Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence (IJCAI 2015)

What Do We Elect Committees For? A Voting Committee Model for Multi-Winner Rules

Piotr Skowron
University of Warsaw
Warsaw, Poland
p.skowron@mimuw.edu.pl

Abstract

We present a new model that describes the process of electing a group of representatives (e.g., a parliament) for a group of voters. In this model, called the voting committee model, the elected group of representatives runs a number of ballots to make final decisions regarding various issues. The satisfaction of voters comes from the final decisions made by the elected committee. Our results suggest that depending on a single-winner election system used by the committee to make these final decisions, different multi-winner election rules are most suitable for electing the committee. Furthermore, we show that if we allow not only a committee, but also an election rule used to make final decisions, to depend on the voters' preferences, we can obtain an even better representation of the voters.

1 Introduction

There are many real-life situations in which we want to select a subset of a given set of candidates. Examples of such situations include selecting a set of movies for an airplane [Skowron *et al.*, 2015], choosing locations for common facilities [Shmoys *et al.*, 1997], short-listing candidates for hiring in a new team, or electing a parliament. These examples differ in their nature and in each of them we might desire the selected group of candidates to exhibit different properties. In this paper we consider the case of selecting a *committee*, also referred to as a *group of representatives*, for the agents. However, even if we restrict our analysis to the concrete case of choosing a committee, we might require different representatives to be elected, depending on their rights and responsibilities in the elected body.

Indeed, different views on what the desired properties of committees are have led to inventing a number of multi-winner election rules [Chamberlin and Courant, 1983; Monroe, 1995; Fishburn, 1981; Gehrlein, 1985; Brams *et al.*, 2007; Kilgour, 2010; Debord, 1992]. In this paper we argue that the desired properties of multi-winner rules strongly depend on what we elect the committee for. One of our main conclusions says that for different single-winner rules used by the committee to make final decisions, different multi-winner rules should be used to elect the decision-making committee.

Specifically, we obtain theoretical justification for the following claims. We argue that multi-winner rules that achieve fully proportional representation, such as Chamberlin–Courant rule [1983] and Monroe rule [1995] are particularly well-suited for electing committees that need to make unanimous decisions. Unanimity is often required in situations where making a wrong decision implies severe consequences, e.g., in case of juries voting on convictions. Indeed, in such cases we are particularly willing to ensure that minorities are represented in the committees to avoid biased decisions.

Further, our analysis suggests that K-top rules, i.e., scoring rules [Young, 1975] that select K candidates with the highest total utilities, are suitable for electing committees that use the random dictatorship rule to make decisions. It might seem that the significance of this observation is compromised by the low applicability of randomized election rules. Such randomized decision-making processes, however, model situations where decisions are made by individual members of the committee but we are uncertain which issues will be considered by which individuals. As an example of such a group of representatives consider a senior program committee for a conference, where individual experts makes final decisions regarding submitted papers.

We observe that the median OWA rule [Skowron *et al.*, 2015] is suitable for electing committees that make final decisions by majority voting. Informally speaking, in the median OWA rule a voter is satisfied with a committee C if she is satisfied with at least half of the members of C. We recall the formal definition of the class of the OWA rules in Section 2.

We derive the above results by introducing and studying a new formal model that allows us to validate the applicability of multi-winner election rules. In our model, hereinafter referred to as the *voting committee model*, we assume that an elected committee runs a number of ballots, in which it makes collective decisions regarding various issues. This setting corresponds to, e.g., a parliament voting on law acts, a jury voting on convictions, a supervisory board voting on strategic business decisions, etc. We assume that the ultimate satisfaction of the voters depends solely on the final decisions made by the committee. Consequently, each voter ranks candidates for the committee according to how likely it is that they vote according to her preferences. This new model allows to numerically assess qualities of committees, depending on what election rule these committees use to make the final decisions. Intuitively, the numerical quality of a committee C estimates how likely it is that C makes decisions consistent with voters' preferences.

Our work extends the literature on properties of multi-winner election rules [Simeone and Pukelsheim, 2006; Barberá and Coelho, 2008; Elkind *et al.*, 2014; Aziz *et al.*, 2015; Gehrlein, 1985; Debord, 1992]. This literature includes, e.g., the works of Barberá and Coelho [2008] and of Aziz et al. [2015], where the authors define properties that "good" multi-winner rules should satisfy. Elkind et al. [2014] argue that the desirability of many natural properties of multi-winner rules must be evaluated in the context of their specific applications. Our paper extends this discussion by showing an intuitive model and concrete examples concerning this model, for which different multi-winner rules are particularly well applicable.

Many multi-winner election systems are defined as functions selecting such committees that optimize certain metrics, usually related to voters satisfaction [Bock *et al.*, 1998; Chamberlin and Courant, 1983; Monroe, 1995; Brams *et al.*, 2007; Kilgour, 2010; Debord, 1992]. These different optimization metrics capture certain desired properties of election systems. Our paper complements these works by presenting a specific metric that is motivated by the analysis of decision-making processes of committees.

Similarly to Fishburn [1981], we explore the idea of comparing multi-winner election systems. Further, our perspective is conceptually close to the ones given by Christian et al. [2007], who study computational problems related to lobbying in direct democracy, where the decisions are made directly by the voters who express their preferences in open referenda (there are no representatives). Our model is also close to the one of Koriyama et al. [2013], who alike assume that the elected committee makes a number of decisions in a sequence of ballots. In such model, they compute the frequency, with which the will of each individual is implemented. They further make some assumptions on how the society evaluates the frequency values, to justify degressive proportionality, the principle of proportional apportionment.

Our methodology is also close in spirit to the one given by Young [1995], who introduced probabilistic models to validate optimality of election rules. In our of such models the decisions are taken by a group of experts—these decisions might be factually either correct or wrong. Young shows the rule that is most adequate in such a setting.

Finally, we note that there exists a broad literature on voting on multi-attribute domains, where agents need to make collective decisions regarding a number of issues [Brams *et al.*, 1998; Lacy and Niou, 2000; Xia *et al.*, 2008; 2010]. We differ from these works by considering an indirect process of decision-making, through an elected committee.

Our contribution is the following: (i) We introduce a new formal model that relates voter satisfaction from a committee to their satisfaction from the committee's decisions. In our model we define the notion of optimality of a multi-winner election rule, given that a certain single-winner rule is used by the elected committee to make the final decisions. (ii) By studying the deterministic variant of our model, we find optimal multi-winner rules for three natural single-winner rules: majority, unanimity, and random dictatorship. (iii) After relaxing the assumptions regarding the determinism of the model, we show that the class of OWA rules contains all optimal rules for most natural single-winner election systems. (iv) We introduce a definition of the indirect election rule that allows voters to elect not only committees, but also the single-winner rules used by these committees to make decisions. We discuss the existence of optimal indirect rules.

2 Preliminaries

In this section we first define the notation and basic notions used in the paper. Next, we recall definitions of several known single-winner and multi-winner election rules that we use in our analysis.

2.1 Notation and Basic Definitions

For each set X, by $\mathbb{1}_X$ we denote the indicator function of X, defined as: $\mathbb{1}_X(x) = 1 \iff x \in X$, and $\mathbb{1}_X(x) = 0 \iff x \notin X$. For simplicity, we write $\mathbb{1}_X$ instead of $\mathbb{1}_X(x)$ if x is clear from the context.

For each finite set X and for each voter i, a *utility function* of i over X, is a function $ut_i : X \to \mathbb{R}$ that quantifies the level of satisfaction of the voter i from the elements of X. We denote the utility that an agent i assigns to a candidate a as $u_{i,a} = ut_i(a)$. We are specifically interested in the *approval model*, where utilities come from the binary set $\{0, 1\}$. In the approval model, we say that a voter i approves of a candidate a if $u_{i,a} = 1$. Otherwise we say that i disapproves of a. We do not make any assumptions on where the utility functions come from. For instance, the utilities can be provided directly by the voters, can be extracted from rankings provided by the voters (cf. the work of Young [1975]), etc. For each two finite sets X and Y, with $|Y| = y$, a *utility profile* of Y over X is a vector of y utility functions. We denote the set of all possible utility profiles of Y over X as $\Pi^y(X)$.

For each set X and each $K \in \mathbb{N}$, by $\mathcal{P}(X)$ we denote the set of all subsets of X and by $\mathcal{P}_K(X)$ the set of all subsets of X of size K. We refer to the elements of $\mathcal{P}_K(X)$ as to *committees* of size K, elected from X. A *K-multi-winner election rule* for a set of voters Y, $|Y| = y$, and a set of candidates X is a function $\mathcal{R}_1 : \Pi^y(X) \to \mathcal{P}_K(X)$ that for a given utility profile of Y over X returns a committee of size K. We denote the set of all K-multi-winner election rules for sets Y and X as $\mathcal{MW}(Y, X, K)$.

In general, a single-winner election rule for a set of voters Y, $|Y| = y$, and a set of candidates X is a function $\mathcal{R}_2 : \Pi^y(X) \to X$ that for a given utility profile returns exactly one candidate from X. Thus, single-winner rules can be seen as 1-multi-winner rules.

In our further analysis we consider single-winner election rules for binary sets of candidates only. We also study randomized single-winner rules that for each utility profile, instead of a single candidate from X, return a lottery over X. Thus, hereinafter we use the following definition. For a set of voters Y, $|Y| = y$, and a binary set of candidates $X = \{d_0, d_1\}$, a *(randomized) single-winner election rule* $\mathcal{R}_2$ is a function $\mathcal{R}_2 : \{d_0, d_1\}^y \to \mathbb{R}$, that for each vector of y binary values, returns the probability that the value d_0 will be selected. We denote a set of all randomized single-winner election rules for sets Y and X as $\mathcal{RSW}(Y, X)$.

2.2 Overview of Single-Winner Rules

A *uniformly random dictatorship* rule selects a candidate d with probability proportional to the number of voters who prefer d over the other candidate. A *majority* rule deterministically selects d_0 if at least half of the voters prefer d_0 over d_1 (this assumes that ties are resolved in favor of d_0). A *unanimity* rule returns deterministically d_0 if and only if all voters prefer d_0 over d_1 (in case of the unanimity rule it is convenient to think of voting for d_0 as of voting for pass and of voting for d_1 as of voting for veto).

2.3 Overview of Multi-Winner Rules

A significant part of results that we provide in the paper concerns OWA rules, the broad class of multi-winner election systems defined by Skowron et al. [2015]. Below, we recall the definition this class of rules and describe several of them.

For each voter i, each committee C, and each number j, $1 \leq j \leq K$, let $u_{i,C}(j)$ denote the utility of the j-th most preferred candidate from C, according to i. In other words: $\{u_{i,C}(1), \dots, u_{i,C}(K)\} = \{u_{i,a} : a \in C\}$, and $u_{i,C}(1) \geq \dots \geq u_{i,C}(K)$. For each voter i, each committee C, and each K-element vector α, let $\alpha(i, C)$ denote the α-satisfaction of i from C: $\alpha(i, C) = \sum_{j=1}^{K} \alpha_j u_{i,C}(j)$. The α-rule is a rule that selects a committee C that maximizes total α-satisfaction of the voters $\sum_i \alpha(i, C)$.

For a given set of candidates X, a set of voters Y, and a size of the committee we want to elect, K, let $\mathcal{OWA}(X, Y, K)$ denote the set of all OWA rules that given votes of Y over X return committees of size K:

$$\mathcal{OWA} = \{\alpha\text{-rule} : \alpha = \langle \alpha_1, \alpha_2, \dots, \alpha_K \rangle, \forall_i \alpha_i \geq 0\}.$$

There are several specifically interesting OWA rules. For instance, if $\alpha = \langle 1, 0, \dots, 0 \rangle$ we get Chamberlin–Courant rule [1983], where each voter "receives satisfaction scores" from her most preferred member of the committee only. If $\alpha = \langle 1, 1, \dots, 1 \rangle$ we get a K-top rule that selects K candidates with the highest total utility. If α has 1 on the k-th positions and 0 on the others, then we get a k-median rule, in which the satisfaction of a voter from a committee C is her satisfaction from the k-th most preferred member of C.

3 The Voting Committee Model

In this section we describe a voting committee model that allows one to assess the qualities of various multi-winner election rules. Let $N = \{1, 2, \dots, n\}$ be a set of *voters*, and let $A = \{a_1, a_2, \dots, a_m\}$ be a set of *candidates*. We assume there is a set $D = \{D_1, D_2, \dots, D_r\}$ of r issues; each issue D_j is a binary set consisting of two alternatives $D_j = \{d_0, d_1\}$. Voters have strict preferences over the alternatives within each issue; by d_i^j we denote the preferred alternative from D_j from the perspective of the voter i. The issues might differ in their importance to different voters. We consider two types of attitudes: a voter i can consider an issue D_j either as *important* or as *insignificant*. We denote the set of all the issues important for voter i as $D_{\mathrm{im}}(i) \subseteq D$.

In the first stage in our model, a committee C of size K is selected through a multi-winner election rule $\mathcal{R}_1$; the selected committee consists of K members. In the second stage, the committee C runs r independent ballots, for each ballot using the same single-winner election rule $\mathcal{R}_2$. In the i-th ballot, $1 \leq i \leq r$, the committee makes a collective decision regarding issue D_i—for D_i the committee can make either decision d_0 or decision d_1. The final outcome of this two-stage process is described by a vector of r decisions.

Intuitively, the first stage of elections might correspond to e.g., parliamentary elections, elections for supervisory or faculty board, etc. The second stage might be viewed as e.g., a sequence of parliamentary ballots regarding various matters, such as financial and monetary economics, education politics, changes in national health-care system, etc.

The ultimate satisfaction of voter i depends solely on the final outcome of the r ballots. Nevertheless, the voters need to select the committee members first and, thus, they need a way of judging the qualities of the committees. Intuitively, voter i considers committee C as good, if for i's important issues, i.e., issues from $D_{\mathrm{im}}(i)$, C is likely to make decisions consistent with i's preferences. More formally, we introduce the following probabilistic model.

For each voter $i \in N$ and each candidate $a \in A$ we set $p_{i,a}$, the *probability of representation of i by a*, to be the probability that a for each i's important issue $D_j \in D_{\mathrm{im}}(i)$ will vote for d_i^j, i.e., according to i's preferences. For the sake of simplicity of notation, we also use $q_{i,a} = 1 - p_{i,a}$. We assume that the voters know these probabilities, and that in the considered utility functions the values that voters assign to the candidates are function of the probabilities of representation only. More formally, we assume there exists an injective function $P : \mathbb{R} \to \mathbb{R}$, such that a voter i assigns utility u to the candidate a, if and only if $u = P(p_{i,a})$.

We are specifically interested in the *approval model*, where there are only two "allowed" values of utilities, and thus two values of probabilities of representation: $p = P^{-1}(1)$ and $q = P^{-1}(0)$. As a distinguished special case of the approval model, we consider a variant in which the values of probabilities of representation come from the binary set $\{0, 1\}$ ($p = 1$ and $q = 0$). In such case, hereinafter referred to as the *deterministic model*, for each voter i there are two types of candidates: the candidates that perfectly represent i, i.e., deterministically vote according to i's preferences in all issues important for i (and these candidates are approved of by i) and the others that perfectly misrepresent i, i.e., deterministically vote contrarily to i's preferences (and so, are disapproved of).

For each voter i, each important issue $D_j \in D_{\mathrm{im}}(i)$ for i, each committee C, and each committee vote $v \in D_j^K$, let $\mathbb{P}_C(v)$ denote the probability that members of C cast vote v:

$$\mathbb{P}_C(v) = \prod_{a \in C} \left(\mathbb{1}_{v[j] = d_i^j} p_{i,a} + \mathbb{1}_{v[j] \neq d_i^j} q_{i,a} \right).$$

Let $\mathcal{R}_2 : \{d_0, d_1\}^K \to \mathbb{R}$ be a randomized single-winner election rule used by the committee to make decisions over issues. By $\mathbb{P}_{\mathcal{R}_2}(i|v)$ we denote the probability that the rule $\mathcal{R}_2$, given vote v, makes decision d_j^i:

$$\mathbb{P}_{\mathcal{R}_2}(i|v) = \mathbb{1}_{d_0 = d_j^i} \mathcal{R}_2(v) + \mathbb{1}_{d_1 = d_j^i} (1 - \mathcal{R}_2(v)).$$

For each committee C, each voter i, and each important issue $D_j \in D_{\mathrm{im}}(i)$, by $\mathbb{P}_{C,\mathcal{R}_2}(i)$ we denote the probability that C makes decision consistent with i's preferences:

$$\mathbb{P}_{C,\mathcal{R}_2}(i) = \sum_{v \in D_j^K} \mathbb{P}_C(v)\mathbb{P}_{\mathcal{R}_2}(i|v).$$

Now, we can define the expected ultimate satisfaction of a voter i from the committee C, as $\mathbb{P}_{C,\mathcal{R}_2}(i)$—this is the expected fraction of issues important for i, for which the committee C would make decisions consistent with i's preferences.

Finally, we can define the central notion of this paper.

Definition 1. *Let $\mathcal{R}_2$ be a single-winner election rule used by the committee to make final decisions, and let K denote the size of the committee to be elected. A committee C is* optimal in the utilitarian sense *if:*

$$C = \mathrm{argmax}_{C' \in A : |C'| = K} \sum_i \mathbb{P}_{C,\mathcal{R}_2}(i). \quad (1)$$

Analogously, we can define committees *optimal in the egalitarian sense*, by replacing "sum" with "max" in Definition 1. For the sake of concreteness, in this paper we focus on the utilitarian case and refer to the optimality in the utilitarian case as to the optimality.

4 Optimality for Known Single-Winner Rules

In this section we show that several known multi-winner election rules can be viewed as optimal in our voting committee model. Each such a rule is optimal for different single-winner election system $\mathcal{R}_2$, used by the selected committee to make final decisions. Thus, our results give intuition regarding the applicability of different multi-winner election rules, depending on for what kind of decision making the committee is elected for.

Definition 2. *Let $\mathcal{R}_2$ be a single-winner election rule used in the second stage of the election model. A multi-winner election rule $\mathcal{R}_1$ is optimal for $\mathcal{R}_2$ if for each preferences of voters it elects an optimal committee.*

For multi-winner approval rules and deterministic model, we will add the additional word "deterministically" to the notion of optimality.

We start with presenting four theorems characterizing four multi-winner election rules in which voters express their preferences by approving subsets of candidates. These rules are optimal for majority, random dictatorship, and unanimity single-winner election systems.

Theorem 1. *Assume K is odd. The $\frac{K+1}{2}$-median election system is deterministically optimal for the majority rule.*

Proof. We calculate $\mathbb{P}_{C,\mathcal{R}_2}(i)$, the ultimate satisfaction of a voter i from a committee C, assuming C uses majority rule to make final decisions. Since the model is fully deterministic, and a committee member $a \in C$ votes according to i's preferences if and only if a is approved by i, a committee C makes decisions consistent with i's preferences, if it contains at least $\frac{K+1}{2}$ members approved by i. Thus, the satisfaction $\mathbb{P}_{C,\mathcal{R}_2}(i)$ of i from C is equal to 1 if C contains at least $\frac{K+1}{2}$ members approved by i, otherwise it is equal to 0.

The same formula defines satisfaction of i from C in the $\frac{K+1}{2}$-median election system. Finally, we note that in the $\frac{K+1}{2}$-median election system, the committee that maximizes voters' total satisfaction is selected, which completes the proof. □

Theorem 2. *The K-approval election system is optimal for the uniformly random dictatorship rule.*

Proof. For a committee C, let $\mathrm{apprv}_i(C)$ denote the number of candidates from C that are approved of by i. Let p and q denote the probability of representation of i by candidates approved of and disapproved of by i, respectively. Naturally, $p > q$. For each important for i issue $D_j \in D_{\mathrm{im}}(i)$ and each committee C let $\mathbb{P}(C, a)$ denote the probability that, during the uniformly random dictatorship ballot regarding D_j, the final decision will be made by the committee member a. Naturally, $\mathbb{P}(C, a) = \frac{1}{K}$. The probability that C will vote according to i's preferences is equal to:

$$\begin{aligned}\mathbb{P}_{C,\mathcal{R}_2}(i) &= \sum_{a \in C} \mathbb{P}(C, a) \cdot (p\mathbb{1}_{i \text{ approves of } a} + q\mathbb{1}_{i \text{ disapproves of } a}) \\ &= \mathrm{apprv}_i(C) \cdot \frac{p}{K} + (K - \mathrm{apprv}_i(C)) \cdot \frac{q}{K} \\ &= \mathrm{apprv}_i(C) \cdot \frac{p-q}{K} + q.\end{aligned}$$

Consequently, a committee C is optimal if it has the maximal value $\sum_i \mathrm{apprv}_i(C)$. Exactly such committee is elected by the K-approval election system. □

Theorems 1, and 2 are quite powerful in a sense that they claim the optimality of the $\frac{K+1}{2}$-median and K-approval multi-winner election rules, for the corresponding single-winner rules, irrespectively of the preferences of the voters over the issues. Unfortunately, this is not always the case, which is expressed by the following two theorems.

Before we proceed further, we introduce two new definitions that describe two extreme classes of voters' preferences over the issues. The intuitive meaning of these two classes can be better expressed if we rename the alternatives d_0 and d_1 to pass and veto, respectively. We say that voters are *rejection-oriented* if for each voter i, and each $D_j \in D_{\mathrm{im}}(i)$, $d_i^j = d_1$ (after renaming, $d_i^j = \mathrm{veto}$), meaning that each voter gets utility only from rejecting important issues. Analogously, we say that the voters are *acceptance-oriented* if for each voter i, and each $D_j \in D_{\mathrm{im}}(i)$, $d_i^j = \mathrm{pass}$.

Theorem 3. *For rejection-oriented voters, Chamberlin–Courant system with approval votes is deterministically optimal for the unanimity system.*

Proof. Here, the reasoning is very similar to the proof of Theorem 1. The ultimate satisfaction of a voter i from a committee C, $\mathbb{P}_{C,\mathcal{R}_2}(i)$, is equal to 1 if C contains at least one candidate approved by i, and is equal to 0 otherwise. This is equivalent to the definition of voters' satisfaction in the Chamberlin–Courant election rule, which completes the proof. □

Theorem 4. *For acceptance-oriented voters, the K-median system with approval votes is deterministically optimal for the unanimity system.*

Theorems 3 and 4 suggest that Chamberlin–Courant's rule and K-median rule are suitable for electing committees that have veto rights. One should choose one of them depending on the voters' satisfaction model. For instance, if passing a wrong decision has much more severe consequences than rejecting a good one (which is captured by modeling the voters as rejection-oriented), a committee should be selected with Chamberlin–Courant's rule, and it should use unanimity rule to make final decisions. If passing a wrong decision has relatively low cost compared to rejecting a good one, the committee should be selected with a K-median rule

Below, we show a more general result that characterizes a class of optimal election systems for most natural single-winner election rules $\mathcal{R}_2 : \{0,1\}^K \to \mathbb{R}$. These "natural" rules are those which satisfy neutrality, anonymity and monotonicity (for the definitions of these properties we refer the reader to the work of Endriss [2015]). Intuitively, the outcomes of such rules depend on the number of votes given to certain alternatives only.

Definition 3. *A nondeterministic rule $\mathcal{R}_2 : \{0,1\}^K \to \mathbb{R}$ is* normal *if there exists a non-decreasing function $\mathcal{R}_n : \mathbb{N} \to \mathbb{R}$ such that $\mathcal{R}_2(v) = \mathcal{R}_n(|\{i : v[i] = 0\}|)$, and such that for each j, $\mathcal{R}_n(j) = 1 - \mathcal{R}_n(K - j)$.*

In Definition 3, the requirement that for each j, $\mathcal{R}_n(j) = 1 - \mathcal{R}_n(K - j)$ enforces that the probability of selecting the value 0 given j votes for 0 is equal to the probability of selecting 1, given j votes for 1, thus, it follows from neutrality.

Theorem 5. *For each normal election system $\mathcal{R}_2$, in the approval model there exists a K-element vector α, such that α-rule with approval votes is optimal for $\mathcal{R}_2$.*

Proof. Let $\mathcal{R}_n$ be as in Definition 3. Let $C_{\ell,i}$ denote the committee that has exactly ℓ members approved of by i, and let $\mathbb{P}(i, s, \ell)$ be the probability that exactly s members of $C_{\ell,i}$ will vote accordingly to i's preferences. We have:

$$\mathbb{P}(i,s,\ell) = \sum_{x=1}^{K} \mathbb{1}_{x \le \ell} \mathbb{1}_{x \le s} \mathbb{1}_{s-x \le K-\ell} \cdot \binom{\ell}{x} p^x q^{\ell-x} \binom{K-\ell}{s-x} q^{s-x} p^{K-\ell-s+x}.$$

We can see that $\mathbb{P}(i, s, \ell)$ does not depend on i. Further, let $\mathbb{P}_{\mathcal{R}_2}(i|s)$ be the probability that the rule $\mathcal{R}_2$ makes decision consistent with i's preferences, assuming s members of $C_{\ell,i}$ votes accordingly to i's preferences.

$$\mathbb{P}_{\mathcal{R}_2}(i|s) = \mathcal{R}_n(s)\mathbb{1}_{d_i^j = d_{j,0}} + (1 - \mathcal{R}_n(K - s))\mathbb{1}_{d_i^j = d_{j,1}}.$$

Since either $d_i^j = d_{j,0}$ or $d_i^j = d_{j,1}$ and since $\mathcal{R}_n(s) = (1 - \mathcal{R}_n(K - s))$, we get that $\mathbb{P}_{\mathcal{R}_2}(i|s) = \mathcal{R}_n(s)$, and thus $\mathbb{P}_{\mathcal{R}_2}(i|s)$ does not depend on i. Consequently, for each important for i issue $D_j \in D_{\mathrm{im}}(i)$ we can calculate the expected ultimate satisfaction $\mathbb{P}_{C_{\ell,i},\mathcal{R}_2}(i)$ of a voter i from a committee $C_{\ell,i}$, so that this value does not depend on i:

$$\mathbb{P}_{C_{\ell,i},\mathcal{R}_2}(i) = \sum_{s=1}^{K} \mathbb{P}(i,s,\ell)\mathbb{P}_{\mathcal{R}_2}(i|s).$$

Because $\mathbb{P}_{C_{\ell,i},\mathcal{R}_2}(i)$ does not depend on i, we will denote it as $\mathbb{P}_{C_\ell,\mathcal{R}_2}$. Naturally, since $p \ge q$, we have $\mathbb{P}_{C_{\ell+1},\mathcal{R}_2} \ge \mathbb{P}_{C_\ell,\mathcal{R}_2}$, for each ℓ. Now, we can see that the following vector:

$$\alpha = \Big\langle \mathbb{P}_{C_1,\mathcal{R}_2},\ (\mathbb{P}_{C_2,\mathcal{R}_2} - \mathbb{P}_{C_1,\mathcal{R}_2}),\ (\mathbb{P}_{C_3,\mathcal{R}_2} - \mathbb{P}_{C_2,\mathcal{R}_2}), \ldots, (\mathbb{P}_{C_K,\mathcal{R}_2} - \mathbb{P}_{C_{K-1},\mathcal{R}_2})\Big\rangle$$

satisfies the requirement from the thesis. Indeed, in the α-rule the satisfaction of a voter from a committee with ℓ approved members is the sum of first ℓ coefficients of α, which is $\mathbb{P}_{C_\ell,\mathcal{R}_2}$. This completes the proof. ☐

The above theorem can be viewed as an evidence of the expressiveness and power of OWA election rules. Unfortunately, OWA election rules are not sufficiently expressive to describe the non-approval model. To get characterization similar to the one given in Theorem 5, but for arbitrary utilities, we would need to consider a more general class of election rules, that is rules in which the satisfaction of a single voter i from a committee C is expressed as a linear combination of products of i' utilities from individuals: $\sum_{C' \subseteq C} \prod_{a \in C'} u_i(a)$ (in contrast to a linear combination of utilities only). Such rules that consider inseparable committees were considered e.g., by Ratliff [2006; 2003]. Nevertheless, for the case of arbitrary utilities, we can get a result similar to the one given in Theorem 2.

Theorem 6. *There exists a utility function $P : \mathbb{R} \to \mathbb{R}$ transforming voters' utilities to representation probabilities, such that K-top election system is optimal for the uniformly random dictatorship rule.*

Proof. Let us define $\mathbb{P}(C, a)$ as in the proof of Theorem 2. The probability that C will vote according to i's preferences is equal to:

$$\mathbb{P}_{C,\mathcal{R}_2}(i) = \sum_{a \in C} \mathbb{P}(C,a) \cdot p_{i,a} = \frac{1}{K} \sum_{a \in C} p_{i,a}.$$

Let P be the identity function. The rule that maximizes the total score of selected candidates, also maximizes their total expected ultimate satisfaction. This completes the proof. ☐

5 Optimality of Indirect Election Rules

In the previous section we studied the optimality of multi-winner election rules, given information on what single-winner election system the committee will use to make final decisions. In this section we show that in our probabilistic model we can compare qualities of the pairs of multi-winner and single-winner systems. Thus, our results not only suggest which multi-winner election rule is suitable for electing a committee, but also indicates which single-winner rule should be used by the committee to make final decisions. We start with introducing the definition of the indirect election rule. This definition uses an interesting concept: it allows the voters to elect not only committees, but also single-winner rules that these committees use to make final decisions.

Definition 4. *An* indirect election rule *is a function $\mathcal{F} : \Pi^n(N) \to \mathcal{P}(A) \times \mathcal{SW}$ that for each utility profile $\pi^n \in \Pi^n(N)$ returns a pair $\mathcal{F}(\pi^n) = (C, \mathcal{R}_2)$, where $C \in \mathcal{P}(A)$*

is the elected committee, and $\mathcal{R}_2 \in \mathcal{SW}$ is the single-winner election rule that C will use when making final decisions. We use the notation $\mathcal{F}(\pi^n)[1] = C$ and $\mathcal{F}(\pi^n)[2] = \mathcal{R}_2$.

Definition 5. *Let $P : \mathbb{R} \to \mathbb{R}$ be a function transforming voters' utilities to representation probabilities. An indirect election rule $\mathcal{F}$* weakly dominates *an indirect election rule $\mathcal{G}$ for P, if for each utility profile $\pi^n \in \Pi^n(N)$, $\mathcal{F}$ returns a committee and a single-winner rule that gives the total expected ultimate satisfaction of the voters at least as high as the one given by a committee and a single-winner rule returned by $\mathcal{G}$:*

$$\sum_i \mathbb{P}_{\mathcal{F}(\pi^n)[1],\mathcal{F}(\pi^n)[2]}(i) \geq \sum_i \mathbb{P}_{\mathcal{G}(\pi^n)[1],\mathcal{G}(\pi^n)[2]}(i). \quad (2)$$

$\mathcal{F}$ strongly dominates *$\mathcal{G}$ for P if it weakly dominates $\mathcal{G}$ and if there exists profile $\pi^n \in \Pi^n(N)$ for which:*

$$\sum_i \mathbb{P}_{\mathcal{F}(\pi^n)[1],\mathcal{F}(\pi^n)[2]}(i) > \sum_i \mathbb{P}_{\mathcal{G}(\pi^n)[1],\mathcal{G}(\pi^n)[2]}(i). \quad (3)$$

An indirect election rule $\mathcal{F}$ is optimal *for P if it weakly dominates every other indirect election rule.*

Definition 6. *For a multi-winner rule $\mathcal{R}_1$ and a single-winner rule $\mathcal{R}_2$, by $\mathcal{R}_1$ followed by $\mathcal{R}_2$ we call the indirect function that selects committee using $\mathcal{R}_1$ and, independently of voters' preferences, always uses $\mathcal{R}_2$ to make final decisions.*

We show how to apply Definition 5 in perhaps the simplest variant, that is in the deterministic approval model. First, we define a new election rule COMB as a combination of $\frac{K+1}{2}$-median rule followed by majority rule with the K-approval rule followed by uniformly random dictatorship rule.

Definition 7. *The rule* COMB *is defined as follows. Let C and C' be committees elected by K-approval rule and by $\frac{K+1}{2}$-median rule, respectively. Let* apprv *be the total approval score of C and let* owa *be the total OWA score of C'. If $\frac{\mathrm{apprv}}{K} > \mathrm{owa}$, then* COMB *returns the pair* $(C, \text{uniformly random dictatorship})$. *Otherwise,* COMB *returns the pair* $(C', \text{majority})$.

In our deterministic model, the new rule COMB strongly dominates both rules that it is derived from.

Proposition 1. *In the deterministic model,* COMB *strongly dominates $\frac{K+1}{2}$-median followed by majority, and K-approval followed by uniformly random dictatorship.*

Proof. Let apprv and owa be defined as in Definition 7. Repeating the analysis from the proofs of Theorems 1 and 2, we get that the total ultimate satisfaction of voters under $\frac{K+1}{2}$-median followed by majority is equal to owa, under K-approval followed by uniformly random dictatorship is equal to $\frac{\mathrm{apprv}}{K}$, and under COMB is equal to $\max(\frac{\mathrm{apprv}}{K}, \mathrm{owa})$. This completes the proof. □

The natural question is whether we can find an optimal rule in our general probabilistic model. Unfortunately, this seems unlikely, as the societal quality of a rule also depends on the underlying voters' utility model. For instance, if for each j, $D_j = \{\mathrm{pass}, \mathrm{veto}\}$, and the voters are rejection-oriented, they prefer rules in which committees can more easily make a collective decision veto. In fact, for the rejection-oriented voters we can show that any multi-winner rule followed by the constant single-winner rule that always returns veto is optimal. Such rules, on the other hand, are highly disliked by the acceptance-oriented voters. Interestingly, if we restrict ourselves to normal single-winner rules to make final decisions, and to the deterministic model, then there exists an optimal indirect rule.

Theorem 7. *In the deterministic model, if we consider only normal single-winner rules used for making the final decisions, then there exists an optimal indirect election rule.*

Proof. The proof is constructive. From Definition 3 we recall that a normal single-winner rule $\mathcal{R}_2$ can be described by K values $\mathcal{R}_n(1), \ldots, \mathcal{R}_n(K)$. We use notation from the proof of Theorem 5. In the deterministic model $\mathbb{P}(i, s, \ell) = \mathbb{1}_{s=\ell}$. Consequently, for a committee C_ℓ with ℓ approved members, we have $\mathbb{P}_{C_\ell,\mathcal{R}_2} = \mathcal{R}_n(\ell)$. Let $\mathbb{1}_{\ell,i}(C)$ denote a function that returns 1 if C contains ℓ elements approved of by i, and 0 otherwise. For a given committee C we can find an optimal single-winner election rule $\mathcal{R}_2$ by solving the following linear program:

$$\text{maximize } \sum_i \mathbb{P}_{C,\mathcal{R}_2}(i) = \sum_i \mathcal{R}_n(\ell)\mathbb{1}_{\ell,i}(C)$$

subject to:

$$\begin{aligned}
&\text{(a)} : \mathcal{R}_n(\ell+1) \geq \mathcal{R}_n(\ell), && 1 \leq \ell \leq K-1\\
&\text{(b)} : 0 \leq \mathcal{R}_n(\ell) \leq 1, && 1 \leq \ell \leq K\\
&\text{(c)} : \mathcal{R}_n(\ell) = 1 - \mathcal{R}_n(K-\ell), && 1 \leq \ell \leq K
\end{aligned}$$

The optimal indirect rule tries all committees and selects such that gives the best solution to the integer program. From the solution of the integer program we can extract values $\mathcal{R}_n(1), \ldots, \mathcal{R}_n(K)$ that describe the optimal single-winner election rule that should be used to make final decisions. □

6 Conclusion

We defined a new model, called the voting committee model, which explores scenarios where a group of representatives is elected to make decisions on behalf of the voters. This model links utilities of the voters from the elected committee to their utilities from the committee's decisions regarding various matters. Intuitively, a satisfaction of voter i from a committee C is proportional to the probability that for the issues important for i, C's decisions are consistent with i's preferences.

Most of our results are for a restricted version of the model, called the approval model, in which each voter distinguish only two types of candidates: the candidates who represent her well and those who do not.

Our main conclusion says that different multi-winner election systems are suitable for choosing committees, depending on the rule used by the elected committee to make decisions. In particular, we justified which known multi-winner rules are suitable for which single-winner ones. Further, under some simplifying assumptions, we showed that the class of OWA rules contains all optimal multi-winner election systems.

We introduced the notion of an indirect rule, in which voters elect a decisive committee along with the single-winner rule to be used. Under some assumptions, we showed that there exists an optimal indirect rule and we described the way how it can be constructed.

There is a number of natural questions and open directions for future work. For instance, it is interesting to find out whether the optimal indirect rule can compute winning committees in polynomial time. Another appealing direction is to extend our model beyond binary domains and to analyze a variety of other single-winner rules.

Acknowledgements. The author was supported by Polish National Science Centre grant Preludium 2013/09/N/ST6/03661.

References

[Aziz *et al.*, 2015] H. Aziz, M. Brill, V. Conitzer, E. Elkind, R. Freeman, , and T. Walsh. Justified representation in approval-based committee voting. In *Proceedings of the 28th Conference on Artificial Intelligence (AAAI-2015)*, 2015.

[Barberá and Coelho, 2008] S. Barberá and D. Coelho. How to choose a non-controversial list with k names. *Social Choice and Welfare*, 31(1):79–96, 2008.

[Bock *et al.*, 1998] H. Bock, W. H. E. Day, and F. R. McMorris. Consensus rules for committee elections. *Mathematical Social Sciences*, 35(3):219–232, 1998.

[Brams *et al.*, 1998] S. J. Brams, D. M. Kilgour, and W. S. Zwicker. The paradox of multiple elections. *Social Choice and Welfare*, 15(2):211–236, 1998.

[Brams *et al.*, 2007] S. J. Brams, D. M. Kilgour, and M. R. Sanver. A minimax procedure for electing committees. *Public Choice*, 132(3–4):401–420, 2007.

[Chamberlin and Courant, 1983] B. Chamberlin and P. Courant. Representative deliberations and representative decisions: Proportional representation and the Borda rule. *American Political Science Review*, 77(3):718–733, 1983.

[Christian *et al.*, 2007] R. Christian, M. Fellows, F. Rosamond, and A. Slinko. On complexity of lobbying in multiple referenda. *Review of Economic Design*, 11(3):217–224, 2007.

[Debord, 1992] B. Debord. An axiomatic characterization of Borda's k-choice function. *Social Choice and Welfare*, 9(4):337–343, 1992.

[Elkind *et al.*, 2014] E. Elkind, P. Faliszewski, P. Skowron, and A. Slinko. Properties of multiwinner voting rules. In *Proceedings of the 13th International Conference on Autonomous Agents and Multiagent Systems (AAMAS-2014)*, May 2014. Also presented in FIT-2013.

[Endriss, 2015] U. Endriss. Approval balloting for multiwinner elections. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, *Handbook of Computational Social Choice.* Cambridge University Press, 2015.

[Fishburn, 1981] P. C. Fishburn. An analysis of simple voting systems for electing committees. *SIAM Journal on Applied Mathematics*, 41(3):499–502, 1981.

[Gehrlein, 1985] W. Gehrlein. The condorcet criterion and committee selection. *Mathematical Social Sciences*, 10(3):199–209, 1985.

[Kilgour, 2010] D. Kilgour. Approval balloting for multiwinner elections. In J. Laslier and R. Sanver, editors, *Handbook on Approval Voting*, pages 105–124. Springer, 2010.

[Koriyama *et al.*, 2013] Y. Koriyama, J. F. Laslier, A. Macé, and R. Treibich. Optimal Apportionment. *Journal of Political Economy*, 121(3):584–608, 2013.

[Lacy and Niou, 2000] D. Lacy and E. M. S. Niou. A problem with referendums. 12(1):5–31, 2000.

[Monroe, 1995] B. Monroe. Fully proportional representation. *American Political Science Review*, 89(4):925–940, 1995.

[Ratliff, 2003] T. C. Ratliff. Some startling inconsistencies when electing committees. *Social Choice and Welfare*, 21(3):433–454, 2003.

[Ratliff, 2006] T. C. Ratliff. Selecting committees. *Public Choice*, 126(3-4):343–355, 2006.

[Shmoys *et al.*, 1997] D. Shmoys, É. Tardos, and K. Aardal. Approximation algorithms for facility location problems (extended abstract). In *Proceedings of the 29th Symposium on Theory of Computing (STOC-1997)*, pages 265–274, 1997.

[Simeone and Pukelsheim, 2006] B. Simeone and F. Pukelsheim. *Mathematics and Democracy: Recent Advances in Voting Systems and Collective Choice*. Studies in Choice and Welfare. Springer Berlin Heidelberg, 2006.

[Skowron *et al.*, 2015] P. Skowron, P. Faliszewski, and J. Lang. Finding a collective set of items: From proportional multirepresentation to group recommendation. In *Proceedings of the 28th Conference on Artificial Intelligence (AAAI-2015)*, 2015.

[Xia *et al.*, 2008] L. Xia, V. Conitzer, and J. Lang. Voting on multiattribute domains with cyclic preferential dependencies. In *Proceedings of the 23rd National Conference on Artificial Intelligence - Volume 1*, Proceedings of the 23rd National Conference on Artificial Intelligence (AAAI-2008), pages 202–207, 2008.

[Xia *et al.*, 2010] L. Xia, V. Conitzer, and J. Lang. Aggregating preferences in multi-issue domains by using maximum likelihood estimators. In *Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems (AAMAS-2010)*, pages 399–406, 2010.

[Young, 1975] H. Young. Social choice scoring functions. *SIAM Journal on Applied Mathematics*, 28(4):824–838, 1975.

[Young, 1995] P. Young. Optimal voting rules. *The Journal of Economic Perspectives*, 9(1):51–64, 1995.