A Non-Induced Interval Matrix Norm*

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Abstract

Farhadsefat, Rohn and Lotfi defined the concept of an induced interval matrix norm. They then raised the question of finding an interval matrix norm which is not induced by any point matrix norm. We introduce such a norm in this paper.

Keywords: interval matrix, induced interval matrix norm, absolute norm
AMS subject classifications: 65F35, 65G30

1 Problem Statement and Results

We use the standard notation of interval analysis suggested in [2]. So, interval quantities will always be typeset in boldface. $\mathbb{R}^{m\times n}$ and $\mathbb{IR}^{m\times n}$ are the set of all real point and real interval matrices, respectively. $|||\cdot|||$ denotes interval matrix norms in $\mathbb{IR}^{m\times n}$ as opposed to point matrix norms in $\mathbb{R}^{m\times n}$ denoted by $\|\cdot\|$. Note also that $A \leq B$ and $|A|$ are to be understood entrywise.

A function $|||\cdot||| : \mathbb{IR}^{m\times n} \to \mathbb{R}$ is called [1] an *interval matrix norm* in $\mathbb{IR}^{m\times n}$ if for each $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{m\times n}$, $\alpha \in \mathbb{R}$ it satisfies (a)-(c):

(a) $|||\boldsymbol{A}||| \geq 0$, and $|||\boldsymbol{A}||| = 0$ if and only if $\boldsymbol{A} = [0, 0]$,

(b) $|||\boldsymbol{A} + \boldsymbol{B}||| \ \leq \ |||\boldsymbol{A}\,||| + |||\,\boldsymbol{B}|||$,

(c) $|||\alpha\boldsymbol{A}||| = |\alpha|\;|||\,\boldsymbol{A}|||$.

The following theorem shows how to construct interval matrix norms from point matrix norms.

*Submitted: August 7, 2013; Revised: October 11, 2013; Accepted:November 27, 2013.

Theorem 1.1 *[1] For any point matrix norm* $\|\cdot\|$ *in* $\mathbb{R}^{m\times n}$*, the function* $|||\cdot|||$: $\mathbb{IR}^{m\times n} \to \mathbb{R}$ *defined by*

$$||| \boldsymbol{A} ||| = \sup\{\| A \| \ \mid A \in \boldsymbol{A}\} \tag{1}$$

is an interval matrix norm in $\mathbb{IR}^{m\times n}$.

The interval matrix norm $|||\cdot|||$ defined by (1) is *induced* by the point matrix norm $\|\cdot\|$. An interval matrix norm induced by some point matrix norm is called simply an induced interval matrix norm [1].

Theorem 1.2 *[1] A norm* $|||\cdot|||$ *in* $\mathbb{IR}^{m\times n}$ *is an induced interval matrix norm if and only if it satisfies*

$$|||\boldsymbol{A}||| = \max\{|||[A, A]\,||| \ \mid A \in \boldsymbol{A}\}$$

for each $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$.

Theorem 1.3 *The function* $|||\cdot||| : \mathbb{IR}^{m\times n} \to \mathbb{R}$*, defined by*

$$||| \boldsymbol{A}||| := \| \underline{A} \| + \| \overline{A} \| \tag{2}$$

is an interval matrix norm, where $\| . \|$ *is any (point) matrix norm.*

Proof. The proof is straightforward.

Definition 1.1 *[1] If a (point) matrix norm satisfies the property*

$$\| \, |A| \, \| = \| A \|$$

for each $A \in \mathbb{R}^{m\times n}$*, then it is called an absolute (point) matrix norm.*

Examples of absolute matrix norms are the 1-norm, the infinity norm and the Frobenius norm [1].

Lemma 1.1 *[1] A norm* $\| . \|$ *in* $\mathbb{R}^{m\times n}$ *is absolute if and only if for each* $A, B \in \mathbb{R}^{m\times n}$, $|A| \le |B|$ *implies* $\| A \| \le \| B \|$.

Here, we provide a characterization of the norm defined in (2).

Theorem 1.4 *Let* $\boldsymbol{A} \subseteq \boldsymbol{B}$, $0 \in \boldsymbol{A}$, $0 \in \boldsymbol{B}$ *and* $\| . \|$ *be an absolute norm. Then,* $|||\boldsymbol{A}||| \le |||\boldsymbol{B}|||$*, where* $|||\cdot|||$ *denotes the norm defined in (2).*

Proof. Since $0 \in \boldsymbol{A}$, we have $\overline{A} \ge 0$, so that $|\overline{A}| = \overline{A}$ and similarly $|\overline{B}| = \overline{B}$. We know that $\boldsymbol{A} \subseteq \boldsymbol{B}$. Therefore, we have $\overline{A} = |\overline{A}| \le |\overline{B}| = \overline{B}$. Since $\|\cdot\|$ is an absolute norm we have $\| \, |\overline{A}| \, \| \le \| \, |\overline{B}| \, \|$. Therefore, by Definition 1 we have

$$\| \overline{A} \| \le \| \overline{B} \| \, . \tag{3}$$

In a similar manner we have $-\underline{A} = |\underline{A}| \le |\underline{B}| = -\underline{B}$ and finally

$$\| \underline{A} \| \le \| \underline{B} \| \, . \tag{4}$$

Formulas (3) and (4) complete the proof.

The following theorem contains our main result.

Theorem 1.5 *The norm* $|||\cdot|||$ *introduced in (2) is not an induced interval matrix norm.*

Proof. The proof is by contradiction. Suppose that the norm in (2) is an induced interval matrix norm. Now let $\boldsymbol{A} = [\underline{A}, \overline{A}]$ be an interval matrix with $\underline{A} = 0$ and $\overline{A} > 0$. By Theorem 1.2 and the definition of our norm (2) we have

$$\begin{aligned} \|\| \boldsymbol{A}\|\| &= \max\{\|\|[A, A]\|\| \mid A \in \boldsymbol{A}\} \geq \|\|[\overline{A}, \overline{A}]\|\| = 2 \| \overline{A} \| \\ &> \| \overline{A} \| = \| \underline{A} \| + \| \overline{A} \| = \|\|\boldsymbol{A}\|\|, \end{aligned}$$

which completes the proof.

Remark 1.1 *Directly from Theorem 1.1 one sees that $\|\|A\|\| = \| A \|$ for each point matrix A provided that $\|\| \cdot \|\|$ is an induced interval matrix norm. Since the norm (2) does not satisfy this relation for non-zero point matrices, it cannot be induced - which is the result of Theorem 1.5.*

It follows from the definition of an induced matrix norm that for each $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{m\times n}$, $\boldsymbol{A} \subseteq \boldsymbol{B}$ implies $\|\|\boldsymbol{A}\|\| \leq \|\|\boldsymbol{B}\|\|$ [1, p. 5]. The following counterexample again proves that the norm (2) is not induced. Here, some of the assumptions of Theorem 1.4 are not satisfied; we have $\boldsymbol{A} \subseteq \boldsymbol{B}$ while $\|\|\boldsymbol{A}\|\| \nleq \|\|\boldsymbol{B}\|\|$.

Example 1.1 *Let $\boldsymbol{A} = [\underline{A}, \overline{A}]$ and $\boldsymbol{B} = [\underline{B}, \overline{A}]$ with $0 < \underline{B} < \underline{A}$ and so, we have $\boldsymbol{A} \subseteq \boldsymbol{B}$. Then, there is some $\alpha > 1$ such that $0 < \underline{B} < \alpha\underline{B} < \underline{A}$ whence*

$$0 < \| \underline{B} \| < \alpha \| \underline{B} \| = \| \alpha\underline{B} \| \leq \| \underline{A} \|$$

for absolute matrix norms $\| \cdot \|$. In particular, $\| \underline{B} \| < \| \underline{A} \|$ and $\| \overline{B} \| = \| \overline{A} \|$. Therefore, $\|\|\boldsymbol{B}\|\| < \|\|\boldsymbol{A}\|\|$, where $\|\| \cdot \|\|$ stands for the norm defined in (2).

Remark 1.2 *For Example 1.1 one can show in the same way that the norm*

$$\|\|\boldsymbol{A}\|\| := \frac{\| \underline{A} \| + \| \overline{A} \|}{2}$$

is not induced, too. Notice that this norm satisfies the property $\|\|A\|\| = \| A \|$ for each point matrix A.

Acknowledgements

The authors would like to offer particular thanks to one of the referees who carefully read the paper and proposed constructive suggestions including Example 1.1 and Remarks 1.1 and 1.2. They are also grateful to Professor Ralph Baker Kearfott for his helpful comments on this paper.

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