Modal Interpretations of Default Logic

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Abstract

In the paper we study a new and natural modal interpretation of defaults. We show that under this interpretation there are whole families of modal nonmonotonic logics that accurately represent default reasoning. One of these logics is used in a definition of possible-worlds semantics for default logic. This semantics yields a characterization of default extensions similar to the characterization of stable expansions by means of autoepistemic interpretation.

We also show that the disjunctive information can easily be handled if disjunction is represented by means of *modal disjunctive defaults* — modal formulas that we use in our interpretation.

Our results indicate that there is no single modal logic for describing default reasoning. On the contrary, there exist whole ranges of modal logics, each of which can be used in the embedding as a "host" logic.

1 Introduction

The default logic of Reiter [1980] is a nonmonotonic formalism based on the paradigm of "negation as failure to prove" and is defined by means of a certain fixed-point construction. It is a formalism in the language of propositional calculus (or, in a more general variant, in the language of first-order logic). In 1982, McDermott [1982], building on the joint work with Doyle [1980], introduced a large class of modal nonmonotonic logics. He proposed a general scheme which, also using "negation as failure to prove" and a fixed-point construction, assigns to each monotone modal logic its nonmonotonic variant. The autoepistemic logic of Moore [1985], an important modal formalism, belongs to the McDermott-Doyle's family of logics (see [Shvarts, 1990]). In recent years there have been numerous attempts to explain and exploit the nature of the relationship between the default logic and modal nonmonotonic logics. There are two main reasons behind the interest in this particular research area. Firstly, modal nonmonotonic logics often have clear, intuitive semantics (for example, list semantics [Moore, 1985], possible-world semantics [Moore, 1984; Konolige, 1988], or preference semantics [Shoham, 1987]), and the default logic lacks one. By embedding the default logic into a modal nonmonotonic logic with an elegant semantics, insights into semantic aspects of default logic can be gained. Secondly, the automated inference methods for the "host" modal *nonmonotonic logic could be* used as a uniform tool for handling default theories.

The default logic was first embedded into a variant of autoepistemic logic by Konolige [1988]. Marek and Truszczynski [1989; 1990] proposed to embed default logic into the nonmonotonic variant of the logic of *necessttation* N — the modal logic that does not contain any modal axiom schemata and uses modus ponens and necessitation as inference rules. Recently, Lin and Shoham [1990] defined yet another, this time bimodal, nonmonotonic logic, which they called the logic of *grounded knowledge (denoted GK),* and provided an interpretation *of* the default logic within logic GK.

Each of these approaches has some disadvantages. We discuss them briefly in the next section. In this paper we consider another interpretation of defaults in the modal language. It is somewhat related to the approach of Siegel [1990] to modal nonmonotonic logic[1]. We argue that our translation avoids the problems of the translations used in the earlier approaches. We show that under this new interpretation, default logic can be faithfully embedded into any of the whole range of modal nonmonotonic logics. As a consequence, possible-worlds semantics for these modal logics yield possible-worlds semantics for default logic, and their automated proof methods (whenever exist) can be applied to default theories. For example, in Section 4, we choose logic S4F, that has a particularly well-structured possible-world semantics, to obtain possible-world semantics for default logic.

An important feature of the new translation is that it easily lends itself to an extension suitable for default

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[1]I would *like to* thank Vladimir Lifschitz who informed me about the work of Siegel after seeing the preliminary version of my paper.

reasonings with indefinite information in the form of disjunction, and generalizes the formalism for such reasonings introduced recently by Gelfond [1990]. In Section 5, we introduce this extension, relate it to an earlier work of [Gelfond, 1990] and [Gelfond *et al.,* 1991], and apply our formalism to discuss the "broken hand" example of Poole [1989].

Our results bring up the following general question: how essential in explaining default reasonings are those aspects of modal logics that are specifically concerned with properties of the modality? Earlier investigations [Konolige, 1988], [Marek and Truszczynski, 1990], [Siegel, 1990], implicitly suggested that there may be some relationship between the properties of the modality and default logic by centering around the question of what is the right modal logic for the embedding to work (logics K45 [Konolige, 1988], N [Marek and Truszczynski, 1990] and T [Siegel, 1990] were considered). But there seem to be no reason for any connection between default reasonings and modal axioms to exist. Default reasonings first use a certain mechanism ("negation as failure to prove") to establish defaults applicable in a given situation and then proceed like in a classical first-order case. In the whole process there is no place where properties of the modality (like standard modal axiom schemata) might intervene.

Our results support this view by showing that there is a significant degree of freedom in the choice of the "host" modal nonmonotonic logic — any out of the whole family of (drastically different) modal nonmonotonic logics will do. An analysis of the translation proposed by Konolige reveals the same "insensitivity" to the choice of modal logic (see [Truszczynski, 1991a]), and Lin and Shoham [1990] make a similar observation in the case of their approach.

2 Previous approaches to the problem of representing default logic as a modal system

Theories in default logic are pairs *(D,W),* where *W* is a collection of formulas in some propositional language L, and *D* is a collection of nonstandard inference rules, called *defaults,* that are of the form

$$\frac{\alpha, M\beta_1, \ldots, M\beta_n}{\gamma}. \qquad (1)$$

To apply default (1) and conclude γ, as in the case of standard inference rules, one first has to establish all its premises. The premises of defaults are of two types. Premise α is treated in a standard way — α has to be proved before the default can be applied. Premises $M\beta_i$ are treated differently. This is the place when we depart from standard monotone systems. We interpret $M\beta_i$ as "β_i is possible" (which is emphasized by the use of the modal operator M) and define this as consistency of β_i with some theory S (the potential collection of consequences of a default theory). Thus, $M\beta_i$ is established if $S \nvdash \neg\beta_i$.

Such treatment of the premises of defaults leads to a nonstandard consequence operator, $\Gamma_{D,W}(S)$ [Reiter, 1980]. For a theory $W \subseteq \mathcal{L}$, $\Gamma_{D,W}(S)$ consists of all facts that can be proven from W by means of propositional calculus and defaults in D, applied as explained above (with S used in consistency checking). If theory S coincides with the set $\Gamma_{D,W}(S)$ of such consequences of W, that is, if

$$S = \Gamma_{D,W}(S), \qquad (2)$$

then S is called an *extension* of a default theory (D,W) and is regarded as a candidate (many theories S may satisfy (2)) for the set of nonmonotonic consequences of (D,W).

A similar approach to nonmonotonicity but within the language of modal logic $\mathcal{L}_L$ — an extension of $\mathcal{L}$ by a modal operator L interpreted as "is known" — was proposed by McDermott and Doyle [1980] and McDermott [1982]. The operator M, mentioned earlier, is simply an abbreviation for $\neg L \neg$. The idea was that modal formulas would play the role of defaults, and the standard consequence operator $Cn_{\mathcal{S}}$ for a selected modal logic $\mathcal{S}$ would replace the nonstandard consequence operator $\Gamma_{D,\cdot}(S)$. To achieve nonmonotonicity, the theory I encoding the knowledge of a reasoning agent was extended by the formulas expressing "negation as failure to prove" that is, by the formulas $\neg L\varphi$ for $\varphi \notin T$. Theory T (playing the same role as S for default logic) is potentially the collection of nonmonotonic consequences of I. If T agrees exactly with consequences in logic $\mathcal{S}$ of $I \cup \{\neg L\varphi : \varphi \notin T\}$ that is, if

$$T = Cn_{\mathcal{S}}(I \cup \{\neg L\varphi : \varphi \notin T\}), \qquad (3)$$

then T is called an $\mathcal{S}$-expansion of I, and is regarded as a candidate (equation (3) may have multiple solutions) for the set of nonmonotonic $\mathcal{S}$-consequences of I.

A variant of nonmonotonic modal logic was recently introduced by Kaminski [1991] and further studied in [Truszczyński, 1991b] (see also [Konolige, 1988]). Instead of extending I by all formulas given by "negation as failure to prove", I is extended only by the results of application of this rule to modal-free formulas of T. Formally, theory T is a ground $\mathcal{S}$-expansion of I if

$$T = Cn_{\mathcal{S}}(I \cup \{\neg L\varphi : \varphi \in \mathcal{L} \setminus T\}). \qquad (4)$$

This approach, by restricting the applicability of the "negation as failure to prove" rule to modal-free formulas, results in nonmonotonic formalisms more robust to updates in the initial theory I. In addition, since modal nonmonotonic logics based on (4) apply "negation as failure" only to modal-free formulas — all that is required in default logic — they seem especially suitable for studying default reasonings.

Before we continue, let us recall one more important notion. A theory $T \subseteq \mathcal{L}_L$ is ***stable*** if it is closed under propositional consequence and necessitation, and if for every $\varphi \notin T$, $\neg L\varphi \in T$. It is well-known that $\mathcal{S}$-expansions and ground $\mathcal{S}$-expansions are stable (for the latter to hold, we require that $\mathcal{S}$ be normal). Stable sets are uniquely determined by their objective part [Moore, 1985; Konolige, 1988]. That is, for each $S \subseteq \mathcal{L}$ there is exactly one stable set T such that $T \cap \mathcal{L} = Cn(S)$, where Cn stands for the consequence operator of propositional calculus. We will denote this unique stable set by $St(S)$.

There have been several attempts to embed default logic into a modal nonmonotonic logic.

(1) Konolige [Konolige, 1988] proposed to interpret a default d given by (1) by the formula

$$emb_K(d) = L\alpha \wedge \neg L \neg \beta_1 \wedge \ldots \wedge \neg L \neg \beta_n \Rightarrow \gamma \qquad (5)$$

This translation can be extended to a default theory (D, W) by setting

$$emb_K(D, W) = W \cup \{emb_K(d): d \in D\}.$$

Under this interpretation no modal nonmonotone logic is known to faithfully capture default reasoning. An additional modification of $emb_K(D, W)$ is needed. This modification, involving a stable theory T, yields a subtheory of $emb_K(D, W)$ *referred to as the reduct of* $emb_K(D, W)$ *with respect to* T. Konolige proved that a consistent and closed under provability theory S is an extension of a default theory (D, W) if and only if $St(S)$ is a ground **K45**-expansion of the reduct of $emb_K(D, W)$ with respect to $St(S)$. For the same interpretation, Marek and Truszczynski [1989] showed a similar result: a consistent and closed under provability theory S is an extension of a default theory (D, W) if and only if $St(S)$ is an **N**-expansion of the reduct of $emb_K(D, W)$ with respect to $St(S)$.

The problem with the approach of Konolige is that the concept of reduct is representation dependent — logically equivalent theories may have reducts that are not logically equivalent. Consequently, in modal systems using the concept of reduct, syntactically different but logically equivalent theories may have different consequences. Moreover, as we will argue later, this translation cannot be used to handle reasonings with disjunction.

(2) In another paper Marek and Truszczynski [1990] introduced a different translation of the default (1):

$$L\alpha \wedge \neg LL \neg \beta_1 \wedge \ldots \wedge \neg LL \neg \beta_n \Rightarrow \gamma. \qquad (6)$$

Under this interpretation consistent extensions of (D, W) correspond exactly to **N**-expansions of the translation of (D, W). This was the first result which embedded default logic into a modal nonmonotonic logic in McDermott-Doyle's family. The main disadvantage of this approach is that the interpretation used does not have a natural, intuitive justification.

(3) Lin and Shoham [Lin and Shoham, 1990] do not use the scheme of McDermott and Doyle. They introduce a modal nonmonotonic logic with two modalities. They define semantics of this logic by modifying the concept of preference semantics and faithfully embed default logic into their logic. The solution is elegant but requires two modalities, which introduces an extra degree of complexity.

3 New modal representation of default logic

The translation we propose is based on the most natural interpretation of a default $d = \frac{\alpha, M\beta_1, \ldots, M\beta_n}{\gamma}$: if α, $M\beta_1, \ldots, M\beta_n$ are known to an agent then the agent knows γ. Such interpretation treats defaults in the same way as standard inference rules are treated — all premises must be known — the only difference is that premises are of two types. This new interpretation can be encoded faithfully in the language of modal logic by the formula

$$emb(d) = L\alpha \wedge LM\beta_1 \wedge \ldots \wedge LM\beta_n \Rightarrow L\gamma. \qquad (7)$$

We extend now this definition to the case of default theories. Let (D, W) be a default theory. A modal theory representing (D, W) is defined by

$$emb(D, W) = \{L\varphi: \varphi \in W\} \cup \{emb(d): d \in D\}. \qquad (8)$$

In the rest of the paper, we assume familiarity with basic concepts of (monotone) modal logic. For all undefined concepts the reader is referred to Huges and Cresswell [1984]. All modal logics considered in this paper contain propositional calculus and use *necessitation* $(\varphi / L\varphi)$ as an additional inference rule. Besides standard modal logic systems like **K**, **T**, **S4**, **S5** and **K45**, we will consider the following two logics:

1. $\mathbf{T}^-$ — the logic containing only axiom schema T,
2. **S4F** — the logic determined by the class of Kripke frames with the accessibility relation of the form $(M_1 \times M_1) \cup (M_1 \times M_2) \cup (M_2 \times M_2)$, where M_1 and M_2 are disjoint (M_1 may be empty) and contain all the worlds of the model. Logic **S4F** contains logic **S4** and is included in logic **S5**. It can be axiomatized by the axiom schemata of **S4** and the followng additional axiom F: $\varphi \wedge ML\psi \Rightarrow L(M\varphi \vee \psi)$ (see [Segerberg, 1971]).

Our first theorem contains the main result of the paper.

Theorem 3.1 *Let (D, W) be a default theory and let $S \subseteq \mathcal{L}$ be consistent and closed under propositional consequence. The following conditions are equivalent:*
1. *S is an extension of (D, W);*
2. $St(S) = Cn_{\mathbf{T}^-}(emb(D, W) \cup \{\neg L\varphi: \varphi \notin St(S)\})$;
3. $St(S) = Cn_{\mathcal{S}}(emb(D, W) \cup \{\neg L\varphi: \varphi \notin St(S)\})$, *for any modal logic $\mathcal{S}$ such that* $\mathbf{T}^- \subseteq \mathcal{S} \subseteq \mathbf{S4F}$;
4. $St(S) = Cn_{\mathbf{S4F}}(emb(D, W) \cup \{\neg L\varphi: \varphi \notin St(S)\})$.

Thus, under the translation (7) — (8) for any of a big range of modal logics ($\mathbf{T}^-$ — **S4F**) its nonmonotonic variant (exactly as introduced in [McDermott and Doyle, 1980; McDermott, 1982]) can be used to express default reasonings, without the need for representation-dependent concepts like reduct.

Let us now consider representing default reasoning in ground modal nonmonotonic logics. We will again use our new translation of defaults. A similar result to the previous one holds: there is a whole range of logics that can be used for an embedding.

Theorem 3.2 *Let $\mathcal{S}$ be any logic such that* $\mathbf{T} \subseteq \mathcal{S} \subseteq$ **S4F**. *A consistent and closed under propositional provability theory S is an extension of a default theory (D, W) if and only if $St(S)$ is a ground $\mathcal{S}$-expansion of $emb(D, W)$.*

This theorem will be used in the next section in a definition of a possible-words semantics for default logic, where logic **S4F**, which has a relatively simple Kripke-model characterization is selected as a modal counterpart of default logic.

Even larger range of equivalent logics, in the sense of faithful representation of default reasoning, can be found if the concept of ground expansion is combined with the concept of the reduct of $emb(D, W)$ In such case any of the logics from the range **T** — **S5** can be used. Similarly, for the original approach of Konolige [1988] (translation emb_K combined with the concept of reduct), we can prove that any of the modal logics in the range **K** — **S5**, and not only logic **K45** used in [Konolige, 1988], allows us to faithfully encode default logic (see [Truszczyński, 1991a] for details).

4 Possible-worlds semantics for default logic

In the previous section we have presented two results embedding default logic into nonmonotonic modal logics of McDermott and their ground variants. In each case we have a significant degree of freedom in choosing the underlying monotone logic. In this section, we make use of this freedom. Our approach is based on Theorem 3.2, and for logic $\mathcal{S}$ we choose **S4F**.

A *(default) model* is a triple $\mathcal{M} = \langle M_1, M_2, V\rangle$, where M_1 and M_2 are sets such that $M_1 \cap M_2 = \emptyset$ and $M_2 \neq \emptyset$, and V is a function assigning to each $m \in M$ a propositional valuation $V(m)$.

We say that a formula φ is true (holds) in a world m of $M_1 \cup M_2$ if φ is true in the valuation $V(m)$ (in symbols: $(\mathcal{M}, m) \models \varphi$). We say that a pair $(\mathcal{M}, m)$, where $m \in M_1$, satisfies default (1) (in symbols, $(\mathcal{M}, m) \models d$) if whenever α is true in all worlds in $M_1 \cup M_2$ and each β_i holds in at least one world of M_2, then γ holds in all worlds in $M_1 \cup M_2$. We say that a pair $(\mathcal{M}, m)$, where $m \in M_2$, satisfies default (1) ($(\mathcal{M}, m) \models d$) if whenever α is true in all worlds in M_2 and each β_i holds in at least one of the worlds in M_2, then γ holds in all worlds in M_2. In a standard way, we define now satisfiability of a default by a model. Namely, $\mathcal{M} \models d$, if $(\mathcal{M}, m) \models d$ for each $m \in M_1 \cup M_2$. In a similar fashion, we define satisfiability of a formula by a model: for $\varphi \in \mathcal{L}$, $\mathcal{M} \models \varphi$ if φ is true in all worlds from $M_1 \cup M_2$.

Note that for each model $\mathcal{M}$ there is an equivalent model $\mathcal{M}' = \langle M_1', M_2', V'\rangle$ (that is, for each default d, $\mathcal{M} \models d$ if and only if $\mathcal{M}' \models d$) such that valuations assigned to worlds in M_i', $i = 1, 2$, are distinct. Consequently, if $\mathcal{L}$ has only finitely many propositional variables, then each model has an equivalent finite model.

In commonsense reasoning we often restrict the class of models by imposing additional conditions. For example, Moore's autoepistemic interpretations of the language $\mathcal{L}_L$ [Moore, 1985] are defined with respect to a theory $T \subseteq \mathcal{L}_L$, called the *modal index* of an interpretation. A valuation v of $\mathcal{L}_L$ is *autoepistemic with respect to* T if $v(L\varphi) = 1$ if and only if $\varphi \in T$. The class of autoepistemic interpretation has been used by Moore to define the concept of *stable expansion* and the nonmonotonic formalism called *autoepistemic logic*.

In this paper, we proceed similarly. Let $S \subseteq \mathcal{L}$. A model $\mathcal{M}$ is called *strongly consistent with* S

$$\{\varphi \in \mathcal{L}: v(\varphi) = 1, \text{ for every } v \in V_2\} \subseteq S.$$

Let (D, W) be a default theory. We say that (D, W) *S-entails* a, where a stands for a default or for a formula from $\mathcal{L}$, if a is satisfied in all models strongly consistent with S in which (D, W) is satisfied. We denote the relation of S-entailment by $\models_S$. We have the following theorem providing a characterization of extensions analogous to Moore's definition of stable expansions by means of autoepistemic interpretations.

Theorem 4.1 *Let (D, W) be a default theory. Let $S \subseteq \mathcal{L}$. Theory S is an extension of (D, W) if and only if*

$$S = \{\varphi \in \mathcal{L}: (D, W) \models_S \varphi\},$$

that is, if S is exactly the set of all formulas true in all S-models of (D, W).

5 Effective disjunction

One of the problems of nonmonotonic formalisms often is that the semantics of the disjunction operator does not accurately capture the intuitive understanding of disjunction [Poole, 1989; Gelfond, 1990). We will show that our embedding of defaults can be used to handle disjunction. In commonsense reasoning we often use a "constructive" or "effective" disjunction — knowing aVb, each belief set an agent will construct will contain *a* or *b* — instead of the classical, nonconstructive interpretation in which we may know aVb without knowing which of the two disjuncts is true. The distinction between the two can easily be achieved in the modal language. The "constructive" disjunction can be expressed as *La* V *Lb* and the "noneffective" one by *L(aV b).*

Let us consider the following example due to Poole [Poole, 1989], Suppose that normally people's left (resp. right) arms are usable and people with broken left (resp. right) arms are exceptions. Suppose also that we remember seeing a friend with a broken arm, but we cannot remember which. Intuitively, we should not conclude that both his arms are usable. A straightforward default encoding of this situation by a default theory (D, *W)* is as follows:

$$D = \{\frac{M\neg ab_l}{u_l}, \frac{M\neg ab_r}{u_r}\},$$

and

$$W = \{b_l \Rightarrow ab_l, b_r \Rightarrow ab_r, b_l \vee b_r\}.$$

Clearly, this default theory has exactly one extension and it contains the formula u; A u_r, contrary to the intuition.

The reason for the inadequacy of the default logic to handle such situations is that default logic does not have a mechanism to deal with effective disjunction. Recently an extension of default logic was proposed in [Gelfond *et al.,* 1991] which commonsense disjunction is expressed by means of a new connective. In this extension of default logic the abovementioned paradox disappears.

Let us now consider the encoding of the above situation using *modal disjunctive defaults* of the form:

$$L\alpha \wedge LM\beta_1 \wedge \ldots \wedge LM\beta_n \Rightarrow L\gamma_1 \vee \ldots \vee L\gamma_k. \quad (9)$$

Using such formulas, the Poole's example can be encoded as follows (recall that $M = \neg L \neg$):

$$I = \{L\neg Lab_l \Rightarrow Lu_l, L\neg Lab_r \Rightarrow Lu_r,$$
$$L(b_l \Rightarrow ab_l), L(b_r \Rightarrow ab_r), Lb_l \vee Lb_r\}.$$

Consider now logic **S4**. In [Shvarts, 1990; Marek *et al.*, 1990] **S4**-expansions are characterized and algorithms to compute them described. Using these results, one can find that the above theory has two **S4**-expansions: one generated by $\{ab_l, b_l, u_r\}$ and the other generated by $\{ab_r, b_r, u_l\}$, which coincides with the intuition. The same result can be obtained for any logic from the range $\mathbf{T}^- - \mathbf{S4F}$.

In fact, more can be shown. Namely, each nonmonotonic logic $\mathcal{S}$, for logics $\mathcal{S}$ such that $\mathbf{T}^- \subseteq \mathcal{S} \subseteq \mathbf{S4}$, and each ground nonmonotonic logic $\mathcal{S}$, for logics $\mathcal{S}$ such that $\mathbf{T} \subseteq \mathcal{S} \subseteq \mathbf{S4F}$, can be used as a nonmonotonic system capable of distinguishing between "effective" and "noneffective" disjunction. These issues will be discussed in detail in the full version of the paper.

Let us briefly mention here that formulas used to represent default proposed by Konolige [1988] and Marek and Truszczynski [1990] cannot be extended to handle disjunction, because they use modal-free formulas to represent consequents of defaults and, thus, cannot distinguish between the "effective" and "noneffective" interpretations of disjunction.

6 Conclusions

In the paper we presented a new and natural interpretation of defaults as modal formulas. We have shown that under this interpretation there are whole families of modal nonmonotonic logics that accurately represent default reasoning. We proposed a semantics for default logic based on the embedding we found.

We also have shown that the disjunctive information can easily be handled within our modal system, disjunctive defaults.

A very important conclusion of this research is that there is no single, distinguished modal logic for describing default reasoning. On the contrary, there exist whole ranges of modal logics, each of which can be used in the embedding as a "host" logic. This shows that, in agreement with the intuition, in order to capture default reasoning the most important step is to translate into a nonmonotonic modal system the principle of "negation as failure *to* prove". *Once this* is made, then the choice of particular modal axiom schemata is of secondary importance, in fact, there is a large degree of freedom in which of them to choose.

7 Proof of Theorem 3.1

Let us define in a precise way the operator $\Gamma_{D,W}(S)$. To this end, let us introduce a formal system $\mathbf{PC} + (D,S)$ as the formal system containing propositional calculus and an additional set D_S of inference rules, where $D_S = \{\frac{\alpha}{\gamma}: \frac{\alpha, M\beta_1, \ldots, M\beta_n}{\gamma} \in D \text{ and } S \not\vdash \neg\beta_i, 1 \leq i \leq n\}$. By $\Gamma_{D,W}(S)$ we denote the set of all formulas that have a proof from W in the system $\mathbf{PC} + (D,S)$. It is well-known that $\Gamma_{D,W}(S)$ can alternatively be defined as the minimal set closed under provability in $\mathbf{PC}+(D,S)$ that is, the minimal set closed under propositional provability and the rules in D_S (this is the original definition of $\Gamma_{D,W}(S)$ given in [Reiter, 1980]).

Throughout the proof we use the following abbreviations: $T = St(S)$ and $I = emb(D,W)$. The *reduct of* $emb(D,W)$ *with respect to* $St(S)$ (in symbols: I_R) is the theory containing $\{L\varphi: \varphi \in W\}$ and formulas $L\alpha \Rightarrow L\gamma$, such that for some $\beta_1, \ldots \beta_n$, $\frac{\alpha, M\beta_1, \ldots, M\beta_n}{\gamma} \in D$ and for all i, $\neg\beta_i \notin S$. $I_R = R(emb(D,W), St(S))$. We will also use the following facts listed below:

P1: For a normal modal logic $\mathcal{S}$, and for $S \subseteq \mathcal{L}$, if $S \subseteq Cn_{\mathcal{S}}(I \cup \{\neg L\varphi: \varphi \in \mathcal{L} \setminus St(S)\})$, then $St(S) \subseteq Cn_{\mathcal{S}}(I \cup \{\neg L\varphi: \varphi \in \mathcal{L} \setminus St(S)\})$ ([Truszczyński, 1991b]). For any modal logic $\mathcal{S}$ containing necessitation, and for $S \subseteq \mathcal{L}$, if $S \subseteq Cn_{\mathcal{S}}(I \cup \{\neg L\varphi: \varphi \notin St(S)\})$, then $St(S) \subseteq Cn_{\mathcal{S}}(I \cup \{\neg L\varphi: \varphi \notin St(S)\})$ ([Marek and Truszczyński, 1990]);

P2: If $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathbf{S5}$ are two modal logics with necessitation, then each $\mathcal{S}$-expansion of I is a $\mathcal{T}$-expansion of I ([McDermott, 1982]);

P3: Logic **S5** is complete with respect to the class of universal Kripke models that is, models with a universal accessibility relation ([Hughes and Cresswell, 1984]);

P4: Each consistent stable set is of the form $\{\varphi: \mathcal{M} \models \varphi\}$, for some universal Kripke model $\mathcal{M}$ [Moore, 1984];

P5: Stable sets are closed under provability in **S5** ([McDermott, 1982]);

P6: A theory $S \subseteq \mathcal{L}$ closed under propositional consequence, is closed under rules in D_S if and only if $emb(D,W) \subseteq St(S)$ if and only if $I_R \subseteq St(S)$ (straightforward to prove).

$1 \Rightarrow 2$. Assume that $S = \Gamma_{D,W}(S)$. Then, if $\varphi \in S$, φ has a proof $\varphi_1, \ldots, \varphi_n (= \varphi)$ from W in the system $\mathbf{PC} + D_S$. We will show, by induction on the length of the proof, that $\varphi \in Cn_{\mathbf{T}^-}(emb(D,W) \cup \{\neg L\varphi: \varphi \notin T\})$. Assume that the claim holds for all formulas in S with proofs of length $< n$. If $\varphi_n (= \varphi)$ is a tautology or is a member of W, then $\varphi \in Cn_{\mathbf{T}^-}(emb(D,W) \cup \{\neg L\varphi: \varphi \notin T\})$. If φ_n is obtained from φ_i, φ_j, where $i, j < n$, by means of modus ponenes, then the induction hypothesis applies and $\varphi_i, \varphi_j \in Cn_{\mathbf{T}^-}(emb(D,W) \cup \{\neg L\varphi: \varphi \notin T\})$. Consequently, $\varphi \in Cn_{\mathbf{T}^-}(emb(D,W) \cup \{\neg L\varphi: \varphi \notin T\})$.

The last possibility is that φ_n is obtained from φ_i, $i < n$, by means of a rule $\frac{\varphi_i}{\varphi_n} \in D_S$. Then, there is a default $d = \frac{\varphi_i, M\beta_1, \ldots, M\beta_m}{\varphi_n} \in D$ such that all $\neg\beta_i \notin S$. Consequently, all $L\neg L\neg\beta_i \in Cn_{\mathbf{T}^-}(emb(D,W) \cup \{\neg L\varphi: \varphi \notin T\})$. By the induction hypothesis, $\varphi_i \in Cn_{\mathbf{T}^-}(emb(D,W) \cup \{\neg L\varphi: \varphi \notin T\})$. Thus, $L\varphi_i \in Cn_{\mathbf{T}^-}(emb(D,W) \cup \{\neg L\varphi: \varphi \notin T\})$. Moreover, $emb(d) = L\varphi_i \wedge L\neg L\neg\beta_1 \wedge \ldots \wedge L\neg L\neg\beta_m \Rightarrow L\varphi_n$ is a member of $Cn_{\mathbf{T}^-}(emb(D,W) \cup \{\neg L\varphi: \varphi \notin T\})$ (recall that $M = \neg L \neg$). Thus, $L\varphi_n \in Cn_{\mathbf{T}^-}(emb(D,W) \cup \{\neg L\varphi: \varphi \notin T\})$. Applying axiom T, we get that $\varphi_n (= \varphi) \in Cn_{\mathbf{T}^-}(emb(D,W) \cup \{\neg L\varphi: \varphi \notin T\})$.

By P1, it follows that $St(S) \subseteq Cn_{\mathbf{T}^-}(emb(D,W) \cup \{\neg L\varphi: \varphi \notin T\})$. To prove the converse inclusion, observe that $St(S)$ is closed under provability in $\mathbf{T}^-$ (by P5), $St(S) \supseteq emb(D,W)$ (since S is an extension of (D,W), S is closed under rules from D_S and P6 applies). Finally,

since $T = St(S)$ and $St(S)$ is stable, $\{\neg L\varphi: \varphi \notin T\} \subseteq St(S)$.
$2 \Rightarrow 3$ and $3 \Rightarrow 4$. These implications follow by P2.
$4 \Rightarrow 1$. The definition of the reduct yields

$$Cn_{\mathbf{S4F}}(I \cup \{\neg L\varphi: \varphi \notin T\}) = Cn_{\mathbf{S4F}}(I_R \cup \{\neg L\varphi: \varphi \notin T\}).$$

Thus, $St(S) = Cn_{\mathbf{S4F}}(I_R \cup \{\neg L\varphi: \varphi \notin St(T)\})$.
Next, we will show that

$$St(S) = Cn_{\mathbf{S5}}(I_R \cup \{\neg L\varphi: \varphi \in \mathcal{L} \setminus T\}).$$

Since T is stable, T is closed under provability in **S5** (by P5). In addition, $\{\neg L\varphi: \varphi \in \mathcal{L} \setminus T\} \subseteq T$. Thus, $Cn_{\mathbf{S5}}(I_R \cup \{\neg L\varphi: \varphi \in \mathcal{L} \setminus T\}) \subseteq T$. To prove the converse inclusion it suffices, by P1, to show that

$$S \subseteq Cn_{\mathbf{S5}}(I_R \cup \{\neg L\varphi: \varphi \in \mathcal{L} \setminus T\}).$$

Consider $\varphi \in S$ and any universal Kripke model $\mathcal{M} = \langle M, R, V \rangle$ such that $\mathcal{M} \models I_R \cup \{\neg L\varphi: \varphi \in \mathcal{L} \setminus T\}$. Let $\mathcal{M}' = \langle M', R', V' \rangle$, $M \cap M' = \emptyset$, be a universal Kripke model such that $T = \{\psi: \mathcal{M}' \models \psi\}$ (apply P4). Define $\mathcal{N} = \langle M'', R'', V'' \rangle$, by letting $M'' = M \cup M'$, $R'' = R \cup (M \times M') \cup R'$ and $V'' = V \cup V'$. It follows from the definition of $\mathcal{N}$ that $\mathcal{N}, m' \models \{\neg L\varphi: \varphi \notin T\}$, for every $m' \in M'$. Thus, also for every $m \in M$, $\mathcal{N}, m \models \{\neg L\varphi: \varphi \notin T\}$. Next, it is easy to see that if $\psi \in \mathcal{L}$ then $\mathcal{N}, m \models \psi$ if and only if $\mathcal{M}, m \models \psi$, and $\mathcal{N}, m \models L\psi$ if and only if $\mathcal{M}, m \models L\psi$. Thus, for every $m \in M$, $\mathcal{N}, m \models I_R$. Since $I_R \subseteq T$, $\mathcal{M}', m \models I_R$, for every $m \in M'$. Consequently, $\mathcal{N}, m \models I_R$, for every $m \in M'$. Summarizing, $\mathcal{N} \models I_R \cup \{\neg L\varphi: \varphi \notin T\}$. By the definitions of the model $\mathcal{N}$ and of the logic **S4F**, it follows that $\mathcal{N} \models \varphi$. Thus, $\mathcal{M} \models \varphi$ and, by P3, $\varphi \in Cn_{\mathbf{S5}}(I_R \cup \{\neg L\varphi: \varphi \in \mathcal{L} \setminus T\})$.

Now, $St(S)$ is a ground **S5**-expansion of I_R and, by a result of Kaminski [Kaminski, 1991], $St(S)$ is a minimal stable theory containing I_R. Consequently, S is a minimal set closed under rules from D_S. Hence, S is an extension of (D, W). □

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