Aggregate Demand-Based Real-Time Pricing Mechanism for the Smart Grid: A Game-Theoretic Analysis

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Abstract

Managing peak energy demand is a critical problem for energy utilities. The energy costs for the peak periods form a major component of their overall costs. Real-time pricing mechanisms have been explored as a means of flattening the demand curve and reducing the energy costs. In this paper, we examine a model of ex-post real-time pricing mechanism that can be used by the utilities for this purpose. In particular, we study a convex piecewise linear cost function that modulates the price of energy based on the aggregate demand of the utility. We provide a game-theoretic analysis of the mechanism by constructing a non-cooperative game among the consumers of a utility wherein the cost to each consumer is decided by the pricing mechanism. We formally characterize the Nash equilibrium and other properties for two settings: (i) consumers have full flexibility in shifting their demand, and (ii) consumers can shift only a fraction of their demand at any time to another time.

1 Introduction

Managing peak energy demand is a challenge faced by many energy utility companies [Strbac, 2008]. The US Energy Information Administration estimates that the peak to average ratio across the continental US is 1.7 and rising. According to International Energy Agency report [IEA, 2003], the cost of energy could have been reduced by approximately 50% by lowering demand by just 5% during the peak hours of the California electricity crisis in 2000/2001. Although, the peak demand period occurs for a small percentage of the overall time, the energy purchase cost to utilities for that period forms a significant proportion of their overall costs.

The key to peak demand management is to encourage consumers to shift their consumption from peak hours to off-peak hours [Palensky and Dietrich, 2011; Spees and Lave, 2007]. From the utility's (energy retailer's) perspective, the ideal scenario would maintain the peak-to-average ratio at 1, i.e. uniform aggregate demand. Real-time pricing mechanisms have been considered as a means of addressing the peak demand problem [Borenstein, 2005]. In real-time pricing, the unit cost of electricity varies with the time of day. Generally, when the total demand (or consumption) is low, the unit cost of electricity is also low since the cost of generating this energy is low, and vice versa. Real-time pricing, when implemented correctly, has the potential to address the peak demand challenge by providing appropriate incentives to the consumers to shift demand to off-peak periods [Barbose *et al.*, 2004].

In particular, the real-time pricing mechanisms that have been extensively studied and in-use across the world are *ex-ante*. Ex-ante refers to the fact that in these mechanisms, the price of electricity is heuristically determined before the concerned time period, based on the prevailing wholesale market prices, and communicated to the consumers by the utilities. It is hoped that the consumers would then shift their demand for that forthcoming time period accordingly, based on their perception of the price and flexibility of consumption. The other form of real-time pricing that has largely been left unexplored for retail purposes is *ex-post*. In this, the price of electricity for a given time period is determined *after* the passage of the time period based on observations of the overall demand in that period, without having to resort to estimates. While such ex-post mechanisms are prevalent in the spot and reserve markets at the ISO level [Zheng and Litvinov, 2011], they haven't been considered for retail pricing because of the concern that consumers may not be able to act upon and shift demand without pricing signals delivered to them beforehand [Hossain *et al.*, 2012].

Nevertheless, in this paper, we revisit ex-post real-time pricing mechanism for consumers (as applied by an utility) from a game-theoretic perspective to derive important observations about consumer behaviors under such schemes, and show the theoretical feasibility of such mechanisms. In particular, we construct a game, called as the *consumption game*, wherein all the consumers of a utility are the players. We assume that all the consumers are rational and intelligent and have individual energy requirements for a time period. With the utility acting as the game designer and aiming to reduce its overall peak demand, the natural way to design the pricing scheme is to set the unit cost of electricity as a function of the aggregate consumption of all the consumers under the utility.

This implies that the cost incurred by a consumer in any time slot depends on the demand of other consumers in that time slot. The goal of any consumer is to consume in such a way that their individual electricity cost throughout the day is minimized while satisfying their individual energy requirements for that day (or some such large period of time).

Specifically, we propose and study ex-post real-time pricing mechanisms in the generic form — a convex K step piece-wise linear function over the aggregate demand. Cost functions for electricity generation and distribution are often heuristically modeled as piece-wise linear functions, that are monotonically increasing with non-decreasing slope [Yasmeen *et al.*, 2012]. It is also to be noted that piece-wise linear functions are convenient interpolation of a small number of points of any cost curve [Aganagic and Mokhtari, 1997]. We consider the following two settings under which we analyze the consumption game:

Flexible Consumption Game (FC Game): In this setting, we assume that all consumers can shift their consumption load from one time slot to another within that period (such as a day) without any restriction.

Partially-Flexible Consumption Game (PFC Game): This is a more practical setting and assumes that all consumers can only shift a fraction of their consumption load at a time slot to any other time slot (in that period or day).

For each of the two game settings, we characterize the equilibrium and efficient consumption profiles. Informally, equilibrium refers to stable consumption profiles whereas efficiency refers to those consumption profiles which result in minimum total cost. Since the cost incurred by each consumer depends on the consumption by other consumers using the real-time pricing scheme, we analyze the Nash equilibrium of the consumption game to understand the most likely consumption behavior of the consumers.

In particular, for the FC game, we first show that the aggregate consumption profile by all consumers would be uniform over time under any pure strategy Nash equilibrium. We also show the relation of the individual consumption profiles with Nash equilibrium of the proposed game. We then formally show that all those consumption profiles where the aggregate consumption by all consumers is uniform turn out to be efficient; however, such profiles need not be in Nash equilibrium. For the PFC game, we are able to reproduce most of the results obtained in FC game under some mild constraints.

Thus, this paper conducts the first study of an ex-post form of real-time pricing mechanism for end consumers in the electricity smart grid, using game theoretic analysis. In the next section, we briefly present relevant literature pertaining to our domain and approach. Then, Section 3 explains our game setting and pricing function in more detail. Following that, Section 4 and Section 5 analyse the FC game and PFC game respectively. Section 6 concludes the paper.

2 Related Work

Demand Response (DR) is increasingly being seen as a suitable means for improving the efficiency of the electricity grid [Albadi and El-Saadany, 2007]. Most DR programs seek to achieve this goal by reducing the total consumption during the peak demand hours, in other words, flattening the demand curve. In our work, we explore a form of real-time pricing which is, in fact, a price based DR program. One example of a deployment of real-time pricing is by the Illinois Power Company to some regions in North America [Allcott, 2009].

Nevertheless, one of the difficulties in measuring the success of DR is to quantify the inconvenience caused to the consumers. This boils down to the problem of modeling the constraints on the consumption behavior of the consumers. There are several ways of representing consumer's preference over the the consumption pattern [Chandan *et al.*, 2014; Li *et al.*, 2011]. In our paper, we have formulated two models which capture the consumption behavior of consumers in an ideal set up and a more practical set up, respectively.

The main contribution of our work is that we study the electricity pricing scheme using game-theoretic techniques, thus following in the same path carved by several recent works that have applied Game Theory and Mechanism Design techniques to address smart grid problems, particularly demand response [Chen *et al.*, 2012; Kota *et al.*, 2012; Jain *et al.*, 2014]. Specifically, formulation of an energy consumption scheduling game among users is carried out by [Mohsenian-Rad *et al.*, 2010]. They also form two optimization problems based on Peak-to-Average Ratio minimization and energy cost minimization and show the relation between them. Similarly, a game theoretic approach to optimize time-of-use pricing strategies is taken by [Yang *et al.*, 2013]. Adoption of storage from a grid perspective (with millions of consumers), with assumption that the effect of changes in individual strategy/consumption on the price is negligible, has been modelled and discussed in [Vytelingum *et al.*, 2011]. In contrast, we focus on utilities determining the price for their customers to reduce peaking, without the above assumption. Thus, our work provides fundamental results for a generic form of ex-post real-time pricing scheme.

3 Game Setting and Pricing Model

Let $N = \{1, 2, \ldots, n\}$ be a set of n consumers with the utility. Let T be the number of discrete time slots in a day. For instance, the value of T is 24 if we consider each hour as one time slot. Let the consumption of the i-th consumer at time t be y_i^t. For each $i \in N$, we denote the daily consumption profile of i-th consumer as $y_i = \{y_i^1, y_i^2, \cdots, y_i^T\}$. Let $Y_i, \forall i \in N$, be the set of all possible consumption profiles of consumer i. Also, for each $i \in N$, y_{-i} represents the profile of consumptions of all players except i. We assume that total consumption by any consumer throughout the day is always strictly positive; that is, $\sum_{t=1}^{T} y_i^t > 0$, $\forall i \in N$ (otherwise, she does not need energy for the day and can be excluded). Let $Y = \times_{i \in N} Y_i$ be the set of all consumption profiles of the consumers. We represent each element in Y as $(y_i, y_{-i}) = (y_1, y_2, \ldots, y_n)$.

Each consumer will have to pay the utility for her electricity consumption. In fact, since we present a real-time electricity pricing scheme based on aggregate demand, the cost to any consumer also depends on the total energy consumed by all the consumers at that time slot. Let us denote the total (aggregate) energy consumption by all players at time t as,

$m^t = \sum_{i=1}^{n} y_i^t, \quad \forall t = 1, 2, \ldots, T.$

The focus of the mechanism is to reduce peak demand rather than overall demand in a day. Therefore, in our setting, we consider that the overall consumption needs of a consumer over the period of a day is fixed (although there exists flexibility regarding the time of consumption within the day). Nevertheless, without any loss of generality, this period of time needn't be one day. It can as well correspond to a portion of a day (e.g., just the daylight hours) or several days, and our results still hold good. Formally, we introduce the following assumption.

Assumption 1 *Conservation of Energy Consumption: A consumer can change her daily consumption profile according to her own convenience; however, the sum of the energy consumption in all those profiles for a particular day must be same. So for each $i \in N$, there exists a constant m_i such that*

$$m_i = \sum_{t=1}^{T} y_i^t, \quad \forall y_i \in Y_i \tag{1}$$

Since the total energy consumption of each consumer on a particular day is fixed, the sum of the energy consumption by all consumers of the utility is also fixed for that day. We refer to this as **Conservation of Energy Consumption**.

We call any consumption profile which satisfies the above assumption, as a feasible consumption profile under this model.

The unit cost of electricity at any time t is determined by a convex K-piece-wise linear function of the aggregate energy consumption m^t. Formally,

$$f(x; A_1, \cdots, A_K, B, \theta_1, \cdots, \theta_{K-1})$$
$$= \begin{cases} A_1 x + B_1, & \text{if } x \leq \theta_1 \\ A_2 x + B_2, & \text{if } \theta_1 < x \leq \theta_2 \\ \cdots \\ A_K x + B_K, & \text{if } \theta_{K-1} \leq x \end{cases} \tag{2}$$

where $A_1, \cdots, A_K, B, \theta_1, \cdots, \theta_{K-1}$ are the parameters of this function with $0 \leq A_1 \leq A_2 \leq \cdots \leq A_K$, and $B_1 = B \geq 0$ and $B_k = B - \sum_{l=1}^{k-1} \theta_l (A_{l+1} - A_l), \forall k = 2, 3, \cdots, K$. Note that $x (= m^t)$ is a real number denoting the aggregate consumption at time t, and hence $x \in \mathbb{R}_{\geq 0}$. For the ease of presentation, let us assume that for a K-piece-wise linear function of this form, $\theta_0 = 0$ and $\theta_K = +\infty$. Clearly $\theta_1, \cdots, \theta_{K-1}$ partition the positive real line $\mathbb{R}_{\geq 0}$ into K segments where the kth segment indicates $\theta_{k-1} \leq x \leq \theta_{k+1}$, $\forall k = 1, 2, \cdots, K - 1$. That is, a K-piece-wise linear function has K segments. The actual values of these parameters will be determined by the utility based on the actual generation and distributional cost of electricity. Figure 3 shows a stylized example of a piece-wise linear function with $K = 4$. Clearly for $K = 1$, Eq. 2 boils down to a linear function in x. For notational convenience, we will only use $f(x)$ instead of $f(x, A_1, \cdots, A_K, B, \theta_1, \cdots, \theta_{K-1})$ where there is no confusion. Following are a few key properties of this function.

Property 1 *The function $f(x)$ is always non-negative for any non-negative x. That is, $f(x) \geq 0, \quad \forall x \geq 0$.*

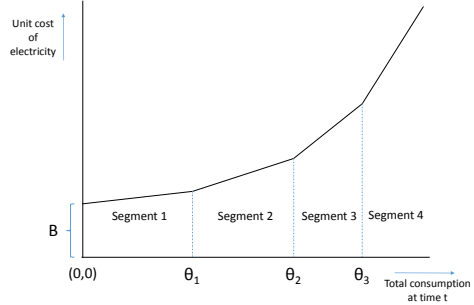**Property 2** *$f(x)$ is monotonically increasing with x.*

Figure 1: An example of a 4-step piece-wise linear function

Property 3 *$f(x)$ is convex over x.*

All of these properties are valid because of the fact that $B \geq 0$ and $0 \leq A_1 \leq A_2 \leq \cdots \leq A_K$. The proof of the above properties is straightforward and can be found in the literature on convex functions [Rockafellar, 1997]; hence, we skip the proofs in the interest of space. Now we present few lemmas that are useful to prove some important results later.

Lemma 1 *Suppose $H \geq 0$ is a constant. Then*
$f(x + H; A_1, \cdots, A_K, B, \theta_1, \cdots, \theta_{K-1})$
$= \bar{H} + f(x; A_1, \cdots, A_K, B, \bar{\theta}_1, \cdots, \bar{\theta}_{K-1})$, *where*
$\bar{H} = f(H; A_1, \cdots, A_K, B, \theta_1, \cdots, \theta_{K-1})$ *and*
$\bar{\theta}_1, \cdots, \bar{\theta}_{K-1}$ *are some parameters to f.*

Proof: We present only sketch of the proof. Clearly, the function $f(x + H; A_1, \cdots, A_K, B, \theta_1, \cdots, \theta_{K-1})$ is a shift of the function $f(x; A_1, \cdots, A_K, B, \theta_1, \cdots, \theta_{K-1})$ by $\bar{H}$. Hence if we set the new parameters as, $\bar{\theta}_l = 0$, if $\theta_l < \bar{H}$ and $\bar{\theta}_l = \theta_l - \bar{H}$, if $\theta_l \geq \bar{H}$, then $f(x + H; A_1, \cdots, A_K, B, \theta_1, \cdots, \theta_{K-1})$
$= \bar{H} + f(x; A_1, \cdots, A_K, B, \bar{\theta}_1, \cdots, \bar{\theta}_{K-1})$. ∎

Lemma 2 *$xf(x)$ is strictly convex.*

Lemma 3 *Consider $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4$ and $x_2 - x_1 = x_4 - x_3$. Then $f(x_2) - f(x_1) \leq f(x_4) - f(x_3)$.*

Proof: Consider that $x_2 - x_1 = x_4 - x_3 = \delta \geq 0$. Let us also assume that x_2 belongs to k^{th} segment and x_3 belongs to the lth segment with respect to the function $f(.)$, where $l \geq k$, as $x_3 \geq x_2$. Hence, $f(x_2) - f(x_1) \leq \delta A_k$, as x_1 can belong to the kth segment or in some lower segment. Again, $f(x_4) - f(x_3) \geq \delta A_l$, as x_4 can belong to the l-th segment or some higher segment. Note that $A_k \leq A_l$ because of $k \leq l$; hence, $f(x_2) - f(x_1) \leq f(x_4) - f(x_3)$. ∎

Now the cost of electricity for consumer i at time slot t is,

$$C_i(y_i, y_{-i}) = A \sum_{t=1}^{T} f(m^t) y_i^t, \quad \forall i = 1, 2, \cdots, n. \tag{3}$$

4 FC Game

Recall that, for the FC game, the consumption loads are completely deferrable. That is, any consumer can shift any amount of her consumption from any time slot to another in that day. Moreover, as the cost of electricity at any time

depends on the total consumption at that time, the cost for each consumer will depend on the consumption of the other consumers. Assuming the consumers are rational and intelligent, each consumer seeks to minimize the cost by choosing her consumption profile by appropriately adjusting her consumption. This strategic interaction among the consumers can be naturally modeled using game theory. Towards this end, based on the above convex K-piecewise linear function, we propose a non-cooperative strategic form game [Myerson, 1997; Narahari *et al.*, 2009] as follows:

Players: The set N of the consumers are the players;

Strategies: For each consumer $i \in N$, the set of possible consumption profiles Y_i are such that they satisfy Eq. 1 (Assumption 1).

Payoffs: For each consumption profile $(y_i, y_{-i}) \in Y$, we define $C_i(y_i, y_{-i}) = A \sum_{t=1}^{T} y_i^t f(m^t)$ as the payoff of consumer $i \in N$. Since C_i is cost, the consumers seek to minimize the cost.

We represent this strategic form game as $\Gamma_C = [N, (Y_i)_{i \in N}, (C_i)_{i \in N}]$. This defines the FC game.

We call a consumption profile $(y_i, y_{-i}) \in Y$ is in *(Strict) Nash equilibrium* if for all $i \in N$ and for each $\bar{y}_i \in Y_i$

$$C_i(y_i, y_{-i}) < C_i(\bar{y}_i, y_{-i}). \qquad (4)$$

We call a consumption profile $(y_i, y_{-i}) \in Y$ *(Strictly) efficient* if for each $(\bar{y}_i, y_{-i}^-) \in Y$

$$\sum_{i \in N} C_i(y_i, y_{-i}) < \sum_{i \in N} C_i(\bar{y}_i, y_{-i}^-). \qquad (5)$$

Note that for implementing such a scheme, the utility company only needs to obtain the actual consumption profiles of each consumer, which is readily available from smart meters [van Gerwen *et al.*, 2006]. There isn't any privacy concern either because the profiles of one consumer are not revealed to any other consumer.

4.1 Characterization of Nash Equilibrium and Efficiency

In this section, we analyze the consumption profiles that are in equilibrium and efficient. We present either full or sketches of the proofs, as needed. All the notations are consistent with the earlier sections.

Theorem 1 *In any Nash equilibrium of the FC game, the total consumption by all the players is uniform*[1].

Proof: We prove this by contradiction. If there is no equilibrium in this game, then the theorem is vacuously true. Otherwise we assume that the total consumption profile $m^1, m^2, \cdots, m^T$ is non-uniform. So we get two time slots t_1 and t_2 such that $m^{t_1} > m^{t_2}$ and consequently $\exists i \in N$ such that $y_i^{t_1} > y_i^{t_2}$. We now change the consumption profile y_i to $\bar{y}_i$ as follows: $\bar{y}_i^t = y_i^t, \forall t \neq t_1, t_2$
$\bar{y}_i^{t_1} = y_i^{t_1} - \epsilon$,
$\bar{y}_i^{t_2} = y_i^{t_2} + \epsilon$, for some ϵ where $0 < \epsilon <$

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[1]By 'total consumption of all players is uniform', we always imply the aggregate of consumption of the players is uniform over T

$min(\frac{m^{t_1} - m^{t_2}}{2}, y_i^{t_1})$.

$$\therefore C_i(y_i, y_{-i}) - C_i(\bar{y}_i, y_{-i})$$
$$= y_i^{t_1} f(m^{t_1}) + y_i^{t_2} f(m^{t_2}) - (y_i^{t_1} - \epsilon) f(m^{t_1} - \epsilon)$$
$$- (y_i^{t_2} + \epsilon) f(m^{t_1} + \epsilon)$$
$$= y_i^{t_1}(f(m^{t_1}) - f(m^{t_1} - \epsilon)) - y_i^{t_2}(f(m^{t_2} + \epsilon) - f(m^{t_2}))$$
$$+ \epsilon(f(m^{t_1} - \epsilon) - f(m^{t_2} + \epsilon))$$

Since $f(.)$ is monotone increasing and $m^{t_1} > m^{t_2}$, for any $\epsilon < \frac{m^{t_1} - m^{t_2}}{2}$ and $\epsilon > 0$, it holds that $f(m^{t_1} - \epsilon) - f(m^{t_2} + \epsilon)$, and consequently, $\epsilon(f(m^{t_1} - \epsilon) - f(m^{t_2} + \epsilon)) > 0$. Now from Lemma 3, $f(m^{t_1}) - f(m^{t_1} - \epsilon) > f(m^{t_2} + \epsilon) - f(m^{t_2})$, since $m^{t_2} < m^{t_2} + \epsilon < m^{t_1} - \epsilon < m^{t_1}$. This implies that $y_i^{t_1}(f(m^{t_1}) - f(m^{t_1} - \epsilon)) > y_i^{t_2}(f(m^{t_2} + \epsilon) - f(m^{t_2}))$. Hence, $C_i(y_i, y_{-i}) > C_i(\bar{y}_i, y_{-i})$, which is a contradiction to our assumption. Hence proved. ∎

Note that the converse of Theorem 1 is not true.

Theorem 2 *In FC game, if each player consumes uniformly, then it is in Nash equilibrium.*

Proof: We present the proof sketch here. Consider a consumption profile $(y_1, y_2, \ldots, y_n)$ such that y_i is uniform for each $i \in N$. Hence we will have, $y_i^1 = y_i^2 = \cdots = y_i^T = p_i, \forall i = 1, 2, \cdots, n$. Consequently the sum of the consumption of all the players is uniform. Assume that for any time t, the total consumption is $m = \sum_{i=1}^{n} p_i$. Now consider the consumption profile (z_i, y_{-i}), $z_i \in Y_i$ of all the players where player i plays non-uniform consumption strategy z_i, $z_i \neq y_i$, and all the remaining players play the uniform consumption strategy. Let the total consumption by all the players except i at any time t be $m_{-i} = \sum_{j=1, j \neq i}^{n} p_j$.

$$C_i(z_i, y_{-i}) - C_i(y_i, y_{-i})$$
$$= \sum_{t=1}^{T} z_i^t f(z_i^t + m_{-i}) - T p_i f(p_i + m_{-i})$$
$$= \sum_{t=1}^{T} z_i^t f_i(z_i^t) + \sum_{t=1}^{T} z_i^t f_{-i}(m_{-i})$$
$$- T p_i f_i(p_i) - T p_i f_{-i}(m_{-i})$$

[From Lemma 1, replacing H by m_{-i} in Lemma 1; and as, f_i and f_{-i} are having different parameters]

$$= \sum_{t=1}^{T} z_i^t f_i(z_i^t) - T p_i f_i(p_i)$$

$$[\because (\sum_{t=1}^{T} z_i^t) f_{-i}(m_{-i}) = T p_i f_{-i}(m_{-i})]$$

$$> T(\sum_{t=1}^{T} (z_i^t/T)) f(\sum_{t=1}^{T} (z_i^t/T)) - T p_i f_i(p_i)$$

[Due to Lemma 2 and Jensen's inequality]

$$= T p_i f_i(p_i) - T p_i f_i(p_i) = 0 \quad [\because \sum_{t=1}^{T} (z_i^t/T) = p_i]$$

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That is, $C_i(z_i, y_{-i}) > C_i(y_i, y_{-i})$. Hence, the uniform consumption profile (y_i, y_{-i}) is in Nash equilibrium. ∎

Theorem 3 *For $K \geq 2$ (in Equation 2), FC game can be in Nash equilibrium, even when all the players are not consuming uniformly.*

Proof: We prove this by constructing a specific consumption profile wherein each player is not consuming uniformly. Let us consider a game with 2 players and with two time slots. That is, $n = 2$ and $T = 2$. Consider the unit cost of electricity is calculated using a two step piece-wise linear function as: $f(x) = 2x$, when $x \leq 10$ and $f(x) = 100x - 980$, when $10 < x \leq 19$ and so on for higher values of x.

From Theorem 1, the total consumption should be uniform in any Nash equilibrium of this game. So we need to construct the example in such a way that the total consumption is uniform, but individual consumption should be non-uniform. Consider a profile, $y_1^1 = 7$, $y_1^2 = 8$ and $y_2^1 = 3$, $y_2^2 = 2$. So, $m^1 = m^2 = 10$. Then total daily costs for the players $c_1 = 7 \times 20 + 8 \times 20 = 300$ and $c_2 = 3 \times 20 + 2 \times 20 = 100$ respectively. We now show that this consumption profile is in Nash equilibrium.

Let us change the consumption profile for player 1 as, $\bar{y_1}^1 = 7 + \epsilon$ and $\bar{y_1}^2 = 8 - \epsilon$, where $0 < \epsilon \leq 8$, keeping the consumption profile for player 2 unchanged. Note that the total daily consumption for player 1 is same because of $\bar{y_1}^1 + \bar{y_1}^2 = y_1^1 + y_1^2 = 15$. So, $\bar{m}^1 = (10 + \epsilon)$ and $\bar{m}^2 = (10 - \epsilon)$. Now the total cost for player 1 is $\bar{c_1} = (7 + \epsilon)(100(10 + \epsilon) - 980) + (8 - \epsilon)2(10 - \epsilon) = 300 + 684\epsilon + 102\epsilon^2 > 300 = c_1$. Similarly, if we increase y_1^2 and decrease y_1^1, the total daily cost for player 1 will increase again. Hence for any $\bar{y_1} \in Y_1$, $\bar{c_1} > c_1$, when consumption profile for player 2 is unchanged.

Similarly keeping the consumption profile for player 1 unchanged, it can be shown in the same way that for any $\bar{y_2} \in Y_2$, $\bar{c_2} > c_2$. Hence the consumption profile $((y_1^1, y_1^2), (y_2^1, y_2^2))$ is in Nash equilibrium; however, both the players are playing non-uniform strategy. ∎

Note that, for FC game with $K = 1$, Theorem 3 is not valid because it can be shown that the game is in Nash equilibrium if and only if each player consumes uniformly.

Theorem 4 *In the FC game, any consumption profile of the players is efficient if and only if the total consumption by all the players is uniform over time.*

Proof: *Claim:* If the total consumption by all the players is uniform over time, then it is an efficient consumption profile. Consider a consumption profile $(y_1, y_2, \ldots, y_n)$ such that the total consumption is uniform. Then the total consumption at any time t is same, call this value m. That is, $\sum_{i=1}^{n} y_i^t = m$, $\forall t = 1, 2, \cdots, T$. Also consider any consumption profile $(\bar{y_1}, \bar{y_2}, \ldots, \bar{y_n})$ wherein the total consumption by all the players is not uniform. Then the total consumption at any time t is $\sum_{i=1}^{n} \bar{y_i}^t = \bar{m}^t$, $\forall t = 1, 2, \cdots, T$. Now suppose $C(y_i, y_{-i}) = \sum_{i=1}^{n} C_i(y_i, y_{-i})$ is the total cost

incurred by all the players over all the time slots. Then,

$$C(\bar{y_i}, y_{-i}) - C(y_i, y_{-i})$$

$$= \sum_{t=1}^{T} \sum_{i=1}^{n} \bar{y_i}^t f(\bar{m}^t) - \sum_{t=1}^{T} \sum_{i=1}^{n} y_i^t f(m)$$

$$= \sum_{t=1}^{T} \bar{m}^t f(\bar{m}^t) - T m f(m)$$

$$[\because \sum_{i=1}^{n} \bar{y_i}^t = \bar{m}^t \text{ and } \sum_{i=1}^{n} y_i^t = m]$$

$$> T(\sum_{t=1}^{T} \bar{m}^t / T) f(\sum_{t=1}^{T} \bar{m}^t / T) - T m f(m)$$

[Applying Jensen's inequality, and using Lemma 2]

$$= T m f(m) - T m f(m) = 0 \quad [\because \sum_{t=1}^{T} \bar{m}^t = T m]$$

So, $C(\bar{y_i}, y_{-i}) > C(y_i, y_{-i})$. This implies that the total cost of all the players when the total consumption by them is not uniform is strictly greater than that of a consumption profile where the total consumption by all the players is uniform. This proves the claim. Equivalently we can prove the converse of this statement. ∎

Corollary 1 *Any Nash equilibrium of FC game is also efficient, but the converse is not necessarily true.*

In summary, we have shown for the FC game that under any Nash equilibrium, the total (aggregate) consumption profile by all consumers is uniform over time. Therefore, as Nash equilibrium corresponds to likely behavior, the aggregate demand of the utility is likely to be flat under such a pricing scheme. We have also shown the existence of Nash equilibrium for this game and that it can be achieved without needing the individual consumption profiles to be uniform over time. This is an important result showing that Nash equilibrium does not need to correspond to the trivial solution. Similarly, we have shown that the aggregate level uniform consumption profiles is also efficient. So for this game, individually best strategies (Nash equilibrium strategies) imply aggregate level flat consumption, which in turn implies best strategy for all (efficient strategy). Next, we move onto the partially-flexible consumption game (PFC game), where we seek to reproduce the same results under minor additional conditions.

5 PFC Game

We define the expected consumption of ith consumer at time t is the most preferred consumption of that consumer for that time, in the absence of any real-time pricing scheme or other direct or indirect incentives to modify her consumption behavior. For each $i \in N$ and for each $t = 1, 2, \cdots, T$, we denote the expected consumption of ith consumer at time t by exp_i^t where $exp_i^t \geq 0$.

In this setup, we assume that the expected consumption of any player at any time slot consists of two components: *primary consumption* and *secondary consumption*. In detail, *primary consumption* refers to that fraction of the consumption

which cannot be shifted from some particular time slot to any other. Similarly secondary consumption refers to loads which can be shifted from one time slot to another provided there is some incentive.

Now we represent the expected consumption as the sum of primary (pri_i^t) and secondary (sec_i^t) consumption. That is,

$$exp_i^t = pri_i^t + sec_i^t, \quad \forall i \in N, \ t \in \{1, 2, \ldots, T\}. \tag{6}$$

Let us assume that $pri_i^t = r_i exp_i^t$ and thus $sec_i^t = (1 - r_i) exp_i^t$, where $0 \leq r_i < 1$, $\forall i, t$. For the sake of simplicity of our analysis, we assume that the fraction of primary consumption is fixed throughout the day for a consumer. Nevertheless, even in the case of r_i varying over time, similar results can be derived. Note that when $r_i = 1 \ \forall i \in N$, it refers to the case when the consumption profile of each consumer is completely fixed with no secondary component. Thus, it reduces to a trivial scenario where each consumer has only one strategy in their strategy set.

We now present a non-cooperative strategic form game based on this model as follows:

Players: The set N of the consumers are the players;

Strategies: For each consumer $i \in N$, the set of possible consumption profiles Y_i are such that

$$y_i^t \geq r_i exp_i^t \tag{7}$$

$$\sum_{t=1}^{T} y_i^t = \sum_{t=1}^{T} exp_i^t \tag{8}$$

$\forall i = 1, 2, \cdots, n$ and $\forall t = 1, 2, \cdots, T$

Payoffs: For each consumption profile $(y_i, y_{-i}) \in Y$, we define $C_i(y_i, y_{-i}) = \sum_{t=1}^{T} f(m^t) y_i^t$ as the payoff of consumer $i \in N$. Since C_i is cost, the consumers want to minimize the cost.

We represent this strategic form game as,
$\Delta = [N, (Y_i)_{i \in N}, (C_i)_{i \in N}]$. This defines our PFC game.

Note that Eq. 8 essentially implies Assumption 1. We call any consumption profile as feasible in this setting if it satisfies Eq. 7 and Eq. 8. Clearly $(exp_1^t, exp_1^t, \ldots, exp_1^t)$ belongs to the set of feasible consumption profiles. Note that this game is a more generic form of the FC game. Specifically, when $r_i = 0$ for each $i \in N$, then PFC game boils down to the FC game model.

5.1 Characterization of Equilibrium and Efficiency

Here we analyze the equilibrium and efficient consumption profiles using the PFC game.

Theorem 5 *Consider the set of all feasible consumption profiles $(y_1, y_2, \ldots, y_n)$ such that $y_i^t > r_i exp_i^t$ when $r_i > 0$ and $exp_i^t > 0$, $\forall i \in N$. Then in any Nash equilibrium of the PFC game, the total consumption by all the players is uniform over time.*

Proof: Due to the space constraints, we only provide the sketch of the proof here. The method is similar to that of Theorem 1. As before, we find time slots t_1 and t_2 where $m^{t_1} > m^{t_2}$ and the existence of $i \in N$ such that $y_i^{t_1} > y_i^{t_2}$. However, the condition on ϵ should now be

$$0 < \epsilon < min\{(y_i^{t_1} - r_i exp_i^{t_1}), \frac{(m^{t_1} - m^{t_2})}{2}\} \tag{9}$$

This is feasible due to the assumption that $y_i^t > r_i exp_i^t$. The rest of the proof is on the same lines as in Theorem 1. ∎

Let us study the significance of the assumption made in this theorem. Clearly when $r_i = 0$, there is no primary (non-deferrable) component in the load of that consumer. But if there is some primary component (implies $r_i > 0$), and the expected consumption of the consumer is non-zero at some time t, then any feasible consumption at time t will always be strictly greater than (instead of '$\geq$' as defined by Eq. 7) the primary consumption at that time. In other words, there have to be some secondary component in any feasible consumption at time t when exp_i^t and r_i are both non-zero to ensure that the total consumption is uniform under Nash equilibrium.

Theorem 6 *Consider the set of all feasible consumption profiles $(y_1, y_2, \ldots, y_n)$ such that $p_i > \max_t(r_i exp_i^t)$, $\forall i \in N$. Then the PFC game is in Nash Equilibrium if all the players are consuming uniformly over time.*

Proof: Again due to space constraints, we only give a sketch. First, note that uniform consumption of each player is a feasible consumption in this case if and only if $p_i > \max_t(r_i exp_i^t)$, $\forall i \in N$. The rest of the proof is similar to that of Theorem 2. ∎

As already described in the proof, the assumption on p_i in the above theorem ensures that individually uniform consumption profiles become feasible with respect to the constraint in Eq. 7. Similar to Theorem 3, we can also prove[2] the following theorem.

Theorem 7 *For $K \geq 2$, and for any r_i with, $0 \leq r_i < 1$, $\forall i \in N$, PFC game can be in Nash equilibrium even when all the players are not consuming uniformly over time.*

Furthermore, we relate the efficient profiles to that of aggregate consumption profiles in the following theorem.

Theorem 8 *Consider the set of all feasible consumption profiles $(y_1, y_2, \ldots, y_n)$ such that $y_i^t > r_i exp_i^t$ when $r_i > 0$ and $exp_i^t > 0$. Also let $m > \max_t(\sum_{i=1}^{n} r_i exp_i^t)$, $\forall i \in N$. Then, a consumption profile of the PFC game is efficient if and only if the total consumption by all the players is uniform.*

Proof: The proof of this theorem is similar to that of Theorem 4, except that the total consumption by all the players can be uniform only if $m > \max_t(\sum_{i=1}^{n} r_i exp_i^t)$. ∎

6 Conclusion and Future Work

In this paper, we studied ex-post real-time pricing for end consumers from a game-theoretic point of view. Our work shows that even in the absence of advance pricing signals in the case of ex-post schemes, it is possible to visualize consumption patterns assuming that the consumers are rational players, thus providing a viable alternative pricing scheme to utilities. For future work, we need to investigate the human factors such as human comfort and inconvenience involved with electricity consumption. Including distributed user-side generation into the model is also a promising advancement.

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[2]In the proof of Theorem 3, we need to assume that $exp_1 = (7, 8)$, $exp_2 = (3, 2)$, and also, $0 < \epsilon \leq 8(1 - r_1)$ in the third paragraph. Following the same approach, we can show that (exp_1, exp_2) is a Nash equilibrium profile in that game.

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