Representation Theorems for Multiple Belief Changes

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Abstract

This paper aims to develop further and systemize the theory of multiple belief change based on the previous work on *the package contraction,* developed by [Fuhrmann and Hansson 1994] and *the general belief changes,* developed by [Zhang 1996]. Two main representation theorems for general contractions are given, one is based on partial meet models and the other on nice-ordered partition models. An additional principle, called *Limit Postulate,* for the general belief changes is introduced which specifies properties of infinite belief changes. The results of this paper provides a foundation for investigating the connection between infinite nonmonotonic reasoning and multiple belief revision.

1 Introduction

Belief change is the process through which a rational agent acquires new beliefs or retracts previously held ones. A very influential work on belief change goes back to Alchourron, Gardenfors and Makinson [Alchourron *et al.* 1985], who developed a formal mechanism for the revision and the contraction of beliefs, which has been now widely referred to as the AGM theory. For a set of existing beliefs, represented by a deductively closed set *K* of propositional sentences, and a new belief, represented by a propositional sentence *A,* three kinds of belief change operations are considered in the AGM theory: expansion, contraction and revision, denoted by A' + *A, K* — *A* and /\ * *A,* respectively. A set of rationality postulates for belief contractions and belief revisions, based on the idea of minimal change, are given and two different tools, partial meet model and epistemic entrenchment ordering, for constructing belief change operations have been developed in [Alchourron *et al.* 1985] and [Gadenfors and Makinson 1988], respectively. Although AGM's belief change operators appear to capture of what is required of an ideal system of belief change, they are not suitable to characterize changes of beliefs with sets of new beliefs, especially with infinite sets. A number of studies on extending and generalizing these operations so as to enable a treatment of belief change by sets of sentences then come out [Fuhrmann 1988] [Niederee 1991] [Rott 1992] [Hansson 1992] [Fuhrmann and Hansson 94] [Zhang 1995] [Zhang 1996]. The extended operators for expansion, contraction and revision are usually called multiple ones while the original operators are referred to as singleton ones. A framework for multiple belief changes is not only interesting but also useful. We will benefit from it at least in the following aspects:

- The new information an agent accepts often involves simultaneously more than one belief, or even infinitely many, especially when the underlying language is extended to the first-order logic.
- It has been found that there are fundamental differences between iterated belief changes and simultaneous belief changes. The revisions of a belief set by a sentence *A* and then by a sentence *B* are by no means identical to the revision simultaneously by the set *{A, B).* A framework for multiple changes will provide a possibility to describe the relationship between two sorts of belief changes (see [Zhang 1995]). A ready example is the supplementary postulates for multiple revisions(see subsection 2.2 of this paper).
- Connections between belief change and non-monotonic reasoning have been widely investigated in the literature [Makinson and Gardenfors 1991] [Brewka 1991] [Nebel 1992] [Gardenfors and Makinson 1994] [Zhang 1996]. The key idea is translating *B e K * A* into *AI-B* and vice versa. As claimed in [Gardenfors and Makinson 1994] this translation makes sense only on the finite level. 'The idea of infinite revision functions seems to make good intuitive sense.' (see [Makinson and Gardenfors 1991]P.190)

There have been several proposals for multiple belief changes ([Fuhrmann 1988] [Rott 1992] [Hansson 1992] [Fuhrmann and Hansson 94] [Zhang 1995] [Zhang 1996]). This paper is by no means to present an alternative one. Instead of that, we attempt to combine these approaches and develop some necessary tools to improve and systemize them. We will outline, in section 2, the two main paradigms of multiple belief changes: *package contraction,* developed by [Fuhrmann and Hansson 1994], and *general belief change operations,* developed by

[Zhang 1996]. It seems, however, that both paradigms fail to capture full characterization of multiple belief changes. The former succeeds in specifying the basic properties of multiple contractions but fails to give the generalization of the supplementary postulates, whereas the latter presents a full extension of AGM's postulates for belief changes but without providing a representation theorem for its framework. In section 3 we will devote to present a representation theorem for the general contraction, partially using the similar result of the package contraction. In section 4 we will argue with a counterexample that the postulates available for general contractions are not strong enough to characterize the multiple contractions. An additional principle, called *Limit Postulate,* is introduced, which reflects the relationship between the contraction by an infinite set and the ones by its finite subsets. The related representation result for Limit Postulate is then given. Section 5 will conclude the paper with a discussion on the application of this research to non-monotonic reasoning.

Unfortunately, space limitations do not allow a full presentation. All the proofs of lemmas and theorems and some of the lemmas which lead to the main results of the present paper are omitted.

Throughout this paper, we consider the first-order language $\mathcal{L}$ with the standard logical connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$, and $\leftrightarrow$. The set of all subsets of $\mathcal{L}$ is denoted by $\mathcal{F}$. If $F = \{A_1, \cdots, A_n\}$ is a set of sentences, $\wedge F$ is an abbreviation of $A_1 \wedge \cdots \wedge A_n$. We shall assume that the underlying logic includes classical first-order logic and is compact. The notation $\vdash$ means classical first-order derivability and Cn the corresponding closure operator. We call a set K of sentences a *belief set*, which means that $K = Cn(K)$. The set of all belief set in $\mathcal{L}$ is denoted by $\mathcal{K}$. The notation $K + F$ will denote $Cn(K \cup F)$.

2 Postulates for Multiple Belief Change

In this section we try to give a survey of the current research on multiple belief change. Two types of multiple contraction and one type of multiple revision are discussed and their relationships is then established.

2.1 Multiple Contraction

[Fuhrmann and Hansson 1994] introduced two types of multiple contraction operations: *package contraction* and *choice contraction*[1]. They may all be viewed as generalizations of AGM contraction operation, but the former seems more acceptable. The so-called package contraction means contracting a belief set by removing all members of a set of sentences from it. For characterizing this operation, *six basic postulates* as generalizations of the corresponding basic postulates for AGM contraction are given as follows:

[1] Similar formalisms are also introduced by [Rott 1992] and [Hansson 1992].

(K[-]1) $K[-]F = Cn(K[-]F)$.

(K[-]2) $K[-]F \subseteq K$.

(K[-]3) If $\phi \not\vdash F$, then $F \cap (K[-]F) = \phi$.

(K[-]4) If $\phi \vdash F$, then $K \subseteq K[-]F$.

(K[-]5) If $A \in K \backslash K[-]F$, then there is some subset S of K such that $K[-]F \subseteq S$ and $S \not\vdash F$, but $S \cup \{A\} \vdash F$.

(K[-]6) If $F_1 \equiv_K F_2$, then $K[-]F_1 = K[-]F_2$.

Here $\Gamma \vdash F$ represents that there is some $A \in F$ such that $\Gamma \vdash A$; $F_1 \equiv_K F_2$ represents $\forall X \subseteq K(X \vdash F_1 \leftrightarrow X \vdash F_2)$.

A model for package contractions based on partial meet method has been constructed and the representation theorem for these basic postulates was also given in [Fuhrmann and Hansson 1994]. Although two tentative generalizations of AGM's supplementary postulates were given in the same paper, unfortunately, one of them was found to be inconsistent with the basic ones(personal communication).

Another kind of multiple contractions, called *the general contraction* introduced by [Zhang 1996], is motivated by a quite different idea. It seems to lay more emphasis on 'contracting' rather than 'removing'. The principal idea of general contractions is contracting a belief set so that the resulting set is consistent with a set of sentences.

Formally, for a given belief set K, a function $K\ominus : \mathcal{F} \rightarrow \mathcal{F}$[2], is a *general contraction operation over* K if it satisfies the following postulates:

$(K \ominus 1)$ $K \ominus F = Cn(K \ominus F)$.

$(K \ominus 2)$ $K \ominus F \subseteq K$.

$(K \ominus 3)$ If $F \cup K$ is consistent, then $K \ominus F = K$.

$(K \ominus 4)$ If F is consistent, then $F \cup (K \ominus F)$ is consistent.

$(K \ominus 5)$ $\forall A \in K(F \vdash \neg A \rightarrow K \subseteq K \ominus F + A)$.

$(K \ominus 6)$ If $Cn(F_1) = Cn(F_2)$, then $K \ominus F_1 = K \ominus F_2$.

$(K \ominus 7)$ $K \ominus F_1 \subseteq K \ominus (F_1 \vee F_2) + F_1$.

$(K \ominus 8)$ If $F_1 \cup (K \ominus (F_1 \vee F_2))$ is consistent, then $K \ominus (F_1 \vee F_2) \subseteq K \ominus F_1$.

Here $F_1 \vee F_2 = \{A \vee B : A \in F_1 \wedge B \in F_2\}$.

Among the above postulates, $(K \ominus 1)$-$(K \ominus 4)$ and $(K \ominus 6)$ are direct generalizations of the AGM postulates $(K-1)$-$(K-4)$ and $(K-6)$, respectively. The postulate $(K \ominus 5)$, being claimed as the generalization of *AGM's Recovery* (the most controversial among the AGM postulates for contractions), seems to be stronger than its original. Here we provide an equivalent property, called *Saturation*, which somewhat supports the postulate.

Lemma 2.1 *If* $\ominus$ *satisfies* $(K\ominus 1)-(K\ominus 4)$, *then* $(K\ominus 5)$ *is equivalent to the following property:*

[2] In the following, K will be omtted from '$K\ominus$' for convenience.

$(\ominus Sat)$ $(K \ominus F + F) \cap K \subseteq K \ominus F$ *(Saturation).*

Saturation expresses the idea that if a piece of knowledge could be kept in the new knowledge base, it does not need to be abandoned when the contraction is conducted.

The postulate $(K \ominus 6)$ seems to be too weak when we consider the relationship between general contractions and package contractions. Instead, the following stronger principle is suggestible:

$(\ominus 6_S)$ If $\forall A \in K(F_1 \vdash \neg A \leftrightarrow F_2 \vdash \neg A)$, then $K \ominus F_1 = K \ominus F_2$.

It is obvious that $(K \ominus 6_S)$ implies $(K \ominus 6)$, but the inverse needs the presence of $(K \ominus 7)$ and $(K \ominus 8)$.

The postulates $(K \ominus 7)$ and $(K \ominus 8)$ are clearly nonintuitive. A slight improvement may be done by giving the following alternatives:

$(K \ominus 7')$ If $F_1 \subseteq F_2$, then $K \ominus F_2 \subseteq K \ominus F_1 + F_2$.

$(K \ominus 8')$ If $F_1 \subseteq F_2$ and $F_2 \cup K \ominus F_1$ is consistent, then $K \ominus F_1 \subseteq K \ominus F_2$.

In fact, $(K \ominus 7)$ and $(K \ominus 8)$ are equivalent to $(K \ominus 7')$ and $(K \ominus 8')$, respectively, by noting the fact that $F_1 \vee F_2 \dashv\vdash Cn(F_1) \cap Cn(F_2)$.

In this paper we will call $(\ominus 1)$-$(\ominus 5)$ and $(\ominus 6_S)$ *the basic postulates* for general contractions, whereas $(\ominus 7')$ and $(\ominus 8')$ *the supplementary postulates.*

An explicit construction of a general contraction has been given in [Zhang 1996] (also see section 4 of this paper), which shows that the set of postulates for general contractions is consistent, but it does not lead to a representation theorem for this sort of multiple contractions.

Despite the differences in motivation for two types of multiple contractions, they are closely related. In fact, the general contraction can be defined by the package contraction and, inversely, the latter can be partially defined in terms of the former. To show this, let us start with two notations. For any $F \in \mathcal{F}$, $\overline{F} = \{A : \exists B_1 \cdots B_n \in F(A = \neg B_1 \vee \cdots \vee \neg B_n)\}$ and $\overline{F} = \{\neg A : A \in F\}$. Then we have

Proposition 2.2 *Let $K \in \mathcal{K}$ and '$[-]$' be a package contraction function over K. Define a general contraction function '$\ominus$' over K as follows: for any $F \in \mathcal{F}$,*

$$K \ominus F \stackrel{\text{Def}}{=} K[-]\,\overline{F}$$

If '$[-]$' satisfies all the basic postulates for package contractions, then '$\ominus$' satisfies all the basic postulates for general contractions.

Proposition 2.3 *Let $\mathcal{F}_\vee = \{F \in \mathcal{F} : \forall A, B \in F(A \vee B \in F)\}$ and $\ominus$ be a general contraction function over K. Define a package contraction function $[-] : \mathcal{F}_\vee \longrightarrow \mathcal{F}_\vee$ over K as follows: For any $F \in \mathcal{F}_\vee$,*

$$K[-]F = K \ominus \overline{F}$$

If $\ominus$ satisfies all the basic postulates for general contractions, then $[-]$ satisfies all the basic postulates for package contractions.

2.2 Multiple Revision

There are few investigations for multiple revision. A reason for this may be that it is widely agreed that revisions can be reduced to contractions. As a kind of generalizations of AGM revision operation, [Zhang 1995] [Zhang 1996] introduced a multiple revision function '$\otimes$', called *general revision.* Formally, for any belief set K, a function $\otimes : \mathcal{F} \longrightarrow \mathcal{F}$ is a *general revision function over* K if it satisfies the following postulates:

$(K \otimes 1)$ $K \otimes F = Cn(K \otimes F)$.

$(K \otimes 2)$ $F \subseteq K \otimes F$.

$(K \otimes 3)$ $K \otimes F \subseteq K + F$.

$(K \otimes 4)$ If $K \cup F$ is consistent, then $K + F \subseteq K \otimes F$.

$(K \otimes 5)$ $K \otimes F$ is inconsistent if and only if F is inconsistent.

$(K \otimes 6)$ If $Cn(F_1) = Cn(F_2)$, then $K \otimes F_1 = K \otimes F_2$.

$(K \otimes 7)$ $K \otimes (F_1 \cup F_2) \subseteq K \otimes F_1 + F_2$.

$(K \otimes 8)$ If $F_2 \cup (K \otimes F_1)$ is consistent, then $K \otimes F_1 + F_2 \subseteq K \otimes (F_1 \cup F_2)$.

In analogy with *Levi identity* and *Harper identity* in the AGM framework, the relationship between the general contraction and the general revision has been established by the following definitions:

(Def $\otimes$) $K \otimes F \stackrel{def}{=} (K \ominus F) + F$.

(Def $\ominus$) $K \ominus F \stackrel{def}{=} (K \otimes F) \cap K$

Theorem 2.4 *[Zhang 1996] If $\ominus(\otimes)$ satisfies $(K \ominus 1)$–$(K \ominus 8)((K \otimes 1)$- $(K \otimes 8))$, then $\otimes(\ominus)$ obtained from $Def\otimes(Def\ominus))$ satisfies $(K \otimes 1) - (K \otimes 8)((K \ominus 1) - (K \ominus 8))$.*

This result enables us to lay more emphasis on the task of characterizing contraction operations.

3 Partial Meet Model for General Contractions

As mentioned above, the general contraction is successful in generalizing the AGM supplementary postulates, but fails to give a representation result whereas the package contraction is just opposite. In this section we try to give a representation theorem for the general contraction.

According to the relationship of package contractions and general contraction, a partial meet model for the general contraction can readily be constructed. The only problem is whether and how this kind of model can be suitably restricted so that the supplementary postulates for general contractions are also satisfied. This section will try to give an answer for it.

In the AGM theory the notation A' J. *A* Represents the set of maximal subsets of *K* that does not imply *A*. This notation can be easily generalized to the following form:

Definition 3.1 *For $K \in \mathcal{K}$ and $F \in \mathcal{F}$, $K' \in K \| F$ if and only if*
1. $K' \subseteq K$;
2. $F \cup K'$ is consistent, and
3. $\forall K'' \subseteq K(K' \subset K'' \rightarrow K' \cup F$ is inconsistent).

It is easy to see that $K\|\{A\} = K \perp \neg A$. The following notations are useful.
$\mathcal{U}_K = \bigcup\{K\|F \ : \ F \in \mathcal{F}\}$ and $\mathbf{U}_K = \{K\|F \ : \ F \in \mathcal{F}\}$

Definition 3.2 *For any $K \in \mathcal{K}$, a selection function for K is a function $S : \mathbf{U}_K \rightarrow 2^{\mathcal{U}_K}$ such that*

$$\forall H \in \mathbf{U}_K(S(H) \subseteq H \wedge (H \neq \phi \rightarrow S(H) \neq \phi))$$

Definition 3.3 *An operation $K\ominus : \mathcal{F} \rightarrow \mathcal{F}$ is a partial meet contraction over K if and only if there exists a select function S such that for any $F \in \mathcal{F}$,*

$$K \ominus (F) = \begin{cases} K & \text{if } F \text{ is inconsistent;} \\ \bigcap S(K\|F) & \text{otherwise.} \end{cases}$$

We will omit K form '$K\ominus$'.

A similar proof of representation theorem for the package contraction leads to the following representation result for the general contraction on the basic postulates.

Theorem 3.4 *For any belief set K, $\ominus$ satisfies all the basic postulates for general contractions if and only if it is a partial meet contraction.*

Definition 3.5 *Let $K \in \mathcal{K}$. A selection function S for K is complete if for all $H \in \mathbf{U}_K$*

$$S(H) = \{K' \in H : \bigcap S(H) \subseteq K'\}$$

A partial meet contraction function is complete if it can be generated by such a selection function.

Definition 3.6 *Let $K \in \mathcal{K}$. A selection function S for K is (transitively) rational when there is a (transitive) relation $\leq$ on $\mathcal{U}_K$ such that for any $H \in \mathbf{U}_K$,*

$$S(H) = \{X \in H : \forall Y \in H(Y \leq X)\}$$

The contraction function generated from such S is called a (transitively) relational partial meet contraction function.

One of the main results in this paper is the following representation theorem.

Theorem 3.7 *(The first representation theorem) For any belief set K, $\ominus$ satisfies postulates $(K \ominus 1) - (K \ominus 8)$ if and only if $\ominus$ is a complete transitively rational partial meet contraction function.*

It should be remarked that there is an important difference between representation theorems of general contractions and of singleton contractions that the completeness of selection functions is needed just one direction in singleton contractions rather than both.

The following lemmas are found to be critical for the proof of the theorem.

Lemma 3.8 *Let $K \in \mathcal{K}$ and S be a complete select function for K. For any $\Delta \in \mathcal{U}_K$ and $F \in \mathcal{F}$, if F is consistent and $\bigcap S(K\|F) \subseteq \Delta$, then $\Delta \in S(K\|F)$ or $\Delta = K$.*

Lemma 3.9 *Let $K \in \mathcal{K}$, $F \in \mathcal{F}$ and Γ be a closed subset of K. If $F \cup K$ is inconsistent but $\Gamma \cup F$ is consistent, then*

$$\bigcap\{\Delta \in K\|F : \Gamma \subseteq \Delta\} + F = \Gamma + F$$

Lemma 3.10 *If $\ominus$ is a complete relational partial meet contraction, then $(K \ominus 7)$ holds.*

Lemma 3.11 *Let S be a complete selection function for K. If $F_1 \subseteq F_2$ and $F_2 \cup (\bigcap S(K\|F_1))$ is consistent, then $(K\|F_2) \cap S(K\|F_1) \neq \phi$*

Lemma 3.12 *Let S be a transitively rational selection function. If $F_1 \subseteq F_2$ and $(K\|F_2) \cap S(K\|F_1) \neq \phi$, then*

$$S(K\|F_2) \subseteq S(K\|F_1)$$

Lemma 3.13 *If $\ominus$ is a complete transitively rational partial meet contraction, then $(K \ominus 8)$ holds.*

There are two limiting cases of singleton contractions: *maxichoice contractions* and *full meet contractions*, being investigated in the AGM theory, which are viewed as the lower and upper bounds of partial meet contractions, and therefore, are useful sometimes for understanding contraction operations sometimes. Here we present two similar representation results for general contractions. Assuming that the maxichoice contractions and the full meet contraction for general contractions are defined as usual, then we have

Proposition 3.14 *A function $\ominus : \mathcal{F} \rightarrow \mathcal{F}$ is a maxichoice contraction over a belief set K if and only if it satisfies $(K \ominus 1)$ -$(K \ominus 5)$, $(K \ominus 6_S)$ and the condition:*
If $A \in K \setminus K \ominus F$, then there exists $B \in K$ such that $F \vdash \neg B$ and $A \rightarrow B \in K \ominus F$.

Proposition 3.15 *A function $\ominus : \mathcal{F} \rightarrow \mathcal{F}$ is a full meet contraction over a belief set K if and only if it satisfies $(K \ominus 1)$ -$(K \ominus 5)$, $(K \ominus 6_S)$ and the condition:*
If $F_1 \subseteq F_2$ and $F_1 \cup K$ is inconsistent, then $K \ominus F_1 \subseteq K \ominus F_2$.

4 Limited Postulation for General Contractions

In this section, we use the approach of nice-ordered partition models, developed by [Zhang 1996], to show why we think that the postulates available for the general contraction are insufficient to characterize infinite belief changes. An additional postulate for the general contraction is introduced and its representation theorem is provided.

In order to construct a model for the general contraction operation, [Zhang 1996] introduced the notion of nice-ordered partition with the motivation of capturing the idea of degrees of reliability of information.

Definition 4.1 *[Zhang 1996] For any belief set K, let $\mathcal{P}$ be a partition of K and $<$ a total-ordering relation* on $\mathcal{P}$. *The triple* $\Sigma = (K, \mathcal{P}, <)$ *is called a total-ordered partition (TOP) of* K. *For any* $p \in \mathcal{P}$, *if* $A \in p$, p *is called the rank of* A, *denoted by* $b(A)$.

A total-ordered partition $\Sigma = (K, \mathcal{P}, <)$ *is a nice-ordered partition (NOP) if it satisfies the following Logical Constraint:*
If $A_1, \cdots, A_n \vdash B$, *then* $\sup\{b(A_1), \cdots, b(A_n)\} \geq b(B)$.

The idea underlying this definition is that different pieces of knowledge in a knowledge base are accepted with different degrees of reliability. It could be supposed that the knowledge base has been grouped according to degrees of reliability of knowledge. The rank of a piece of knowledge represents the group it belongs to. According to the above definition the less rank a sentence is, the higher degree of reliability it has. The relationship between NOP and epistemic entrenchment has been studied in [Zhang 1996]. Roughly speaking, the latter can be viewed as a special case of the former in some sense.

An explicit construction for multiple contraction functions is then given as follows:

Definition 4.2 *[Zhang 1996] Let* $\Sigma = (K, \mathcal{P}, <)$ *be an NOP of a belief set* K. *Define a function* $\dot\ominus : \mathcal{F} \to \mathcal{F}$, *called NOP contraction over* K, *as follows: for any* $F \in \mathcal{F}$
i). If $F \cup K$ *is consistent , then* $K \dot\ominus F = K$; *otherwise,*
ii). $B \in K \dot\ominus F$ *if and only if* $B \in K$ *and there exists* $A \in K$ *such that* $F \vdash \neg A$ *and*

$$\forall C \in K(A \vdash C \wedge F \vdash \neg C \to (b(A \vee B) < b(C) \vee \vdash A \vee B))$$

In particular, when F *is finite, then* $B \in K \dot\ominus F$ *if and only if*

$$B \in K \text{ and } b(\neg(\wedge F) \vee B) < b(\neg(\wedge F)) \vee \vdash \neg(\wedge F) \vee B \tag{1}$$

The following result expresses the fact that an NOP contraction must be a general contraction and, therefore, the set of postulates for general contractions is consistent.

Theorem 4.3 *[Zhang 1996] IF* $\dot\ominus$ *is an NOP contraction over* K, *then it satisfies* $(K \ominus 1) - (K \ominus 8)$.

A natural question is now whether a general contraction function must be an NOP contraction. For the finite case the following lemma gives an affirmative answer.

Lemma 4.4 *If '*$\ominus$*' is a general contraction function over a belief set* K, *then there exists a unique NOP,* $\Sigma = (K, \mathcal{P}, <)$, *of* K *such that for any finite set* F *of sentences,* $B \in K \ominus F$ *iff* $B \in K$ *and*

$$b(\neg(\wedge F) \vee B) < b(\neg(\wedge F)) \vee \vdash \neg(\wedge F) \vee B$$

For the infinite case, however, the answer is negative. Before we illustrate why, let us first introduce a variant of NOP contraction functions.

Definition 4.5 *Let* $\Sigma = (K, \mathcal{P}, <)$ *be an NOP of a belief set* K. *Define a contraction function* $\hat\ominus : \mathcal{F} \to \mathcal{F}$ *over* K *as follows: for any* $F \in \mathcal{F}$
i). If $F \cup K$ *is consistent, then* $K \hat\ominus F = K$, *otherwise;*
ii). $B \in K \hat\ominus F$ *if and only if* $B \in K$ *and there exists* $A \in K$ *such that* $F \vdash \neg A$ *and*

$$\forall C \in K(A \vdash C \wedge F \vdash \neg C \to (b(C \vee B) < b(C) \vee \vdash C \vee B))$$

It is easy to see that two types of NOP contractions agree on finite arguments. The following example shows that they branch in the infinite case.

Example 4.1 We consider a propositional language $\mathcal{L}$ only including propositional variables $p_0, p_1, \cdots$. Let $K = Cn(\{p_0, p_1, \cdots\})$. Constructing an NOP $\mathcal{P}$ of K as follows:
1. $\mathcal{P} = \{P_{-\infty}\} \cup \{P_{-n} : n \in \omega\}$, where

i) $P_{-\infty} = Cn(\phi)$.

ii) For any $A \in K$ such that $A \notin Cn(\phi)$,

- a) If $A = p_{i_1} \wedge \cdots \wedge p_{i_n}$, then $A \in P_{-\min\{i_1, \cdots, i_n\}}$, specially, $p_i \in P_{-i}$; otherwise,
- b) If A' is the unique complete disjunctive normal form of A, then each disjunctive branch of A' is of the form:

$$*p_{i_1} \wedge \cdots \wedge *p_{i_k}$$

  where $*$ is empty or negative '$\neg$'. Since $A \in K$, there exists a disjunctive branch of A' which belongs to K. Suppose that all such branches have been assigned into $P_{-j_1}, \cdots, P_{-j_m}$, respectively, then $A \in P_{-\max\{j_1, \cdots, j_m\}}$.

2. For any $n \in \omega$, $P_{-\infty} < P_{-n}$. For any $m, n \in \omega$, $P_{-m} < P_{-n}$ if and only if $m > n$.

It is not difficult to show that $\Sigma = (K, \mathcal{P}, <)$ is an NOP of K.

According to definitions of $\dot\ominus$ and $\hat\ominus$, we have
(1). If $F = \{\neg p_{i_0}, \cdots, \neg p_{i_n}\} (i_0 < \cdots < i_n)$, then $K \dot\ominus F = K \hat\ominus F = Cn(\{p_{i_{n+1}}, p_{i_{n+2}}, \cdots\} \cup F) \cap K$.
(2). If $F = \{\neg p_{i_0}, \neg p_{i_1}, \cdots\} (i_0 < i_1 < \cdots)$, then $K \dot\ominus F = K \hat\ominus F = Cn(F) \cap K$.

Now we change Σ into Σ' in this way of bringing all the formulas with the form $p_0 \vee p_i (i = 1, 2, \cdots)$ into $P_{-\infty}$ and doing the corresponding changes in order to make the resulting partition an NOP.

It is not difficult to check that if $F = \{\neg p_{i_0}, \neg p_{i_1}, \cdots\}$ $(0 < i_0 < i_1 < \cdots)$, then (with respect to the NOP Σ'):

$$K \dot\ominus F = Cn(F) \cap K \tag{2}$$

$$K \hat\ominus F = Cn(F \cup \{p_0\}) \cap K \tag{3}$$

The above example and Lemma 4.4 tell us the fact that the available postulates for general contractions are not sufficient to uniquely determine an NOP contraction as a general contraction function is given. We need some additional principles to describe the relationship between a belief set contracted by a infinite set of sentences and by its finite subsets. In fact, we can easily see from the above example that there is a situation in which an infinite set is inconsistent with all the contractions by its finite subsets (noting that it must consistent with the contraction by itself). In this case the available postulates for general contractions are insufficient to restrict the relation between the contraction by the infinite set and the ones by its finite subsets.

A reasonable thought is to assume that the contraction by a infinite set is some limiting case of the ones by its finite subset. This motivates us to introduce the following *Limit Postulate* for the general contraction:

Let $C_F = \{\bar{F} : \bar{F} \subseteq Cn(F) \text{ and } \bar{F} \text{ is finite }\}$,

$$(K \ominus LP) \qquad K \ominus F = \bigcup_{\bar{F} \in C_F} \bigcap_{\substack{\bar{F} \subseteq \bar{F}' \\ \bar{F}' \in C_F}} K \ominus \bar{F}'$$

In other words, $B \in K \ominus F$ if and only if there exists a finite subset $\bar{F}$ of $Cn(F)$ such that for each finite subset $\bar{F}'$ of $Cn(F)$, if $\bar{F} \subseteq \bar{F}'$, then $B \in K \ominus \bar{F}'$.

An equivalent assumption for the general revision is readily given in terms of $(Def\ominus)$ and $(Def\otimes)$ as follows (an easy check is necessary):

$$(K \otimes LP) \quad K \otimes F = \bigcup_{\bar{F} \in C_F} \bigcap_{\substack{\bar{F} \subseteq \bar{F}' \\ \bar{F}' \in C_F}} K \otimes \bar{F}'$$

Before we switch to present the representation theorem for the Limit Postulate we have to make a choice between the two types of NOP contractions (see Definition 4.2 and 4.5). By comparing Equations 2 and 3 we find that $\hat{\ominus}$ is more acceptable than $\dot{\ominus}$. Note that even though the proposition p_0 has the lowest degree of reliability in the old knowledge base, the agent believes extremely in $p_0 \vee p_i (i = 1, 2, \cdots)$. Therefore, when the fact tells that some p_i's are false, he will conclude that p_0 is true. In the remaining part of this paper, an NOP contraction will only refer to $\hat{\ominus}$.

Theorem 4.6 *If $\hat{\ominus}$ is an NOP contraction function, then $\hat{\ominus}$ satisfies $(K \ominus LP)$.*

Theorem 4.7 *Let $\ominus$ be a general contraction function over K. If $\ominus$ satisfies $(K \ominus LP)$, then there exists an NOP, $\Sigma = (K, \mathcal{P}, <)$, of K such that $\ominus$ is exactly the NOP contraction generated by Σ.*

The Lemma 4.4 and the above theorems lead to another main result of this paper:

Theorem 4.8 *(The second representation theorem) For any belief set K, $\ominus$ satisfies $(K \ominus 1) - (K \ominus 8)$ and $(K \ominus LP)$ if and only if $\ominus$ is an NOP contraction over K.*

5 Conclusions and Future Work

Two representation theorems for the general contraction are presented in this paper. The first one illustrates that the AGM partial meet contraction can be smoothly extended to the case of multiple changes. The second one persuades us that confining our attention only to the generalization of postulates for singleton belief change would fail to get a full characterization of infinite belief changes. This result also provides a powerful tool to investigate the connection between belief revision and non-monotonic reasoning. In [Zhang *et al.* 1997], we introduced a non-monotonic logic the semantic of which bases on the theory of multiple belief revision and one of the inference rules in which, *finite supracompactness,* is just the counterpart of the Limit Postulate.

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