Constructive Tightly Grounded Autoepistemic Reasoning*

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Abstract

The key concept of autoepistemic logic introduced by Moore is a stable expansion of a set of premises, i.e., a set of beliefs adopted by an agent with perfect introspection capabilities on the basis of the premises. Moore's formalization of a stable expansion, however, is non-constructive and produces sets of beliefs which are quite weakly grounded in the premises. A new more constructive definition of the sets of beliefs of the agent is proposed. It is based on classical logic and enumerations of formulae. Considering only a certain subclass of enumerations, *L-hierarchic* enumerations, an attractive class of expansions is captured to characterize the sets of beliefs of a fully introspective agent. These L-hierarchic expansions are stable set minimal, very tightly grounded in the premises and independent of the syntactic representation of premises. Furthermore, Reiter's default logic is shown to be a special case of autoepistemic logic based on L-hierarchic expansions.

1 Introduction

Nonmonotonic reasoning is one of the most important and active areas of research in knowledge representation and reasoning. Autoepistemic logic introduced by Moore [1985] appears to be one of the best available tools for studying nonmonotonic reasoning as recent results [Elkan, 1990; Konolige, 1989; Marek and Trusiczyriski, 1989] on the relationship between autoepistemic logic and other forms of nonmonotonic reasoning suggest that it offers a unifying approach to a large part of nonmonotonic reasoning.

Autoepistemic logic is a modal logic with an operator *L* which is read 'is believed'. It was originally introduced as a reconstruction of McDennott and Doyle's [l980] nonmonotonic logic to avoid some peculiarities of this logic. Autoepistemic logic models the beliefs of an ideally rational agent who is capable of perfect introspection. The interesting question is to determine the set of beliefs of the agent given a set of formulae as the initial assumptions of the agent. The agent's rationality is interpreted as requiring that the beliefs of the agent have to be logical consequences of the initial assumptions and the beliefs of the agent. The agent is capable of using both positive introspection (if x is a belief, BO is *Lx)* and negative introspection (if x is not a belief, then *~Lx* is). The agent is also ideal: if a formula is a logical consequence of the beliefs of the agent, then it belongs to the set of beliefs of the agent.

The informal description of the set of beliefs of the agent given above is circular: the set of beliefs is defined using the set of beliefs. Moore [1985] offers a formal definition where the sets of beliefs, the *stable expansions* of the initial assumptions, are defined as the fixed points of an operator given with the aid of the logical consequence relation used by the agent. Moore's formalization is elegant but in connection with some sets of premises it produces sets of beliefs which are quite weakly grounded in the premises. Another problem is that Moore's formalization is non-constructive. It yields no direct method of enumerating or constructing a set of beliefs of the agent given a set of premises. It merely states a condition to be satisfied by any proper set of beliefs based on the premises.

We propose a new more constructive definition of the set of beliefs of the agent. It is based on classical logic and enumerations of formulae. It produces sets of beliefs which are a proper subclass of the stable expansions defined by Moore. In fact, the class coincides with iterative expansions defined by Marek and Truszczynski [1989]. Iterative expansions have many desirable properties. In particular, autoepistemic logic based on iterative expansions can be regarded as a generalization of the other leading nonmonotonic logic, default logic [Reiter, 1980]. The basic problem of iterative expansions is that they are not necessarily stable set minimal which suggests that autoepistemic logic based on iterative expansions is not a proper generalization of default logic. This problem is analysed and shown be the result of the possibility of introspecting a formula before introspecting its subformulae when constructing an iterative expansion. When using the new definition where the enumeration specifies the order of introspection tighter groundedness can be ensured by restricting the order of the formulae in the

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enumeration, i,e., the order of introspection.

In this paper we propose a new class of expansions based on a special subclass of enumerations, *L-kierarchic* enumerations, as the sets of beliefs adopted by an ideal agent with perfect introspection capabilities. L-hierarchic expansions turn out to be more tightly grounded than iterative expansions because they are always stable set minimal. However, autoepistemic logic based on L-hierarchic expansions still remains a generalization of default logic.

The outline of the paper is as follows. First we introduce antoepistemic logic. Then we present a short survey of the groundedness notions proposed previously. After that we introduce the enumeration based expansions and show that Reiter's default logic is a special case of antoepistemic logic based on L-hierarchic expansions.

2 Autoepistemic logic

We can view autoepistemic logic being induced by some underlying logic CL whose language is $\mathcal{L}$. In our case CL is the propositional logic. We build an autoepistemic logic CL_{ae} on top of CL. We extend $\mathcal{L}$ by adding a monadic operator L and obtain an autoepistemic language $\mathcal{L}_{ae}$ which is the language of CL_{ae}. $\mathcal{L}_{ae}$ is defined recursively as $\mathcal{L}$ but containing an extra formation rule: if $\phi \in \mathcal{L}_{ae}$, then $L\phi \in \mathcal{L}_{ae}$.

Autoepistemic logic models an ideal agent's reasoning about his own beliefs. The agent reasons according to a consequence relation based on the underlying logic and can reflect on his own beliefs. The consequence relation of the underlying logic CL is extended to the richer language $\mathcal{L}_{ae}$ simply by treating $L\phi$ formulae as atomic. Thus the consequence relation $\models$ used by the agent is defined as follows. A formula ϕ is a consequence of a set of formulae Σ ($\Sigma \models \phi$) if ϕ is true in every interpretation in which every formula in Σ is true. The interpretations treat $L\phi$ formulae as atomic formulae.

The key objects in autoepistemic logic are the sets of total beliefs of the agent given a set of premises as the agent's initial assumptions. These sets are called *stable expansions* of the premises and they are the fixed points of an operator defined with the aid of the underlying consequence relation in the following way.

Definition 2.1 *[Moore, 1985] Δ is a stable expansion of Σ iff Δ satisfies the following fixed point equation.*

$$\Delta = \{\phi \in \mathcal{L}_{ae} \mid \Sigma \cup L(\Delta) \cup \neg L(\overline{\Delta}) \models \phi\} \quad (1)$$

where $L(\Delta) = \{L\phi \mid \phi \in \Delta\}$, $\neg\Delta = \{\neg\phi \mid \phi \in \Delta\}$, and $\overline{\Delta} = \mathcal{L}_{ae} - \Delta$. Thus $\neg L(\overline{\Delta}) = \{\neg L\phi \mid \phi \in \mathcal{L}_{ae} - \Delta\}$.

3 Groundedness

It can be argued that stable expansions are too weakly grounded in the premises. We start by an example clarifying the problem.

Example 1 Consider the following premise where p is an atomic formula.

$$Lp \rightarrow p \quad (2)$$

The premise has two stable expansions: one containing p and the other not containing p. The stable expansion containing p can be considered too weakly grounded because the agent's belief p is based on the fact that the agent believes p, thus obtains Lp by positive introspection and from this and the premise deduces p. This kind of a belief based on a circular argument is rather weakly grounded in the premises and a stronger form of groundedness is often required. ■

In this chapter we survey the various groundedness notions proposed in the literature and show in what respects they are not satisfactory.

Konolige [1988] presents two stronger notions of groundedness leading to *moderately grounded and strongly grounded expansions.* His basic motivation is to find a class of stable expansions which would capture extensions in default logic [Reiter, 1980] under a suitable translation of default logic to autoepistemic logic. To eliminate circularly based beliefs Konolige introduces the concept of a moderately grounded expansion. He shows that a set of formulae Δ moderately grounded in Σ is in fact a stable expansion of Σ which is minimal in the following sense: there is no stable set[1] S which contains Σ such that $S \cap \mathcal{L} \subset \Delta \cap \mathcal{L}$, i.e. formulae of S without the L operator are a proper subset of formulae of Δ without the L operator. Such expansions are said to be *stable set minimal* for Σ.

Moderately grounded expansions (or stable set minimal expansions) do not quite capture extensions in default logic and, moreover, are still rather weakly grounded as can be seen from the following example also discussed by Konolige [1988].

Example 2 Consider the following set of premises Σ where p, q are atomic.

$$\{Lp \rightarrow p, \neg Lp \rightarrow q\} \quad (3)$$

Σ has two stable expansions: one containing p but not q and one containing q but not p. Both are stable set minimal but the first one contains the belief p which is based on the belief p, i.e., the same circular argument as in the previous example. ■

To capture extensions in default logic Konolige proposes the notion of an expansion strongly grounded in the premises. The notion is defined only for autoepistemic formulae in a normal form:

$$\neg L\alpha \vee L\beta_1 \vee \ldots \vee L\beta_n \vee \gamma \quad (4)$$

where α, β_i, and γ are all in $\mathcal{L}$. Strongly grounded expansions are also moderately grounded. However, Marek and Truszczynski [1989] have discovered that strongly grounded expansions do not capture extensions in default logic. Marek and Truszczynski [1989] report that Konolige has introduced a corrected version of the definition of strongly grounded expansions. Neither definition of strong groundedness is satisfactory because different expansions could be grounded in different subsets of the premises and strongly grounded expansions depend on the syntactic representation of premises. This

[1] A set S is stable if it is closed under propositional consequence and satisfies the conditions (1) if $\phi \in S$ then $L\phi \in S$ and (2) if $\phi \in \overline{S}$ then $\neg L\phi \in S$.

means that classically equivalent premises do not have the same strongly grounded expansions. For example, $(Lp \rightarrow p) \wedge (\neg Lp \rightarrow p)$ is equivalent to p in propositional logic but the former has none and the latter has one strongly grounded expansion (according to the corrected definition of strong groundedness).

Marek and Truszczynski [1989] present an interesting notion of groundedness in their study of the relationship of default logic and autoepistemic logic. It is based on a monotone operator A defined as follows.

$$A(\Sigma) = Cn(\Sigma \cup L(\Sigma)) \tag{5}$$

where $Cn(\Sigma) = \{\phi \mid \Sigma \models \phi\}$. The operator A is equipped with a context parameter $\Delta \subseteq \mathcal{L}_{ae}$ and given a set of premise $\Sigma \subseteq \mathcal{L}_{ae}$ the operator A^{Δ} is iterated in the following way.

$$A_0^{\Delta}(\Sigma) = Cn(\Sigma \cup \neg L(\overline{\Delta})) \tag{6}$$

$$A_{n+1}^{\Delta}(\Sigma) = A(A_n^{\Delta}(\Sigma)) \tag{7}$$

$$A^{\Delta}(\Sigma) = \bigcup_{n=0}^{\infty} A_n^{\Delta}(\Sigma) \tag{8}$$

Marek and Truszczynski [1989] note that $A^{\Delta}(\Sigma)$ is the *set of formulae provable from* $\Sigma \cup \neg L(\overline{\Delta})$ *using propositional logic and the necessitation rule.* They define a new class of expansions based on the operator A.

Definition 3.1 *(Marek and Truszczynski [1989])* Δ *is an* iterative expansion *of* Σ *iff* $\Delta = A^{\Delta}(\Sigma)$.

They prove that iterative expansions are in fact stable expansions. The class of iterative expansions is an attractive candidate for the sets of beliefs adopted by a fully introspective agent. In particular, because in a later paper Marek and Truszczynski [1990] show that iterative expansions generalise nicely reasoning in Reiter's default logic. They show that under the translation tr_T of the defaults D in a default theory (D, W)

$$tr_T\left(\frac{\alpha : \beta_1, \ldots, \beta_n}{\gamma}\right) = (L\alpha \wedge \neg LL\neg\beta_1 \wedge \ldots \wedge \neg LL\neg\beta_n) \rightarrow \gamma \tag{9}$$

default logic can be seen as a fragment of autoepistemic logic based on iterative expansions, i.e., E is an extension of a default theory (D, W) iff $E = \Delta \cap \mathcal{L}$ for an iterative expansion Δ of $tr_T(D, W)$.

Marek and Truszczynski [1990] already observed a basic groundedness problem of iterative expansions and gave the following example which shows that iterative expansions are not necessarily stable set minimal. Thus iterative expansions may be regarded too weakly grounded.

Example 3 Consider a set

$$\Sigma = \{\neg L\neg Lp \rightarrow p\} \tag{10}$$

where p is atomic. The set Σ has two stable expansions: one containing p and the other not. Both are iterative but only the second one is minimal. Technically the problem seems to be that the definition of $A^{\Delta}(\Sigma)$ allows the adoption of a belief (in the context Δ) which is justified (indirectly) on the basis of the same belief. This is what happens in the first iterative expansion. If p is put to the context Δ but $\neg Lp$ is not, then by negative introspection we have that $\neg L\neg Lp \in A_0^{\Delta}(\Sigma)$ and $p \in A_0^{\Delta}(\Sigma)$. Thus $Lp \in A_1^{\Delta}(\Sigma)$. Hence $p \in A^{\Delta}(\Sigma)$ and $Lp \in A^{\Delta}(\Sigma)$. It cannot be the case that $\neg Lp \in A^{\Delta}(\Sigma)$ (unless $A^{\Delta}(\Sigma)$ is inconsistent, which it is not). Thus the justification of p is indirectly based on the belief p. ■

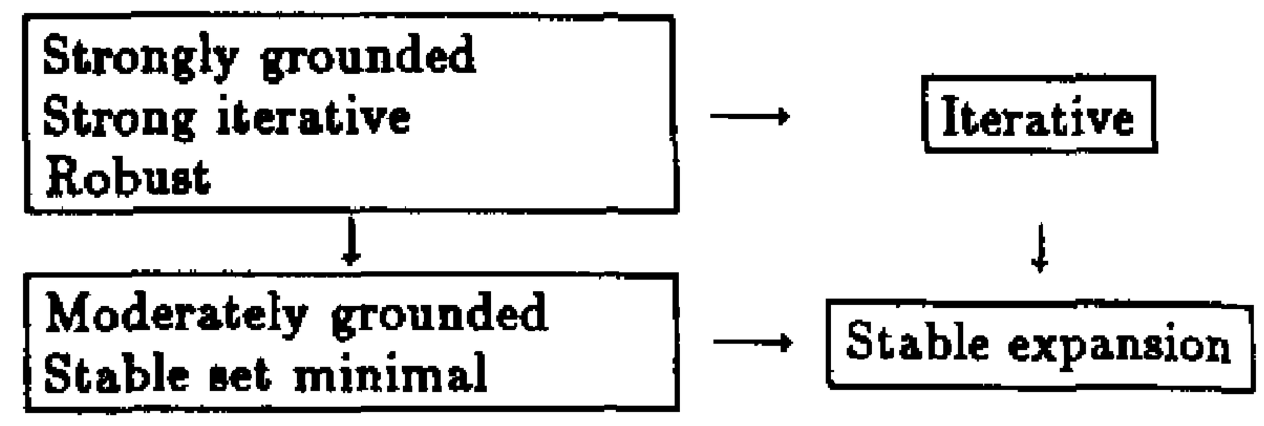

Figure 1: Relationships of groundedness concepts.

Marek and Truszczynski [1989] show how to capture extensions in default logic using two new classes of expansions: ***strongly iterative expansions*** (a strengthened form of iterative expansions) and ***robust expansions.*** These classes of expansions are equivalent with Konolige's strongly grounded expansions and are also defined only for premises in the normal form (4). They suffer from the same unnatural properties as discussed in connection with Konolige's strongly grounded expansions, especially, the dependence on the syntactic representation of premises.

Figure 1 displays a summary of the various groundedness conditions discussed above and their relationships. The arrow is interpreted as implies. E.g., strong groundedness implies stable set minimality.

It seems clear that to get tighter grounded expansions the construction of an expansion should start from the formulae without the L operator. Marek and Truszczynski [1989] present such a construction which builds a stable set out of its L-free part. This method does not seem to be applicable to building expansions from premises, however, because expansions cannot be constructed using such a layered method where, roughly speaking, all formulae having the same depth of L operators are decided at the same time. E.g., consider the set of premises $\{\neg Lq \rightarrow q, \neg LLp \rightarrow q\}$.

Another related idea is to restrict the set of formulae subject to direct introspection and accomplish further introspection by applying some modal logic. Tiomkin and Kaminski [1990] propose ***nonmonotonic ground logics*** based on modal logics *T*, *S4*, and *S5* where only negative introspection restricted to L-free formulae is applied directly. The set of beliefs is defined using the following fixed point equation

$$\Delta = \{\phi \mid \Sigma \cup \{\neg L\psi \mid \psi \in \mathcal{L} - \Delta\} \vdash_S \phi\} \tag{11}$$

where $\vdash_S$ is the derivability relation of the underlying modal logic (*T*, *S4*, *S5*). It seems, however, that these ***ground S-expansions*** are not necessarily stable expansions. E.g., the premise $\neg Lp \rightarrow p$ has no stable expansions. But all the underlying logics *T*, *S4*, and *S5* contain the axiom scheme T ($L\phi \rightarrow \phi$) which produces an S-expansion containing p. So to stay within stable expansions the axiom scheme T has to be abandoned. Konolige's [1988] moderately grounded expansions are an example of these kinds of ground S-expansions. They are based on the modal logic *K45.* Unfortunately, the expansions are rather weakly grounded as discussed in Example 2. In fact, it seems that even ground expansions based on the modal logic *K* have similar problems of weak groundedness as discussed in Example 2. Consider the set of premises $\{\neg Lp \rightarrow q, \neg L\neg Lp \rightarrow p\}$. It seems that it has two ground k-expansions: one containing p and the other containing q. However, the first one is quite weakly grounded in the premises.

4 Enumeration Based Expansions

Our aim is to find a proper characterisation of the sets of beliefs of an agent with full introspection capabilities. These sets should be stable expansions of the premises but more tightly grounded in the premises. At least they should be stable set minimal and, in addition, multiple stable set minimal expansions should be excluded in the situations like in Example 2. Iterative expansions suggested by Marek and Trussceynski [I989] are a step in the right direction but are still too weakly grounded as they are not necessarily stable set minimal. Strong iterative expansions (which are the same as strongly grounded and robust expansions) are stable set minimal but are not satisfactory, especially, because of their dependence on the syntactic representation of premises. Another goal is to find a more constructive basis for defining expansions.

We propose a new more constructive definition of the sets of beliefs of an introspective agent which produces tightly grounded sets of beliefs. The definition is based on enumerations of the formulae of L_{ae}. Davis [I980] used this idea in connection with McDermott and Doyle's [I980] nonmonotonic logic to characterize the intersection of fixed points in McDermott and Doyle's logic. The author [Niemela, 1988] has extended the work of Davis and shown that also individual fixed points in McDermott and Doyle's logic can be characterised using enumerations of the formulae.

The idea is to build a set of beliefs from premises Σ by applying introspection to formulae in the order given by an enumeration ε. A set $\mathbf{B}^\varepsilon(\Sigma)$ is constructed which contains all the results of introspection in the order ε starting from the premises Σ. The set $\mathbf{B}^\varepsilon(\Sigma)$ together with Σ induces the set of beliefs $\mathbf{SE}^\varepsilon(\Sigma)$ of a fully introspective agent having initial assumptions Σ after introspecting formulae in the order ε.

Definition 4.1 *Let $\Sigma \subseteq \mathcal{L}_{ae}$. Let $\varepsilon = \psi_1, \psi_2, \ldots$ be an enumeration of all the formulae of $\mathcal{L}_{ae}$. Let $\mathbf{B}^\varepsilon_0(\Sigma) = \emptyset$. Define $\mathbf{B}^\varepsilon_{i+1}(\Sigma)$ for $i = 0, 1, \ldots$ as follows:*

$$\mathbf{B}^\varepsilon_{i+1}(\Sigma) = \begin{cases} \mathbf{B}^\varepsilon_i(\Sigma) \cup \{L\psi_{i+1}\} & \text{if } \Sigma \cup \mathbf{B}^\varepsilon_i(\Sigma) \models \psi_{i+1} \\ \mathbf{B}^\varepsilon_i(\Sigma) \cup \{\neg L\psi_{i+1}\} & \text{otherwise} \end{cases}$$

Finally let

$$\mathbf{B}^\varepsilon(\Sigma) = \bigcup_{i=0}^{\infty} \mathbf{B}^\varepsilon_i(\Sigma)$$
$$\mathbf{SE}^\varepsilon(\Sigma) = \{\phi \in \mathcal{L}_{ae} \mid \Sigma \cup \mathbf{B}^\varepsilon(\Sigma) \models \phi\}.$$

We would like $\mathbf{SE}^\varepsilon(\Sigma)$ to be a stable expansion of Σ induced by the enumeration ε. Unfortunately, this not the case for all enumerations.

Example 4 Consider a set of premises

$$\Sigma = \{\neg Lp \rightarrow q\} \tag{12}$$

and take an enumeration $\varepsilon = q, p, \ldots$ Then $\mathbf{B}^\varepsilon_1(\Sigma) = \{\neg Lq\}$ as $\Sigma \not\models q$ and $\mathbf{B}^\varepsilon_2(\Sigma) = \{\neg Lq, \neg Lp\}$ as $\Sigma \cup \mathbf{B}^\varepsilon_1(\Sigma) \not\models p$. But now $q \in \mathbf{SE}^\varepsilon(\Sigma)$ and $\neg Lq \in \mathbf{SE}^\varepsilon(\Sigma)$ but $Lq \notin \mathbf{SE}^\varepsilon(\Sigma)$. Thus $\mathbf{SE}^\varepsilon(\Sigma)$ does not satisfy the fixed point equation (1) of a stable expansion. ■

To stay within stable expansions we must require enumerations to be acceptable in the following sense.

Definition 4.2 *An enumeration ε is* Σ-acceptable *if there is no i and no formula ϕ such that $\neg L\phi \in \mathbf{B}^\varepsilon_i(\Sigma)$ but $\Sigma \cup \mathbf{B}^\varepsilon_i(\Sigma) \models \phi$.*

Theorem 4.3 *For all Σ-acceptable enumerations ε, $\mathbf{SE}^\varepsilon(\Sigma)$ is a stable expansion of Σ.*

Proof. We prove that $\mathbf{SE}^\varepsilon(\Sigma) = \{\phi \mid \Sigma \cup \mathbf{B}^\varepsilon(\Sigma) \models \phi\}$ is a stable expansion of Σ by showing that $\mathbf{B}^\varepsilon(\Sigma) = L(\mathbf{SE}^\varepsilon(\Sigma)) \cup \neg L(\overline{\mathbf{SE}^\varepsilon(\Sigma)})$.

If $L\psi_i \in \mathbf{B}^\varepsilon(\Sigma)$, then $\Sigma \cup \mathbf{B}^\varepsilon_{i-1}(\Sigma) \models \psi_i$ and as $\mathbf{B}^\varepsilon_{i-1}(\Sigma) \subseteq \mathbf{B}^\varepsilon(\Sigma)$, $\psi_i \in \mathbf{SE}^\varepsilon(\Sigma)$ and thus $L\psi_i \in L(\mathbf{SE}^\varepsilon(\Sigma))$. If $\neg L\psi_i \in \mathbf{B}^\varepsilon(\Sigma)$, then $\Sigma \cup \mathbf{B}^\varepsilon_{i-1}(\Sigma) \not\models \psi_i$. Assume that $\Sigma \cup \mathbf{B}^\varepsilon(\Sigma) \models \psi_i$. Then there exists a $k \geq i$ such that $\Sigma \cup \mathbf{B}^\varepsilon_k(\Sigma) \models \psi_i$ but $\neg L\psi_i \in \mathbf{B}^\varepsilon_k(\Sigma)$, which is a contradiction as ε is Σ-acceptable. Thus $\psi_i \notin \mathbf{SE}^\varepsilon(\Sigma)$ and $\neg L\psi_i \in \neg L(\overline{\mathbf{SE}^\varepsilon(\Sigma)})$. This shows that $\mathbf{B}^\varepsilon(\Sigma) \subseteq L(\mathbf{SE}^\varepsilon(\Sigma)) \cup \neg L(\overline{\mathbf{SE}^\varepsilon(\Sigma)})$.

If $\neg L\psi_i \in \neg L(\overline{\mathbf{SE}^\varepsilon(\Sigma)})$, then $\Sigma \cup \mathbf{B}^\varepsilon(\Sigma) \not\models \psi_i$. This implies that $\Sigma \cup \mathbf{B}^\varepsilon_{i-1}(\Sigma) \not\models \psi_i$ as $\mathbf{B}^\varepsilon_{i-1}(\Sigma) \subseteq \mathbf{B}^\varepsilon(\Sigma)$. Thus $\neg L\psi_i \in \mathbf{B}^\varepsilon(\Sigma)$. If $L\psi_i \in L(\mathbf{SE}^\varepsilon(\Sigma))$, then $\Sigma \cup \mathbf{B}^\varepsilon(\Sigma) \models \psi_i$. Assume that $\Sigma \cup \mathbf{B}^\varepsilon_{i-1}(\Sigma) \not\models \psi_i$. Then there exists a $k \geq i$ such that $\Sigma \cup \mathbf{B}^\varepsilon_k(\Sigma) \models \psi_i$ but $\neg L\psi_i \in \mathbf{B}^\varepsilon_k(\Sigma)$, which is a contradiction as ε is Σ-acceptable. Thus $\Sigma \cup \mathbf{B}^\varepsilon_{i-1}(\Sigma) \models \psi_i$ and $L\psi_i \in \mathbf{B}^\varepsilon(\Sigma)$. Hence $\mathbf{B}^\varepsilon(\Sigma) = L(\mathbf{SE}^\varepsilon(\Sigma)) \cup \neg L(\overline{\mathbf{SE}^\varepsilon(\Sigma)})$. □

Σ-acceptable enumerations offer a constructive way to define stable expansions. The expansion is directly built from the premises and the enumeration. It should be noted that even the acceptability condition is in a sense local to each $\mathbf{B}^\varepsilon_i(\Sigma)$ as only those formula ψ_j have to be checked for which $j \leq i$ because they are the only formulae for which $\neg L\psi_j \in \mathbf{B}^\varepsilon_i(\Sigma)$ could hold.

However, enumerations are not capable of capturing every stable expansion. Consider the premise $Lp \rightarrow p$ discussed in Example 1. It has two stable expansions: one containing p and the other not containing p. Only the second one is characterisable using enumerations because in enumeration based expansions Lp cannot be used in deriving p directly. This leads one to assume that enumerations might produce minimal expansions. However, this is not the case as shown by the following example.

Example 5 Consider the set

$$\Sigma = \{\neg L\neg Lp \rightarrow p\} \tag{13}$$

where p is atomic. Σ has two stable expansions which are both characterisable using enumerations. The first one not containing p using an enumeration $p, \neg Lp, \ldots$ and the second containing p using an enumeration $\neg Lp, p, \ldots$ Let us look what happens in the second enumeration ε. $\mathbf{B}_1^\varepsilon(\Sigma) = \{\neg L \neg Lp\}$ as $\Sigma \not\models \neg Lp$. But then $\mathbf{B}_2^\varepsilon(\Sigma) = \{\neg L\neg Lp, Lp\}$ as $\Sigma \cup \mathbf{B}_1^\varepsilon(\Sigma) \models p$ ∎

Thus the enumeration method fails to capture stable set minimal expansions exactly in the same situation where iterative method fails. This is no coincidence because enumeration based expansions and iterative expansions are closely related. Enumerations produce always iterative expansions and at least for finite sets of premises for every iterative expansion there is a corresponding enumeration producing it.

Theorem 4.4 *For all Σ-acceptable enumerations ε, $\mathbf{SE}^\varepsilon(\Sigma)$ is an iterative expansion of Σ.*

Proof. We have to show that for $\Delta = \mathbf{SE}^\varepsilon(\Sigma)$, $A^\Delta(\Sigma) = \Delta$ holds. First we show that for all $i = 0, 1, 2, \ldots$

$$\mathbf{B}_i^\varepsilon(\Sigma) \subseteq L(A_i^\Delta(\Sigma)) \cup \neg L(\overline{\Delta}). \quad (14)$$

For $i = 0$ this holds trivially. If $L\psi_i \in \mathbf{B}_i^\varepsilon(\Sigma)$, then $\Sigma \cup \mathbf{B}_{i-1}^\varepsilon(\Sigma) \models \psi_i$. By the induction hypothesis $\Sigma \cup L(A_{i-1}^\Delta(\Sigma)) \cup \neg L(\overline{\Delta}) \models \psi_i$ which implies that $\psi_i \in A_i^\Delta(\Sigma) = Cn(A_{i-1}^\Delta(\Sigma) \cup L(A_{i-1}^\Delta(\Sigma)))$ because $\Sigma \cup \neg L(\overline{\Delta}) \subseteq A_n^\Delta(\Sigma)$ for all $n = 0, 1, \ldots$ Thus $L\psi_i \in L(A_i^\Delta(\Sigma))$. If $\neg L\psi_i \in \mathbf{B}_i^\varepsilon(\Sigma)$, then $\psi_i \notin \Delta$ as ε is Σ-acceptable. Thus $\neg L\psi \in \neg L(\overline{\Delta})$. Hence $\mathbf{B}_i^\varepsilon(\Sigma) \subseteq L(A_i^\Delta(\Sigma)) \cup \neg L(\overline{\Delta})$.

This result implies $\Delta \subseteq A^\Delta(\Sigma)$ in the following way. Let $\phi \in \Delta$. Then there exists i such that $\Sigma \cup \mathbf{B}_{i-1}^\varepsilon(\Sigma) \models \phi$. By (14) $\Sigma \cup L(A_{i-1}^\Delta(\Sigma)) \cup \neg L(\overline{\Delta}) \models \phi$ which implies that $\phi \in A_i^\Delta(\Sigma) \subseteq A^\Delta(\Sigma)$.

We show that for all $i = 0, 1, 2, \ldots$ $A_i^\Delta(\Sigma) \subseteq \Delta$ which implies $A^\Delta(\Sigma) \subseteq \Delta$ and thus $A^\Delta(\Sigma) = \Delta$. Let $\phi \in A_0^\Delta(\Sigma)$. Then $\Sigma \cup \neg L(\overline{\Delta}) \models \phi$. So $\phi \in \Delta = \{\phi \mid \Sigma \cup L(\Delta) \cup \neg L(\overline{\Delta}) \models \phi\}$. Let $\phi \in A_i^\Delta(\Sigma)$. Then $A_{i-1}^\Delta(\Sigma) \cup L(A_{i-1}^\Delta(\Sigma)) \models \phi$. By the induction hypothesis $A_{i-1}^\Delta(\Sigma) \subseteq \Delta$ and thus $L(A_{i-1}^\Delta(\Sigma)) \subseteq L(\Delta) \subseteq \Delta$. Thus $\phi \in \Delta$. □

Theorem 4.5 *For each iterative expansion Δ of a finite set of premises Σ there is a Σ-acceptable enumeration ε such that $\Delta = \mathbf{SE}^\varepsilon(\Sigma)$.*

Proof. It can be shown that a Σ-acceptable enumeration ε which induces Δ can be constructed from Δ in the following way if Σ is finite.

1. Let the first formulae in ε be the formulae $\psi_i \notin \Delta$ for which $L\psi_i \in Sf^L(\Sigma)$[2].
2. The next formulae in ε are those $\psi_i \in \Delta$ for which $L\psi_i \in Sf^L(\Sigma)$ in the order they are produced by the iteration of A^Δ.
3. All the rest of the formulae in $\mathcal{L}_{ae}$ are ordered such that if $L\psi_i \in Sf^L(\{\psi_j\})$, then $i < j$.

[2] $Sf^L(\Sigma)$ is the set of the $L\chi$ subformulae of the formulae in Σ.

The set Σ has to be finite for the conditions 1 and 2 to be satisfied by an enumeration of formulae. The question whether there exists a Σ-acceptable enumeration ε which induces Δ for any iterative expansion Δ of an infinite set of premises Σ remains open. □

Thus enumeration based expansions provide an alternative way of defining iterative expansions. Example 5 reveals also why iterative expansions are not necessarily stable set minimal. It is possible to introspect a formula ϕ without first introspecting the subformulae of ϕ which can effect the result of the introspection of ϕ. In the case of Example 5 it is possible to introspect $\neg Lp$ and obtain $\neg L\neg Lp$ without first checking the status of p.

So it seems as a very natural requirement to demand that in order to get tightly grounded beliefs all subformulae of a given formula ϕ that can effect the result of the introspection of ϕ must be examined before ϕ. This new groundedness requirement can be incorporated easily to the definition of enumeration based stable expansions. We require the enumerations to be *L-hierarchic*. This means that a formula ϕ can appear in the enumeration only after all ψ such that $L\psi$ is a subformula of ϕ have appeared in the enumeration. It is unclear how to incorporate this kind of a requirement into the definition of iterative expansions.

Definition 4.6 *Let ε be an enumeration $\psi_1, \psi_2, \ldots$ of all the formulae in $\mathcal{L}_{ae}$. ε is* L-hierarchic *if for all ψ_i, ψ_j holds that if $L\psi_i \in Sf^L(\{\psi_j\})$, then $i < j$.*

Definition 4.7 *$\mathbf{SE}^\varepsilon(\Sigma)$ is an* L-hierarchic expansion *of Σ if ε is a Σ-acceptable L-hierarchic enumeration.*

It turns out that the simple requirement on the order of the formulae in the enumerations guarantees that L-hierarchic expansions are stable set minimal.

Theorem 4.8 *Every L-hierarchic expansion $\mathbf{SE}^\varepsilon(\Sigma)$ of Σ is a stable set minimal stable expansion of Σ (a moderately grounded expansion of Σ).*

Proof. We denote $\mathbf{SE}^\varepsilon(\Sigma)$ by Δ. Assume that there exists a stable set S containing Σ such that $S \cap \mathcal{L} \subseteq \Delta \cap \mathcal{L}$. We prove that $\Delta = S$ by showing that for all $i = 1, 2, \ldots$, $\psi_i \in \Delta$ iff $\psi_i \in S$. This proves that Δ is stable set minimal for Σ.

If $\psi_1 \in S$, then as ε is L-hierarchic $\psi_1 \in \mathcal{L}$ and $\psi_1 \in \Delta$. If $\psi_1 \in \Delta$, then $\Sigma \models \psi_1$ as ε is Σ-acceptable. Thus $\psi_1 \in S$ as $\Sigma \subseteq S$ and S is stable.

If $\psi_i \in \Delta$, then $\Sigma \cup \mathbf{B}_{i-1}^\varepsilon(\Sigma) \models \psi_i$ as ε is Σ-acceptable. By the induction hypothesis $\mathbf{B}_{i-1}^\varepsilon(\Sigma) \subseteq S$. Thus $\psi_i \in S$ as $\Sigma \subseteq S$ and S is stable. Let $\psi_i \notin \Delta$. The formula ψ_i can be transformed into a normal form $\psi_i' = d_1 \wedge \ldots \wedge d_n$ where each d_i is of the form

$$\gamma \vee L\alpha_1 \vee \ldots \vee L\alpha_m \vee \neg L\beta_1 \vee \ldots \vee \neg L\beta_l$$

where $\gamma \in \mathcal{L}$ and $\alpha_i, \beta_i \in \mathcal{L}_{ae}$. It holds that $\psi_i \in \Delta$ iff $\psi_i' \in \Delta$. So $\psi_i' \notin \Delta$. Thus there exists a disjunction $d_j \notin \Delta$. As Δ is a stable set, $d_j \notin \Delta$ iff $\gamma \notin \Delta, \alpha_1 \notin \Delta, \ldots, \alpha_m \notin \Delta, \beta_1 \in \Delta, \ldots, \beta_l \in \Delta$. The normal form transformation does not go into the $L\chi$ formulae. Thus $Sf^L(\{\psi_i\}) = Sf^L(\{\psi_i'\})$ and so each α_i and β_i is some ψ_j such that $j < i$ because the enumeration is L-hierarchic. Thus by the induction hypothesis

$\alpha_1 \notin S, \ldots, \alpha_m \notin S, \beta_1 \in S, \ldots, \beta_l \in S$. As $\gamma \in \mathcal{L}$ and $\gamma \notin \Delta$, $\gamma \notin S$. Thus $d_j \notin S$ which implies that $\psi_i' \notin S$. As ψ_i is equivalent to ψ_i', $\psi_i \notin S$. Hence $\Delta = S$ □

Thus an L-hierarchic expansion of Σ is a stable set minimal iterative expansion which is not dependent on the syntactic representation of premises. E.g., $\{Lp \rightarrow p, \neg Lp \rightarrow p\}$ and $\{p\}$ have both the same unique L-hierarchic expansion. So L-hierarchic expansions are not necessarily strongly grounded (or strongly iterative/robust).

It turns out that in the case of T-clauses which are of the form

$$(L\alpha_1 \wedge \ldots \wedge L\alpha_m \wedge \neg LL\beta_1 \wedge \ldots \wedge \neg LL\beta_n) \rightarrow \gamma \quad (15)$$

where $\alpha_i, \beta_i, \gamma \in \mathcal{L}$ all iterative expansions of every finite set of premises are also L-hierarchic expansions. It can be shown that the conditions in the proof of Theorem 4.5 for an enumeration inducing an iterative expansion are also satisfied by an L-hierarchic enumeration for each iterative expansion of every finite set of T-clauses. So for T-clauses iterative and L-hierarchic expansions coincide. This shows that L-hierarchic expansions are adequate to capture extensions in Reiter's default logic using the translation proposed by Marek and Trusscsynski (9) because this translation produces T-clauses.

Theorem 4.9 *$E \subseteq \mathcal{L}$ is an extension of a finite default theory $(D, W)^3$ iff $E = \mathrm{SE}^\epsilon(tr_T(D, W)) \cap \mathcal{L}$ for a $tr_T(D, W)$-acceptable L-hierarchic enumeration ϵ.*

Thus autoepistemic logic based on L-hierarchic expansions is a generalization of default logic which provides an alternative definition of an extension (Reiter [1980] gave originally a fixed point characterization). On the other hand, default logic is a special case of this kind of autoepistemic logic where only formulae of the form $(L\alpha \wedge \neg LL\neg\beta_1 \wedge \ldots \wedge \neg LL\neg\beta_n) \rightarrow \gamma$ are allowed.

The question of a general semantics approach to L-hierarchic expansions remains open. In the case of T-programs which are T-clauses where $\alpha_i, \beta_i, \gamma \in \mathcal{L}$ are literals Marek and Trusscsynski [1990] have given a semantic characterisation of consistent iterative expansions and as in this case iterative and L-hierarchic expansions coincide this semantics applies also to consistent L-hierarchic expansions of T-programs.

5 Conclusions

We have introduced a new class of expansions in autoepistemic logic called *L-hierarchic expansions.* L-hierarchic expansions are very promising candidates for the sets of conclusions derived by an ideally rational agent with full introspection capabilities on the basis of given premises. This new class of expansions has a number of attractive properties. Instead of a fixed point characterisation an L-hierarchic expansion is constructed directly from the premises and an enumeration of the formulae of the language. The construction is given in terms of classical logic, e.g., no modal logic is needed. L-hierarchic expansions are independent of the syntactic representation of premises. They are very tightly grounded in the premises. They are iterative expansions [Marek and Trusscsynski, 1989) and, in addition, always stable set minimal. Furthermore, autoepistemic logic based on L-hierarchic expansions captures default reasoning: it is shown that Reiter's default logic is a special case of this new logic. One of the most important open problems is the question of existence of a semantic characterisation of L-hierarchic expansions in the general case.

[3] For a finite default theory (D, *W)* *D* U *W* is a finite set.

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