\![\nu Z.f]\!]_M(e)$) captures the least (resp. greatest) solution of $X = [\![f]\!]_M(e[Z \mapsto X])$ within 2^S.

Semantical definition of PμTL formulae also yields the model checking algorithm.

Theorem 1. *The model checking problem of PμTL is in* ***UP*** $\cap$***co-UP***.

Indeed, the proof is analogous to the non-probabilistic version [Jurdziński, 1998; Wilke, 2002] and the only noteworthy difference lies from handling $\mathsf{X}^{\sim p}$- subformulae, opposing to $\Box$- and $\Diamond$- subformulae, which could be proceeded in (deterministic) polynomial time.

In what follows, we directly denote by $[\![f]\!]_M$ in the case that f is a closed formula (i.e., each variable of f is bound), and we also denote by $M, s \models f$ if $s \in [\![f]\!]_M$.

Below we give some example properties:

(1) The formula $\nu Z.(a \wedge \mathsf{X}^{>0.8} Z)$ describes that there exists an a-region, where each state has less than 0.2 probability to escape from it immediately (i.e., in one step).

(2) $\nu Z.(a \wedge \mathsf{X}^{>0} \mathsf{X}^{>0} Z)$ says that there is a cycle in the Markov chain, such that a holds at least in every even step.

(3) $M, s \models \mu Z.(a \vee \mathsf{X}^{\geq 0.6} Z)$ if some a-state is reachable from s, but at each step, one just has some probability (not less than 0.6) to go on with the right direction.

(4) The PμTL formula $\mu Z.(b \vee (a \wedge \mathsf{X}^{\geq 1} Z))$ holds if $a \mathsf{U} b$ holds along each path. It is stronger than the property described by the PCTL formula $a \mathsf{U}^{\geq 1} b$. For the latter allows the existence of a-cycles.

(5) As a more complicated example, the formula $\nu Z_1.(a \vee \mu Z_2.(a \vee \mathsf{X}^{>0} Z_2) \wedge \mathsf{X}^{\geq 1} Z_1)$ just tells the story that "a will be surely encountered", as described by $\mathsf{F}^{\geq 1} a$ with PCTL.

Given a PμTL formula f and a bound variable Z, we use $\mathcal{D}_f(Z)$ to denote the subformula which binds Z in f. For example, let $f = \mu Z_1(a \wedge \nu Z_2.(b \wedge \mathsf{X}^{>=0.3} Z_2) \vee \mathsf{X}^{>0.6} Z_1)$, then we have $\mathcal{D}_f(Z_1) = f$ and $\mathcal{D}_f(Z_2) = \nu Z_2.(b \wedge \mathsf{X}^{\geq 0.3} Z_2)$.

We say that a PμTL formula f is *guarded*, if the occurrence of each bound variable Z in $\mathcal{D}_f(Z)$ is in the scope of some X-operator. The following theorem could be proven in a same manner as that in [Walukiewicz, 2000].

Theorem 2. *For each PμTL formula f, there is a guarded formula f' such that $[\![f']\!]_M(e) = [\![f]\!]_M(e)$ for every M and e.*

Thus, in what follows, we always assume that each PμTL formula is guarded.

4 Expressiveness

In this section, we will give a comparison between PμTL and PCTL, and we are only concerned about closed PμTL formulae. For a PμTL formula f and a PCTL formula g, we say that f and g are *equivalent* if $[\![f]\!]_M = [\![g]\!]_M$ for every Markov chain M, denoted as $f \equiv g$.

First of all, we will show that some PμTL formula could not be equivalently expressed by any PCTL formula.

Theorem 3. *Let $f = \nu Z.(a \wedge \mathsf{X}^{\geq 0.5} Z)$, then $g \not\equiv f$ for every PCTL formula g.*

Proof. To show this, we need first construct two families of Markov chains, namely, $M_0, M_1, \ldots,$ and $M'_0, M'_1, M'_2, \ldots$.

For the first group, let $M_n = (\{s_0, s_1, \ldots, s_n\}, \boldsymbol{T}_n, L_n)$, where: $\boldsymbol{T}_n(s_0, s_0) = 1$ and $\boldsymbol{T}_n(s_{i+1}, s_i) = 1$ for each $i < n$ (hence $\boldsymbol{T}_n(s_i, s_j) = 0$ for any other s_i, s_j). In addition, $L_n(s_0) = \emptyset$ and $L_n(s_i) = \{a\}$ for each $0 < i \leq n$.

For the second ones, let $M'_n = (\{s'_0, s'_1, \ldots, s'_n\}, \boldsymbol{T}'_n, L'_n)$ where: $\boldsymbol{T}'_n(s'_n, s'_n) = \boldsymbol{T}'_n(s'_n, s'_{n-1}) = 0.5$, $\boldsymbol{T}'_n(s_0, s_0) = 1$, and

$\boldsymbol{T}'_n(s'_{i+1}, s'_i) = 1$ for every $i < n - 1$. In addition, $L'_n(s'_0) = \emptyset$ and $L'_n(s'_i) = \{a\}$ for each $0 < i \leq n$.

Given a PCTL formula g, let $N(g)$ be the maximal nesting depth of temporal-operators of g. According to [Baier and Katoen, 2008, Thm. 10.45], we have that $M'_n, s'_n \models g$ if and only if $M_n, s_n \models g$ whenever $n \geq N(g)$.

Observe the fact that $M'_n, s'_n \models f$ and $M_n, s_n \not\models f$ for every $n \geq 1$. Assume that there exists some PCTL formula g fulfilling $f \equiv g$, then we have

$$\begin{aligned} & M'_{N(g)}, s'_{N(g)} \models f \iff M'_{N(g)}, s'_{N(g)} \models g \\ \iff & M_{N(g)}, s_{N(g)} \models g \iff M_{N(g)}, s_{N(g)} \models f \end{aligned}$$

and hence it results in a contradiction. □

Conversely, the following theorem reveals that there also exists some PCTL formula that could not be equivalently expressed by any PμTL formula.

Theorem 4. *Let $f = \mathsf{F}^{\geq 0.5}a$, then $g \not\equiv f$ for every (closed) PμTL formula g.*

Proof. Let $M = (\{s_1, s_2, s_3\}, \boldsymbol{T}, L)$ be the (family of) Markov chain(s) where: $L(s_1) = L(s_2) = \emptyset$, $L(s_3) = \{a\}$, $\boldsymbol{T}(s_1, s_1) = x, \boldsymbol{T}(s_1, s_2) = y, \boldsymbol{T}(s_1, s_3) = z$, and $\boldsymbol{T}(s_2, s_2) = \boldsymbol{T}(s_3, s_3) = 1$, with $x, y, z \in (0, 1)$ and $x + y + z = 1$.

For every PCTL and/or closed PμTL formula g, we let $P_x(g)$ be the proposition that "for the fixed x, there are infinitely many y making $M, s_1 \models g$ and there are infinitely many y making $M, s_1 \not\models g$". We now show that if g is a closed PμTL formula, then there exists some $x_g < 1$ such that $P_x(g)$ does not hold whenever $x \in (x_g, 1)$.

- Such x_g can be arbitrarily chosen if $g = \perp$, $g = \top$, $g = a$ or $g = \neg a$.
- In the case that $g = g_1 \wedge g_2$, assume by contradiction that such x_g does not exist, then it implies that for every $x \in (0, 1)$, there exists some $x' > x$ such that $P_{x'}(g)$ holds. Observe that $M, s_1 \models g$ implies both $M, s_1 \models g_1$ and $M, s_1 \models g_2$; and $M, s_1 \not\models g$ implies either $M, s_1 \not\models g_1$ or $M, s_1 \not\models g_2$. Thus, we can infer that either x_{g_1} or x_{g_2} does not exist, which violates the induction hypothesis.
- Proof for the case of $g = g_1 \vee g_2$ is similar to the above.
- If $g = \mathsf{X}^{\sim p}g'$ and $p \in (0, 1)$, whenever $x \in (\max\{p, 1 - p\}, 1)$, since $\sim \in \{>, \geq\}$, then $M, s_1 \models g$ iff $M, s_1 \models g'$ because $y + z < p$ in such situation. In this case, we may just let $x_g = \max\{x_{g'}, p, 1 - p\}$.
- If $g = \mathsf{X}^{\geq 1}g'$, then we need to distinguish two cases: 1) There exist $x, y \in (0, 1)$ such that $M, s_1 \models g$ holds, then we can immediately infer that both $M, s_2 \models g'$ and $M, s_3 \models g'$. In addition, observe that truth values of g' on s_2 and s_3 are irrelevant to x and y. It implies that in such case $M, s_1 \models g$ iff $M, s_1 \models g'$, and hence, we may just let $x_g = x_{g'}$. 2) There is no such x and y having $M, s_1 \models g$ holds, in such situation, x_g can be any number in $(0, 1)$.
- If $g = \mathsf{X}^{>0}g'$, then the proof is similar to the above.
- When $g = \mathsf{X}^{\geq 0}g'$ (or $g = \mathsf{X}^{>1}g'$), things would be trivial, because g could be reduced to $\top$ (resp. $\perp$) in such case.
- If $g = \mu Z.g'$, we let $g_0 = \perp$ and $g_{i+1} = g'[Z/g_i]$. Since that M is a 3-state Markov chain, then g and $\bigvee_{i \leq 3} g_i$ share the same truth value at every state of M. This indicates that all least fix-points could be eliminated w.r.t. such Markov chain.
- When $g = \nu Z.g'$, the preprocessing is almost similar, but we just replace g with $\bigwedge_{i \leq 3} g_i$ where $g_0 = \top$.

Now, for the PCTL formula $f = \mathsf{F}^{\geq 0.5}a$, such x_f does not exist, because, for every $x \in (0, 1)$ we have: $M, s_1 \models f$ provided that $y \in [(1 - x)/2, 1)$; and $M, s_1 \not\models f$ if $y \in (0, (1 - x)/2)$. This implies that $P_x(f)$ holds for every $x \in (0, 1)$, and hence f cannot be equally expressed by any PμTL formula. □

Note that the value 0.5 in the previous two theorems can be generalized to any other probability $p \in (0, 1)$.

We also provide a comparison on the *qualitative* fragments of PCTL and PμTL. Probabilities occurring in such fragments can only be 0 or 1.

Theorem 5. *Every qualitative PCTL formula can be equally expressed by a qualitative PμTL formula.*

Proof. We will give a constructive translation procedure, which takes a qualitative PCTL formula g and outputs an equivalent qualitative PμTL formula $\widetilde{g}$. Inductively:

1. $\widetilde{g} = \perp$ if $g = \perp$, or its root operator is $\mathsf{X}^{>1}$, $\mathsf{U}^{>1}$ or $\mathsf{R}^{>1}$; $\widetilde{g} = \top$ if $g = \top$, or its root operator is $\mathsf{X}^{\geq 0}$, $\mathsf{U}^{\geq 0}$ or $\mathsf{R}^{\geq 0}$.
2. $\widetilde{g} = \widetilde{g_1} \wedge \widetilde{g_2}$ if $g = g_1 \wedge g_2$; and $\widetilde{g} = \widetilde{g_1} \vee \widetilde{g_2}$ if $g = g_1 \vee g_2$.
3. $\widetilde{g} = \mathsf{X}^{>0}\widetilde{g'}$ if $g = \mathsf{X}^{>0}g'$; and $\widetilde{g} = \mathsf{X}^{\geq 1}\widetilde{g'}$ if $g = \mathsf{X}^{\geq 1}g'$.
4. $\widetilde{g} = \mu Z.(\widetilde{g_2} \vee (\widetilde{g_1} \wedge \mathsf{X}^{>0}Z))$ if $g = g_1\mathsf{U}^{>0}g_2$; and $\widetilde{g} = \nu Z.(\widetilde{g_2} \wedge (\widetilde{g_1} \vee \mathsf{X}^{\geq 1}Z))$ if $g = g_1\mathsf{R}^{\geq 1}g_2$.
5. $\widetilde{g} = \nu Z.(\widetilde{g_2} \vee (\widetilde{g_1} \wedge \widetilde{\mathsf{F}^{>0}g_2} \wedge \mathsf{X}^{\geq 1}Z)) = \nu Z.(\widetilde{g_2} \vee (\widetilde{g_1} \wedge \mu Z'.(\widetilde{g_2} \vee \mathsf{X}^{>0}Z') \wedge \mathsf{X}^{\geq 1}Z))$ if $g = g_1\mathsf{U}^{\geq 1}g_2$; and $\widetilde{g} = \mu Z.(\widetilde{g_2} \wedge (\widetilde{g_1} \vee \widetilde{\mathsf{G}^{\geq 1}g_2} \vee \mathsf{X}^{>0}Z)) = \mu Z.(\widetilde{g_2} \wedge (\widetilde{g_1} \vee \nu Z'.(\widetilde{g_2} \wedge \mathsf{X}^{\geq 1}Z') \vee \mathsf{X}^{>0}Z))$ if $g = g_1\mathsf{R}^{>0}g_2$.

The proof of equivalence could be done by induction on the structure of the formula. □

Note that Thm. 5 holds because we are only concerned about finite models in this paper. Interested readers may show that it is not true for infinite Markov chains.

Theorem 6. *The qualitative PμTL formula $f = \nu Z.(a \wedge \mathsf{X}^{>0}\mathsf{X}^{>0}Z)$ cannot be expressed in qualitative PCTL.*

Proof. Construct a series of Markov chains $M''_2, M''_3, \ldots$ such that each M''_n is the Markov chain $(\{s''_0, s''_1, \ldots, s''_n\}, \boldsymbol{T}''_n, L''_n)$, where $\boldsymbol{T}''_n(s''_0, s''_0) = 1$ and $\boldsymbol{T}''_n(s''_{i+1}, s''_i) = 1$ for each $i < n$. In addition, $L''_n(s''_i) = \{a\}$ for each $i \neq 1$, and $L''_n(s''_1) = \emptyset$.

For a given PCTL formula g, let $\hat{g}$ be the LTL formula obtained from g by discarding all probability quantifiers, e.g., we have $\hat{g} = a\mathsf{U}(b \vee \mathsf{G}\neg a)$ if $g = a\mathsf{U}^{\geq 0.3}(b \vee \mathsf{G}^{>0.6}\neg a)$. Since that from s''_n the Markov chain M''_n has exactly one infinite path $\pi_n = s''_n, \ldots, s''_1, (s''_0)^\omega$, then for each $n \geq 2$ we have $M''_n, s''_n \models g$ if and only if $\pi_n \models \hat{g}$. It is shown in [Wolper, 1983] that $M''_n, s''_n \models \hat{g}$ iff $M''_{n+1}, s''_{n+1} \models \hat{g}$ in the case of $n \geq N'(\hat{g}) = N'(g)$, where $N'(g)$ and $N'(\hat{g})$ are the nesting depth of X-operator of g and $\hat{g}$, respectively. Thus, we have

$M''_n, s''_n \models g$ iff $M''_{n+1}, s''_{n+1} \models g$ in such situation. This implies that $\nu Z.(a \wedge \mathsf{X}^{>0}\mathsf{X}^{>0}Z)$ has no equivalent qualitative PCTL expression, because we cannot simultaneously have $M''_n, s''_n \models f$ and $M''_{n+1}, s''_{n+1} \models f$ for each $n \geq 2$. □

Note that the conclusion of Thm. 6 is also pointed out in [Cleaveland *et al.*, 2005], and we here provide a detailed proof. Indeed, this proof also works for general PCTL formulae, and hence the property $\nu Z.(a \wedge \mathsf{X}^{>0}\mathsf{X}^{>0}Z)$ even cannot be expressed by any PCTL formula.

5 Automata Characterization

In this section, we will define a new type of automata recognizing (pointed) Markov chains, called *probabilistic alternating parity automata* (PAPA, for short), and such automata could be viewed as the probabilistic extension of those defined in [Wilke, 2002].

A PAPA A is a tuple (Q, q_0, δ, Ω) where: Q is a finite set of *states*, $q_0 \in Q$ is the *initial state*, δ is the *transition function* to be defined later, and $\Omega : Q \rightsquigarrow \mathbb{N}$, is a partial function of *coloring*; in what follows, we say a state is *colored* if Ω is defined for the state.

The notion of *transition conditions* over Q is inductively defined as follows:

1. $\bot$ and $\top$ are transition conditions over Q.
2. For every $a \in \mathcal{A}$, the literals a and $\neg a$ are transition conditions over Q.
3. If $q \in Q$, then q is a transition condition over Q.
4. If $q \in Q$ and $p \in [0, 1]$, then $\bigcirc^{\sim p} q$ is a transition condition over Q, where $\sim \in \{\geq, >\}$.
5. If $q_1, q_2 \in Q$ then both $q_1 \vee q_2$ and $q_1 \wedge q_2$ are transition conditions over Q.

The transition function δ assigns each state $q \in Q$ a transition condition over Q.

We denote by R_A the *derived graph* of A, its vertex set is just Q, and there is an edge from q_1 to q_2 iff q_2 appears in $\delta(q_1)$. We say that A is *well-structured*, if for every path $q_1, q_2, \ldots, q_n$ that forms a cycle (i.e., $q_1 = q_n$) in R_A, we have that: 1) there exists some $1 \leq i < n$ such that $\delta(q_i) = \bigcirc^{\sim p} q_{i+1}$ with some $p \in [0, 1]$; 2) there exists some $1 \leq j < n$ such that q_j is colored. In what follows, we are only concerned about well-structured PAPA.

Given a pointed Markov chain (M, s_0) with $M = (S, \boldsymbol{T}, L)$ and $s_0 \in S$, a *run* of A over (M, s_0) is a $Q \times S$-labeled tree (T, λ) fulfilling: $\lambda(v_0) = (q_0, s_0)$ for the root vertex v_0; and for each internal vertex v of T with $\lambda(v) = (q, s)$ we require that

- $\delta(q) \neq \bot$, and if $\delta(q) = \top$ then v has no child;
- $a \in L(s)$ if $\delta(q) = a$, and $a \notin L(s)$ if $\delta(q) = \neg a$;
- if $\delta(q) = q_1 \wedge q_2$ then v has two children v_1 and v_2 respectively having $\lambda(v_1) = (q_1, s)$ and $\lambda(v_2) = (q_2, s)$;
- if $\delta(q) = q_1 \vee q_2$ then v has one child v' with $\lambda(v') \in \{(q_1, s), (q_2, s)\}$;
- v has one child v' having $\lambda(v') = (q', s)$, if $\delta(q) = q'$;
- if $\delta(q) = \bigcirc^{\sim p} q'$ then v has a set of children $v_1, \ldots, v_n$ such that $\lambda(v_i) = (q', s_i)$, where $\sum_{i=1}^{n} \boldsymbol{T}(s, s_i) \sim p$.

For an infinite branch $\tau = v_0, v_1, \ldots$ of T, let n_τ be the number

$$\max\{\, n \mid \text{there are infinitely many } i \text{ s.t. } \Omega(\mathrm{proj}_1(\lambda(v_i))) = n\}$$

where $\mathrm{proj}_1(q, s) = q$. A run (T, λ) is *accepting* if n_τ is an even number, for every infinite branch τ of T. A pointed Markov chain (M, s_0) is *accepted* by A if A has an accepting run over it. We denote by $\mathscr{L}(A)$ the set consisting of pointed Markov chains accepted by A.

Theorem 7. *Given a closed PμTL formula f, there is a PAPA A_f such that: $M, s \models f$ iff $(M, s) \in \mathscr{L}(A_f)$, for each pointed Markov chain (M, s).*

Proof. We just let $A_f = (Q_f, q_f, \delta_f, \Omega_f)$, where:

- $Q_f = \{q_g \mid g \text{ is a subformula of } f\}$, and hence $q_f \in Q_f$;
- δ_f is defined as follows:
  - $\delta_f(q_\bot) = \bot$ and $\delta_f(q_\top) = \top$;
  - $\delta_f(q_a) = a$ and $\delta_f(q_{\neg a}) = \neg a$;
  - $\delta_f(q_{g_1 \wedge g_2}) = q_{g_1} \wedge q_{g_2}$ and $\delta_f(q_{g_1 \vee g_2}) = q_{g_1} \vee q_{g_2}$;
  - $\delta_f(q_{\mathsf{X}^{\sim p} g}) = \bigcirc^{\sim p} q_g$;
  - $\delta_f(q_{\mu Z.g}) = q_g$ and $\delta_f(q_{\nu Z.g}) = q_g$;
  - $\delta_f(q_Z) = q_{\mathscr{D}_f(Z)}$.
- Ω_f is defined at every state q_Z with $Z \in \mathcal{Z}$ fulfilling: If Z is a μ-variable (resp. ν-variable), then $\Omega_f(q_Z)$ is the minimal odd (resp. even) number which is greater than every $\Omega_f(q_{Z'})$ such that $\mathscr{D}_f(Z')$ is a subformula of $\mathscr{D}_f(Z)$.

It could be directly examined that A_f is well-structured since f is guarded. The proof of equivalence can be similarly done as that in [Wilke, 2002] — the only different induction step is to deal with transitions being of $\bigcirc^{\sim p} q$ (in that paper, the corresponding cases are $\Box q$ and $\Diamond q$). Actually, we can see that if a PAPA (Q, q, δ, Ω) corresponds to the PμTL formula g, then the PAPA $(Q \cup \{q'\}, q', \delta[q' \mapsto \bigcirc^{\sim p} q], \Omega)$ must correspond to $\mathsf{X}^{\sim p} g$. □

6 Satisfiability Decision

It is known from Section 5 that the Satisfiability problem of PμTL could be reduced to the Emptiness problem of PAPA. In this section, we will further reduce it to parity game solving.

A *parity game* G is a tuple (V, E, C), where: V is a finite set of *locations*, and V could be partitioned into two disjoint sets V^0 and V^1; $E \subseteq V \times V$ is the set of *moves*, required to be total; and $C : V \rightsquigarrow \mathbb{N}$ is a partial function of *coloring*, and we say a location v is *colored*, if $C(v)$ is defined. In addition, for the game G, we require that each loop involves at least one colored location.

Two players — player 0 and player 1, are respectively in charge of V^0 and V^1 when G is being played. A *play* of G starting from $v_0 \in V$ is an infinite sequence of locations $v_0, v_1, \ldots$ made by player 0 and player 1 — for every $i \in \mathbb{N}$, the location v_{i+1} is chosen by player 0 (resp. player 1) with $(v_i, v_{i+1}) \in E$ whenever $v_i \in V^0$ (resp. $v_i \in V^1$).

Player 0 (resp. player 1) *wins* the play $v_0, v_1, \ldots$ if the maximal color occurring infinitely often in it is even (resp. odd) — and we say that a color c *occurs* in this play if there is some v_i with $C(v_i) = c$.

A *winning strategy* for player i is a mapping $H_i : V^* \cdot V^i \to V$, such that for every play $v_0, v_1, \ldots$, player i always wins if $v_{j+1} = H_i(v_0, \ldots, v_j)$ whenever $v_j \in V^i$. In addition, H_i is *memoryless* if $H_i(v_0, \ldots, v_j)$ agrees with $H_i(v_j)$ for every j.

Theorem 8 ([Gurevich and Harrington, 1982; Zielonka, 1998; Jurdziński, 1998]). *For a parity game G, from every location, there is exactly one player having a winning strategy. The problem of deciding the winner at a location is in **UP** $\cap$**co-UP**. In addition, if a player has a winning strategy then she also has a memoryless one from the same location.*

We use $\mathscr{W}_i(G)$ to denote the set consisting of all locations from which player i has a winning strategy.

Given a PAPA $A = (Q, q, \delta, \Omega)$, a *gadget* D of A is a finite directed acyclic digram (P, γ) where $P \subseteq Q$, $\gamma \subseteq P \times P$, and for each $q \in P$:

1. if $\delta(q) = q'$, then $q' \in P$ and $(q, q') \in \gamma$;
2. if $\delta(q) = q_1 \wedge q_2$ then $q_1, q_2 \in P$, and $(q, q_1), (q, q_2) \in \gamma$;
3. if $\delta(q) = q_1 \vee q_2$ then there is some $i \in \{1, 2\}$ such that $q_i \in P$ and $(q, q_i) \in \gamma$,
4. q has no successor for the other cases.

For convenience, we sometimes directly write $q \in D$ whenever $D = (P, \gamma)$ and $q \in P$. We denote by $\mathcal{D}(A)$ the set consisting of all gadgets of A. Since we require that each PAPA A is well-structured, then $\mathcal{D}(A)$ must be a finite set.

Given a sequence of gadgets $D_1, D_2, \ldots$ such that $D_i = (P_i, \gamma_i)$, an *infinite path* within it is a sequence of states $q_{1,1}, \ldots, q_{1,\ell_1}, q_{2,1}, \ldots, q_{2,\ell_2}, \ldots$ such that each $(q_{i,j}, q_{i,j+1}) \in \gamma_i$ and $\delta(q_{i,\ell_i}) = \bigcirc^{\sim p_i} q_{i+1,1}$ for some $p_i \in [0, 1]$. We say such an infinite path is *even* (resp. *odd*) if the maximal color (w.r.t. Ω) occurring infinitely often is even (resp. odd).

We say that a gadget $D = (P, \gamma)$ is *incompatible* if there exist $q_1, q_2 \in P$ and $\delta(q_1) = a$, $\delta(q_2) = \neg a$ for some $a \in \mathcal{A}$; or there is some $q \in P$ with $\delta(q) = \bot$. Otherwise, we say that D is *compatible*.

Let D be a gadget and $\Gamma = \{D_1, \ldots, D_k\}$ be a set of gadgets, we denote by $\Gamma \Vdash D$ if there exist k positive numbers $x_1, \ldots, x_k$ such that: $\sum_{i=1}^{k} x_i \leq 1$, and for each $q \in D$ with $\delta(q) = \bigcirc^{\sim p} q'$, we have $\sum_{q' \in D_i} x_i \sim p$. We in what follows call $x_1, \ldots, x_k$ the *enabling condition*. Note that the relation $\Vdash$ could be decided by solving a linear system of inequality.

According to automata theory, we may construct a deterministic (word) parity automaton $\widetilde{A} = (\widetilde{Q}, \widetilde{q}, \widetilde{\delta}, \widetilde{\Omega})$ were $\widetilde{\delta} : \widetilde{Q} \times \mathcal{D}(A) \to \widetilde{Q}$ and $\widetilde{\Omega}$ is a total coloring function. It takes a gadget sequence as input, and accepts it if every gadget in it is compatible and every infinite path within it is even.

Then, we may create a parity game $G_A = (V_A, E_A, C_A)$ for the PAPA A, in detail:

- $V_A = V_A^0 \cup V_A^1$, where $V_A^0 = 2^{\mathcal{D}(A) \times \widetilde{Q}}$ and $V_A^1 = \mathcal{D}(A) \times \widetilde{Q}$.
- $E_A = \{(\{(D_1, \widetilde{q_1}), \ldots, (D_k, \widetilde{q_k})\}, (D_i, \widetilde{q_i})) \mid 1 \leq i \leq k\} \cup \{((D, \widetilde{q}), \{(D_1, \widetilde{q_1}), \ldots, (D_k, \widetilde{q_k})\}) \mid (D_1, \ldots, D_k) \Vdash D$, and each $\widetilde{q_i} = \widetilde{\delta}(\widetilde{q}, D_i)\}$.
- $C_A(D, \widetilde{q}) = \widetilde{\Omega}(\widetilde{q})$, hence every location in V_A^1 is colored.

Theorem 9. *Let the PAPA $A = (Q, q, \delta, \Omega)$, then $\mathscr{L}(A) \neq \emptyset$ if and only if there is some $D \in \mathcal{D}(A)$ with $q \in D$ such that $\{(D, \widetilde{\delta}(\widetilde{q}, D))\} \in \mathscr{W}_0(G_A)$.*

Intuitively, player 0 could extract a winning strategy from an accepting run of A over any pointed Markov chain; and conversely, one can construct a pointed Markov chain accepted by A according to the (memoryless) winning strategy of player 0. Interested readers may find the detailed proof in the accompanied report [Liu *et al.*, 2015].

As a consequence of Thm. 7, Thm. 8 and Thm. 9 we have the following main conclusion of this section.

Theorem 10. *Both the* Emptiness *problem of PAPA and the* Satisfiability *problem of PμTL are decidable, and both of them are in 2**EXPTIME**.*

Indeed, from Thm. 7 one can get a PAPA whose scale is linear in the size of the input formula, and an n-state PAPA could be converted to a parity game with scale $2^{2^{O(n)}}$. From standard game theory (see [Jurdziński, 1998; Wilke, 2002], and see [Schewe, 2008] for an improved bound), and with a similar analysis of [Wilke, 2002] (see also the analysis of the coloring number in that paper), one can infer that this problem is in 2**EXPTIME**.

7 Discussion

In this paper, we present the logic PμTL, a simple and succinct probabilistic extension of μTL. We have compared the expressiveness of these two kinds of logics: In general, PμTL captures 'local' and 'stepwise' probabilities; whereas PCTL could describe 'global' probabilities in the system. Hence, these two logics are orthogonal and complementary, and one can obtain a more powerful and expressive logic by combing them together, as done in [Castro *et al.*, 2015]. i.e., we may use formulae like $(\mu Z.(a \vee \mathsf{X}^{\geq 0.8} Z))\mathsf{U}^{\geq 0.6}(\nu Z'.(b \wedge \mathsf{F}^{>0.3} Z'))$. Model checking algorithm of such an extension can be acquired from those of the underlying logics.

In this paper, we have also investigated the decision problem of PμTL, the key issue and the most challenging part is to deal with probabilistic quantifiers when doing reduction to parity games, which is a highly nontrivial extension of the non-probabilistic case. As a cost, we have only now got an algorithm with double-exponential time complexity for solving it — in contrast, the Satisfiability problem for the standard μTL is in **EXPTIME**.

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