Interval Pseudo-Inverse Matrices and Interval Greville Algorithm*

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Abstract

This paper investigates interval pseudo-inverse matrices. We state an Interval Greville algorithm and extensions with bisections for calculation of interval pseudo-inverse matrices and give the examples of interval pseudo-inversion application for estimation of solutions of systems of linear equations, and show applications for estimations of solutions and pseudo-solutions in a least squares sense.

Keywords: interval analysis, interval pseudo-inversion, interval Greville algorithm, pseudo-solution
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1 Introduction

For any square interval matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ an interval inverse matrix is determined as the minimal interval matrix $\boldsymbol{A}^{-1} \in \mathbb{IR}^{n\times n}$ so that $\boldsymbol{A}^{-1} \supset \left\{A^{-1} : A \in \boldsymbol{A}\right\}$ [2]. If $\forall A \in \boldsymbol{A} \;\; \exists A^{-1}$ then $\boldsymbol{A}$ is regular. If condition $\rho\left(\mathrm{mid}\left(\boldsymbol{A}\right)\cdot\Delta\right) < 1$ holds, where $\mathrm{mid}\left(\boldsymbol{A}\right)$ is the midpoint matrix of $\boldsymbol{A}$, $\Delta = \mathrm{rad}\left(\boldsymbol{A}\right)$ is the radius matrix, $\rho(\cdot)$ is the spectral radius, then $\boldsymbol{A}$ is called strongly regular [6]. Some algorithms for computation of interval inverse matrices can also be found in [5].

Our main topics of investigation are the development of algorithms for bounds computation of interval inverse matrices and the determination of conditions for simple bounds identification. This paper generalizes and extends the difinition of interval inverse matrices to interval pseudo-inverse matrices for non-regular square or rectangular matrices.

2 Interval Pseudo-Inverse Matrices

For a real rectangular matrix $A \in \mathbb{R}^{m\times n}$ of any rank, the pseudo-inverse matrix (also known as the Moore-Penrose inverse matrix or the Moore-Penrose generalized

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inverse matrix) $A^+ \in \mathbb{R}^{n\times m}$ is the only matrix satisfying the four conditions of Moore-Penrose [1]:

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad (AA^+)^T = AA^+, \quad (A^+A)^T = A^+A. \tag{1}$$

It exists for any matrix. In the case of a regular square matrix it equals the usual inverse matrix.

For any interval matrix $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$ we define the interval pseudo-inverse matrix $\boldsymbol{A}^+ \in \mathbb{IR}^{n\times m}$ as the minimal interval matrix so that $\boldsymbol{A}^+ \supset \{A^+ : A \in \boldsymbol{A}\}$. $\boldsymbol{A}^+$ includes all real pseudo-inverse matrices A^+ for all $A \in \boldsymbol{A}$ [7]. Contrary to the interval inverse matrix we do not need any assumptions of regularity on the matrix $\boldsymbol{A}$. Hence the interval pseudo-inverse matrix is defined for any interval matrix $\boldsymbol{A}$.

Unfortunately there are no algorithms for its computation. There is an interval algorithm for the computation of a real pseudo-inverse matrix [9], but it is not suitable for interval pseudo-inverse matrix computation.

In many applications we need enclosure for $\boldsymbol{A}^+$ instead of an exact interval pseudo-inverse matrix. Such an enclosure can be computed by an interval modification of the well-known Greville algorithm for real pseudo-inverse matrix computation [1].

3 Basic Interval Greville Algorithm

This algorithm was first introduced in [7]. Let $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$ with $\boldsymbol{a}_k$ as its k-th column, where $k = 1, \dots, n$. Let $\boldsymbol{A}_k$ be the submatrix of $\boldsymbol{A}$ constructed from the first k columns of $\boldsymbol{A}$,

$$\boldsymbol{A}_k = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_k \end{pmatrix},$$

where $\boldsymbol{a}_i$ is the i-th column of $\boldsymbol{A}$, $i = 1, \dots, n$. If $k = 1$ then $\boldsymbol{A}_1 = \boldsymbol{a}_1$. For $k = 2, \dots, n$ it is clear that

$$\boldsymbol{A}_k = \begin{pmatrix} \boldsymbol{A}_{k-1} & \boldsymbol{a}_k \end{pmatrix}.$$

Algorithm 1.
Call: IntPseudoInverse($\boldsymbol{A}$).
Input: $\boldsymbol{A}$ – interval matrix.
Output: $\boldsymbol{A}^+$ – enclosure of pseudo-inverse matrix.

Step $k = 1$. Assume

$$\boldsymbol{d}_1 = \|\boldsymbol{a}_1\|^2 = \sum_{i=1}^{m} \boldsymbol{a}_{i1}^2.$$

We have that

$$\boldsymbol{A}_1^+ = \begin{cases} 0, & \text{if } \overline{\boldsymbol{d}_1} = 0, \\ \dfrac{\boldsymbol{a}_1^T}{\boldsymbol{d}_1}, & \text{if } \underline{\boldsymbol{d}_1} > 0, \\ \square\left(0 \bigcup \dfrac{\boldsymbol{a}_1^T}{\boldsymbol{d}_1}\right), & \text{otherwise,} \end{cases}$$

where $0 \in \mathbb{IR}^m$ is the null interval vector and $\square$ is the inteval hull of union of interval vectors.

Steps $k = 2, \dots, n$. Matrix $\boldsymbol{A}_k^+$ can be computed by the formula

$$\boldsymbol{A}_k^+ = \begin{pmatrix} \boldsymbol{A}_{k-1}^+(I - \boldsymbol{a}_k \boldsymbol{f}_k) \\ \boldsymbol{f}_k \end{pmatrix},$$

where I is the unitary matrix of the order m, and

$$\boldsymbol{c}_k = (I - \boldsymbol{A}_{k-1}\boldsymbol{A}_{k-1}^{+})\boldsymbol{a}_k,$$

$$\boldsymbol{d}_k = \|\boldsymbol{c}_k\|^2,$$

$$\boldsymbol{f}_k = \begin{cases} \dfrac{\boldsymbol{c}_k^T}{\boldsymbol{d}_k}, & \text{if } \underline{\boldsymbol{d}_k} > 0, \\ \dfrac{\boldsymbol{a}_k^T(\boldsymbol{A}_{k-1}^{+})^T\boldsymbol{A}_{k-1}^{+}}{1 + \|\boldsymbol{A}_{k-1}^{+}\boldsymbol{a}_k\|^2}, & \text{if } \overline{\boldsymbol{d}_k} = 0, \\ \square\left(\dfrac{\boldsymbol{c}_k^T}{\boldsymbol{d}_k} \bigcup \dfrac{\boldsymbol{a}_k^T(\boldsymbol{A}_{k-1}^{+})^T\boldsymbol{A}_{k-1}^{+}}{1 + \|\boldsymbol{A}_{k-1}^{+}\boldsymbol{a}_k\|^2}\right), & \text{else.} \end{cases}$$

◁

$\boldsymbol{A}_n^{+}$ is the final estimation of the matrix $\boldsymbol{A}^{+}$.

4 Discussion of the Algorithm

The resulting interval pseudo-inverse matrix $\boldsymbol{A}_n^{+}$ may contain infinite bounds in some cases. The probability of this situation is increased for wide matrices and matrices with large sizes. If for any step k we will have

$$\boldsymbol{f}_k = \big([-\infty, +\infty] \quad \dots \quad [-\infty, +\infty]\big)$$

the work of the algorithm can be interrupted, and the result is the matrix

$$\boldsymbol{A}^{+} = \begin{pmatrix} [-\infty, +\infty] & \dots & [-\infty, +\infty] \\ \vdots & \vdots & \vdots \\ [-\infty, +\infty] & \dots & [-\infty, +\infty] \end{pmatrix},$$

showing that the result can be any matrix of $\mathbb{R}^{n\times m}$, not a particularly helpful insight. This matrix would not be useful for subsequent calculations.

We need a way to compute the degree of usability of the matrix $\boldsymbol{A}^{+}$. An accuracy criterion can be based on satisfaction of the Moore-Penrose conditions and defined as

$$\begin{aligned} \boldsymbol{t} \;&=\; \|\boldsymbol{A}\boldsymbol{A}^{+}\boldsymbol{A} - \boldsymbol{A}\| + \|\boldsymbol{A}^{+}\boldsymbol{A}\boldsymbol{A}^{+} - \boldsymbol{A}^{+}\| \\ &\quad + \|(\boldsymbol{A}\boldsymbol{A}^{+})^T - \boldsymbol{A}\boldsymbol{A}^{+}\| + \|(\boldsymbol{A}^{+}\boldsymbol{A})^T - \boldsymbol{A}^{+}\boldsymbol{A}\|. \end{aligned}$$

Another criterion is the width of $\boldsymbol{A}^{+}$.

Use of an interval hull is not necessary if we can work with generalized interval arithmetic. In this case we have to use the union of a finite number of boxes.

Even for real matrices the result can be an interval matrix because of rounding errors. The proposed algorithm can be applied to the computation of ε-extensions $\boldsymbol{A}_\varepsilon$ of real matrices A. It can be used for unstable real matrix pseudo-inversion detection. When $\boldsymbol{A}_\varepsilon^{+}$ is wide or the accuracy criterion $\boldsymbol{t}$ has a large upper bound, the pseudo-inversion operation is unstable.

The recurrent interval Greville algorithm can be applied not by columns but by rows. That is recommended if $m > n$, that is, if the count of rows is greater than the count of columns. The matrix

$$\boldsymbol{A}^{+} \bigcap \left((\boldsymbol{A}^T)^{+}\right)^T \tag{2}$$

can be taken as the result.

The proposed interval Greville algorithm has the property of monotonicity: if $\boldsymbol{A} \subset \boldsymbol{B}$ then $\boldsymbol{A}^+ \subset \boldsymbol{B}^+$. This follows from the monotonicity of interval arithmetic operations.

5 Modified Greville Algorithm with Bisections

The basic Greville algorithm can lead to huge overestimations of pseudo-inverse matrices, but we can obtain better results using monotonicity of operations. Let us see any bisection of the given matrix $\boldsymbol{A}$ into two interval matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ by any coordinate. Because $\boldsymbol{A}_1 \bigcup \boldsymbol{A}_2 = \boldsymbol{A}$, $\boldsymbol{A}_1^+ \subset \boldsymbol{A}^+$ and $\boldsymbol{A}_2^+ \subset \boldsymbol{A}^+$ we have that $\boldsymbol{A}^+ \subset \square \left(\boldsymbol{A}_1^+ \bigcup \boldsymbol{A}_2^+\right)$. We will not lose any solution A^+, but the bounds of the estimation can be better.

We can construct any division of $\boldsymbol{A}_i, i = 1, \dots, p$, such as

$$\bigcup_{i=1}^{p} \boldsymbol{A}_i = \boldsymbol{A},$$

and then

$$\boldsymbol{A}^+ \subset \square \bigcup_{i=1}^{p} \boldsymbol{A}_i^+.$$

Let $T \in \mathbb{Z} \geq 0$ be the depth (count) of bisections. On every step $0 \leq t \leq T$ we use bisection by the widest coordinate. The recursive algorithm can be applied to improve estimation of interval pseudo-inverse matrices.

Algorithm 2.
Call: IntPseudoInverseBisect($\boldsymbol{A}$, T, t).
Input: $\boldsymbol{A}$ – interval matrix, T – depth of bisections, t – current depth.
Output: $\boldsymbol{A}^+$ – estimation of the interval pseudo-inverse matrix.

Step 1. If $t = T$ then we return A^+ computed by the basic interval Greville algorithm:

$$A^+ = \text{IntPseudoInverse}(\boldsymbol{A}).$$

Step 2. If $t < T$ then $(i^*, j^*) = \text{argmax}_{i,j}\, \text{width}\, A_{ij}$. We bisect by the element (i^*, j^*) into matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$.

Step 3. Then we make two recursive calls to this algorithm:

$$\boldsymbol{A}_i^+ = \text{IntPseudoInverseBisect}(\boldsymbol{A}_i, T, t+1), \quad i = 1, 2.$$

Step 4. Return $\square \left(\boldsymbol{A}_1^+ \bigcup \boldsymbol{A}_2^+\right)$.

◁

If the depth $T = 0$ then the modified algorithm is basic interval Greville algorithm. Obviously, the algorithm with a greater value of T can give tighter results. For every increment of T, the computation time increases by a factor of approximately 2 because two times as many matrices are computed.

Instead of bisection by the widest coordinate we can bisect by any other strategy, for example by the smallest coordinate or randomly. Also we can use intersections of results computed with different bisection strategies.

6 Examples of Interval Pseudo-Inverse Matrices

The proposed algorithms were realized in the C++ language using the C++ Builder XE2 environment. The developed computer program was used for numerical experiments on a computer with an Intel Celeron 1.86 GHz CPU, 1 Gb RAM and the Microsoft Windows Vista operating system.

Let us see some examples of computation of interval pseudo-inverse matrices. Our examples show how the results depend on the parameter T of Algorithm 2.

1. Let $\boldsymbol{A} = [1, 2] \in \mathbb{IR}$ be given. The true result is

$$\boldsymbol{A}^+ = [0.5, 1].$$

Table 1 contains estimation and accuracy of computations. We see from it that accuracy of estimation of the interval pseudo-inverse matrix increases when the depth T increases. For $T = 15$ we already have very good estimation.

Table 1: Results for Example 1

Depth T	Estimation of $\boldsymbol{A}^+$	$\boldsymbol{t}$	width $(\boldsymbol{A}^+)$	Time, min:sec
0	$[0.2500, 2.0000]$	$[0, 137.1875]$	1.7500	00:00
1	$[0.3750, 1.5000]$	$[0, 55.7969]$	1.1250	00:00
5	$[0.4921, 1.0313]$	$[0, 17.3699]$	0.5392	00:00
10	$[0.4997, 1.0010]$	$[0, 15.7992]$	0.5013	00:00
15	$[0.4999, 1.0001]$	$[0, 15.7501]$	0.5002	00:00
20	$[0.4999, 1.0001]$	$[0, 15.7501]$	0.5002	00:22

2. Let

$$\boldsymbol{A} = \begin{pmatrix} [2, 4] & [-2, 1] \\ [-1, 2] & [2, 4] \end{pmatrix} \in \mathbb{IR}^{2\times 2}$$

be a well-known matrix taken from [3]. This matrix is regular, and its interval inverse matrix can be computed exactly using interval determinant and adjoint matrix [8]:

$$\boldsymbol{A}^{-1} = \begin{pmatrix} [1/6, 1] & [-1/2, 1] \\ [-1, 1/2] & [1/6, 1] \end{pmatrix}$$

Results are given in Tables 2 and 3. These tables show that the basic interval Greville algorithm and its extension with bisections when the depth T is small give unsatisfactory results. Increasing T leads to increased accuracy. We also note that the time for computation with $T = 20$ is large enough.

3. Let us consider the matrix from the system of linear equations [4]

$$\begin{pmatrix} [2, 3] & [0, 1] \\ [1, 2] & [2, 3] \end{pmatrix}.$$

Results are given in Tables 4 and 5. Interpretation of results is the same as for Example 2. Contrary to the previous example, we obtain a useful result even for depth $T = 5$.

4. Let us consider the matrix

$$\begin{pmatrix} [1, 2] & [2, 3] \\ [1, 2] & [-1, 1] \\ [2, 3] & [0, 1] \end{pmatrix}.$$

Table 2: Estimation of the interval pseudo-inverse matrix for Example 2

Depth T	Estimation of $\boldsymbol{A}^+$	Time, min:sec
0	$\begin{pmatrix} [-\infty,+\infty] & [-\infty,+\infty] \\ [-\infty,+\infty] & [-\infty,+\infty] \end{pmatrix}$	00:00
1	$\begin{pmatrix} [-\infty,+\infty] & [-\infty,+\infty] \\ [-\infty,+\infty] & [-\infty,+\infty] \end{pmatrix}$	00:00
5	$\begin{pmatrix} [-\infty,+\infty] & [-\infty,+\infty] \\ [-\infty,+\infty] & [-\infty,+\infty] \end{pmatrix}$	00:00
10	$\begin{pmatrix} [-0.6505, 24.1336] & [-13.7949, 21.4130] \\ [-21.4130, 13.7949] & [0.0426, 12.6689] \end{pmatrix}$	00:00
15	$\begin{pmatrix} [0.0734, 3.5609] & [-2.2261, 4.4174] \\ [-4.4174, 2.2261] & [0.0999, 3.3209] \end{pmatrix}$	00:03
20	$\begin{pmatrix} [0.1413, 1.8900] & [-1.1394, 2.1210] \\ [-2.1210, 1.1394] & [0.1298, 1.8744] \end{pmatrix}$	02:06

Table 3: Accuracy of the interval pseudo-inverse matrix for Example 2

Depth T	$\boldsymbol{t}$	width $(\boldsymbol{A}^+)$
0	$[0, +\infty]$	$+\infty$
1	$[0, +\infty]$	$+\infty$
5	$[0, +\infty]$	$+\infty$
10	$[0, 80007421.7958]$	35.2078
15	$[0, 172622.8187]$	6.6434
20	$[0, 25138.7173]$	3.2604

Table 4: Estimation of the interval pseudo-inverse matrix for Example 3

Depth T	Estimation of $\boldsymbol{A}^+$	Time, min:sec
0	$\begin{pmatrix} [-\infty,+\infty] & [-\infty,+\infty] \\ [-\infty,+\infty] & [-\infty,+\infty] \end{pmatrix}$	00:00
1	$\begin{pmatrix} [-\infty,+\infty] & [-\infty,+\infty] \\ [-\infty,+\infty] & [-\infty,+\infty] \end{pmatrix}$	00:00
5	$\begin{pmatrix} [0.2240, 13.2170] & [-7.7032, 0.3201] \\ [-15.7515, -0.0134] & [0.0624, 9.7063] \end{pmatrix}$	00:00
10	$\begin{pmatrix} [0.2881, 3.0290] & [-1.8939, 0.0679] \\ [-3.7079, -0.0703] & [0.2260, 2.8724] \end{pmatrix}$	00:00
15	$\begin{pmatrix} [0.3117, 1.7873] & [-1.0653, 0.0324] \\ [-1.9130, -0.1000] & [0.3102, 1.7664] \end{pmatrix}$	00:03
20	$\begin{pmatrix} [0.3227, 1.3076] & [-0.7342, 0.0159] \\ [-1.3567, -0.1082] & [0.3258, 1.3134] \end{pmatrix}$	02:19

Table 5: Accuracy of the interval pseudo-inverse matrix for Example 3

Depth T	$\boldsymbol{t}$	width $(\boldsymbol{A}^{+})$
0	$[0, +\infty]$	$+\infty$
1	$[0, +\infty]$	$+\infty$
5	$[0, 2772398.2393]$	15.7381
10	$[0, 15187.8851]$	3.6376
15	$[0, 2594.6570]$	1.8129
20	$[0, 1046.6976]$	1.2482

This matrix is rectangular, so an interval inverse matrix is undefined for it. Results are shown in Tables 6 and 7. We note the increase in the accuracy of estimation when T is increased, as in Example 3.

Table 6: Estimation of the interval pseudo-inverse matrix for Example 4

Depth T	Estimation of $\boldsymbol{A}^{+}$	Time, min:sec
0	$\begin{pmatrix} [-\infty, +\infty] & [-\infty, +\infty] \\ [-\infty, +\infty] & [-\infty, +\infty] \\ [-\infty, +\infty] & [-\infty, +\infty] \end{pmatrix}^T$	00:00
1	$\begin{pmatrix} [-\infty, +\infty] & [-\infty, +\infty] \\ [-\infty, +\infty] & [-\infty, +\infty] \\ [-\infty, +\infty] & [-\infty, +\infty] \end{pmatrix}^T$	00:00
5	$\begin{pmatrix} [-\infty, +\infty] & [-\infty, +\infty] \\ [-\infty, +\infty] & [-\infty, +\infty] \\ [-\infty, +\infty] & [-\infty, +\infty] \end{pmatrix}^T$	00:00
10	$\begin{pmatrix} [-235.8145, 0.6544] & [0.0145, 230.6975] \\ [-87.1795, 111.6408] & [-108.9520, 85.3497] \\ [-0.8928, 273.3233] & [-266.8085, 1.2718] \end{pmatrix}^T$	00:00
15	$\begin{pmatrix} [-8.4930, 0.2954] & [0.0939, 9.5901] \\ [-3.3600, 3.9459] & [-4.4933, 3.8531] \\ [-0.0017, 10.0504] & [-10.7529, 0.2457] \end{pmatrix}^T$	00:06
20	$\begin{pmatrix} [-2.3703, 0.1954] & [0.1644, 3.1050] \\ [-0.9243, 1.4099] & [-1.6295, 1.2288] \\ [0.0879, 3.1175] & [-3.3554, 0.1615] \end{pmatrix}^T$	03:46

5. Let us consider the matrix with crisp values

$$\begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}.$$

Table 7: Accuracy of the interval pseudo-inverse matrix for Example 4

Depth T	$\boldsymbol{t}$	width $(\boldsymbol{A}^+)$
0	$[0, +\infty]$	$+\infty$
1	$[0, +\infty]$	$+\infty$
5	$[0, +\infty]$	$+\infty$
10	$[0, 950238500568, 4360]$	274.21601
15	$[0, 2238047.1275]$	10.9985
20	$[0, 34090.8096]$	3.5168

Even the basic interval Greville algorithm gives a good result:

$$\begin{pmatrix} [0.0499, 0.0501] & [0, 0] & [0.0499, 0.0501] \\ [0.1500, 0.1500] & [0, 0] & [0.1500, 0.1500] \end{pmatrix}.$$

Computation errors in this case are insignificant.

6. If we consider an interval extension of the matrix from Example 5

$$\begin{pmatrix} [0.9999, 1.001] & [2.9999, 3.0001] \\ [-0.0001, 0.0001] & [-0.0001, 0.0001] \\ [0.9999, 1.001] & [2.9999, 3.0001] \end{pmatrix},$$

we obtain useless result for the basic and modified algorithms:

$$\begin{pmatrix} [-\infty, \infty] & [-\infty, \infty] & [-\infty, \infty] \\ [-\infty, \infty] & [-\infty, \infty] & [-\infty, \infty] \end{pmatrix}.$$

The reason for such a result is the discontinuous nature of pseudo-inversion due to existence of real matrices of different ranks within the interval matrix.

7 Application to Estimation of Linear Equations Systems Solutions and Pseudo-Solutions

Interval pseudo-inversion can be used to estimate solutions or pseudo-solutions of systems of linear equations. Although is not suitable for optimal estimation, it can be efficiently used for an iterative improvement of estimation by the interval Gauss-Seidel method, for instance.

It is known [1] that solutions or minimal pseudo-solutions (in the least squared sense) of a system $Ax = b$ in the real case can be found from the relation

$$x^* = A^+ b, \tag{3}$$

where A and b are a known matrix and a vector, respectively, and x is an unknown vector. Relation (3) allows us to compute a vector x^* such that $\|Ax - b\|_2$ and $\|x\|_2$ are minimal. For an interval system of linear algebraic equations

$$\boldsymbol{A}x = \boldsymbol{b}$$

we have to find an $\boldsymbol{x}$ which contains the set of all solutions x for any $A \subset \boldsymbol{A}$ and $b \subset \boldsymbol{b}$. Hence,

$$\boldsymbol{x}^* = \boldsymbol{A}^+\boldsymbol{b} \tag{4}$$

contains all solutions and minimal pseudo-solutions if the system $Ax = b$ is inconsistent. Formula (4) gives an analytic enclosure of the solution to an interval linear system.

Let us consider the example from [4]:

$$\begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix}.$$

The minimal interval box containing all solutions is

$$\boldsymbol{x}_{\min} = \begin{pmatrix} [-120,90] \\ [-60,240] \end{pmatrix}.$$

Interval pseudo-inverse matrices for this matrix $\boldsymbol{A}$ are given in Table 4. Results are given in Table 8. The interval pseudo-inverse matrix computed with parameter $T = 20$ is close to interval solution and can be used as good initial approximation.

Table 8: Estimation of the set of solutions

Depth T	Estimation of $\boldsymbol{x}^*$	width $(\boldsymbol{x}^*)$	Time, min:sec
0	$\begin{pmatrix} [-\infty,+\infty] \\ [-\infty,+\infty] \end{pmatrix}$	$+\infty$	00:00
5	$\begin{pmatrix} [-1848.7500, 1662.8295] \\ [-1886.4302, 2329.5039] \end{pmatrix}$	4215.9341	00:00
10	$\begin{pmatrix} [-454.5175, 379.7517] \\ [-431.3799, 689.3674] \end{pmatrix}$	1120.7472	00:00
15	$\begin{pmatrix} [-255.6499, 222.2492] \\ [-210.9405, 423.9334] \end{pmatrix}$	634.8739	00:03
20	$\begin{pmatrix} [-176.1879, 160.7234] \\ [-143.2525, 315.2092] \end{pmatrix}$	458.4617	02:19

Formula (4) also can solve systems with a rectangular interval matrix $\boldsymbol{A}$. Let us consider the system

$$\begin{pmatrix} [1,2] & [2,3] \\ [1,2] & [-1,1] \\ [2,3] & [0,1] \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} [-20,20] \\ [10,90] \\ [0,100] \end{pmatrix}.$$

This system is overdetermined. Instead of its solution, we have to find a pseudo-solution. Interval pseudo-inverse matrices for this matrix $\boldsymbol{A}$ are given in Table 6. Results are shown in Table 9. The vector on the right side is wide, so the result is also wide, even for $T = 20$.

The value $\|\boldsymbol{Ax} - \boldsymbol{b}\|_2$ estimates the computation accuracy. If its lower bound is greater than zero, the system is inconsistent. So the system has no solutions for any $A \subset \boldsymbol{A}$ and $b \subset \boldsymbol{b}$.

Table 9: Estimations of the set of solutions

Depth T	Estimation of $\boldsymbol{x}^*$	width $(\boldsymbol{x}^*)$	Time, min:sec
0	$\begin{pmatrix}[-\infty,+\infty]\\ [-\infty,+\infty]\end{pmatrix}$	$+\infty$	00:00
10	$\begin{pmatrix}[-12651.7095, 42096.2877]\\ [-41100.4630, 12422.5933]\end{pmatrix}$	$54747, 9972$	00:00
15	$\begin{pmatrix}[-472.4241, 1530.0130]\\ [-1671.4830, 563.1493]\end{pmatrix}$	2234.6323	00:06
20	$\begin{pmatrix}[-130.5874, 486.0415]\\ [-544.2885, 188.8328]\end{pmatrix}$	733.1211	03:46

8 Conclusion

This article proposes basic and modified Greville algorithms for the computation of estimations of interval pseudo-inverse matrices, which can be used for analytic construction of upper bounds of solutions of systems of linear equations with matrices of any rank. Our algorithms also can be used for initial approximation of solution bounds and further application of known methods such as Gauss-Seidel. Interval pseudo-inverse matrices are also useful for 2-norm optimization problems. In the linear case, they are equivalent to problems of computation of pseudo-solutions of systems of linear equations.

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