An interpretation of consistent belief functions in terms of simplicial complexes

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Abstract

In this paper we pose the study of consistent belief functions (cs.b.f.s) in the framework of the geometric approach to the theory of evidence. As cs.b.f.s are those belief functions whose plausibility assignment is a possibility distribution, their study is a step towards a unified geometric picture of a wider class of fuzzy measures. We prove that, analogously to consonant belief functions, cs.b.f.s form a simplicial complex, and point out the similarity between the consistent complex and the complex of singular belief functions, i.e. belief functions whose core is a proper subset of their domain. Finally, we argue that the notion of complex brings together the possibilistic and probabilistic approximation problems by introducing a convex decomposition of b.f.s in terms of "consistent coordinates" on the complex, closely related to the pignistic transformation.

1 Introduction

The *theory of evidence* (Shafer 1976) is one the most popular approaches to uncertainty description. The notion of belief function (b.f.) was originally introduced by A. Dempster (Dempster 1968) in terms of multi-valued maps, but equivalent alternative definitions can be given in terms of random sets (Nguyen & Wang 1997), compatibility relations, inner measures (Fagin & Halpern 1988), and credal sets. In robust Bayesian statistics there is a large literature on the study of convex sets of probability distributions (Cozman 1999; Berger 1990; Seidenfeld & Wasserman 1993). Melkonyan et al. (Melkonyan & Chambers 2006), for example, recently used results from convex geometry to obtain representations of the prior and posterior degrees of imprecision in terms of width functions and difference bodies.

Instead of working in the probability simplex, it is possible to reason on a different level of abstraction by representing belief measures as points of a Cartesian space (Cuzzolin 2007b). As a b.f. $b : 2^\Theta \rightarrow [0,1]$ is completely specified by its $N-1$, $N = 2^{|\Theta|}$ belief values

$$\{b(A) \; \forall A \subseteq \Theta, A \neq \emptyset\}$$

and can then be seen as a vector

$$v = [v_A = b(A), \emptyset \subsetneq A \subseteq \Theta]'$$

of $\mathbb{R}^{N-1}$ (where $'$ denotes the transpose of a matrix) which live in a simplex called "belief space".

This "geometric approach" was originally motivated by the approximation problem: new probabilistic approximations of belief functions have been inferred by geometric considerations (Cuzzolin 2007e), and new properties of classical ones investigated (Cuzzolin 2007a). However, it can also be seen as the symptom of a strict relationship between combinatorics and subjective probability. This link has never been systematically explored, even though some work has been recently done in this direction, specially by M. Grabish, Yao (Yao & Lingras 1998), and Barthelemy (Barthelemy 2000). For instance, new models for the theory of evidence based on the Moebius inverses of plausibility and commonality functions can be formulated (Cuzzolin 2007d). In this perspective we have recently started to study the geometric properties of *consonant* belief functions (co.b.f.s) (Cuzzolin 2004b). Consonant and *consistent* belief functions (cs.b.f.s) (Dubois & Prade 1990; Joslyn & Klir 1992; Baroni 2004) are the counterparts in the theory of evidence of *possibility measures* (Dubois & Prade 1988).

1.1 Contribution and outline

In this paper we move forward to analyze the convex geometry of consistent belief functions as an additional step towards a unified geometric picture of a wider class of uncertainty measures.

After introducing the basic notions of theory of evidence and possibility theory, and the role of consistent b.f.s as counterparts of plausibility distributions in the ToE (Section 2), we briefly recall in Section 3 the geometric approach to b.f.s. In Section 4 we prove that, analogous to the case of consonant b.f.s, the space of consistent b.f.s forms a *simplicial complex* (Dubrovin, Novikov, & Fomenko 1986). The similarity between consistent complex $\mathcal{CS}$ and the complex of *non-combinable* belief functions $Sing$ is illustrated and commented in Section 5. Finally, in Section 6 we consider the consistent approximation problem in the framework of the consistent simplex, and show that each b.f. can be given a set of "consistent" coordinates which are strictly related to the pignistic transformation (Smets & Kennes 1994).

2 Two uncertainty theories

2.1 Belief functions

In the *theory of evidence* (Shafer 1976) a *basic probability assignment* (b.p.a.) over a finite set *(frame of discernment)* Θ is a function $m : 2^\Theta \rightarrow [0,1]$ on its power set $2^\Theta = \{A \subseteq \Theta\}$ such that

$$m(\emptyset) = 0, \quad \sum_{A \subseteq \Theta} m(A) = 1, \quad m(A) \geq 0 \; \forall A \subseteq \Theta.$$

Subsets of Θ associated with non-zero values of m are called *focal elements* (f.e.s), and their intersection *core*:

$$\mathcal{C}_b \doteq \bigcap_{A \subseteq \Theta \,:\, m(A) \neq 0} A.$$

The *belief function* (b.f.) $b : 2^\Theta \rightarrow [0,1]$ associated with a b.p.a. m on Θ is defined as

$$b(A) = \sum_{B \subseteq A} m(B).$$

Conversely, the unique b.p.a. m_b associated with a given belief function b can be recovered by means of the *Moebius inversion formula*

$$m_b(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} b(B). \qquad (1)$$

A dual mathematical representation of the evidence encoded by a belief function b is the *plausibility function* (pl.f.) $pl_b : 2^\Theta \rightarrow [0,1]$, where

$$pl_b(A) \doteq 1 - b(A^c) = 1 - \sum_{B \subseteq A^c} m_b(B) = \sum_{B \cap A \neq \emptyset} m_b(B).$$

In the theory of evidence a probability function is simply a special belief function assigning non-zero masses to singletons only (*Bayesian* b.f.): $m_b(A) = 0$ for $|A| > 1$. *Consonant* belief functions, i.e. b.f.s whose focal elements are *nested*, are characterized by the following Proposition.

Proposition 1. *If b is a b.f. with pl.f. pl_b, then b is consonant iff*

$$pl_b(A) = \max_{x \in A} pl_b(x)$$

for all non-empty $A \subseteq \Theta$.

A b.f. is said to be *consistent* if its core is non-empty. Consonant b.f.s are obviously consistent, but the vice-versa does not hold.

2.2 Possibility measures

Possibility theory (Dubois & Prade 1988) concerns instead *possibility measures*, i.e. functions $Pos : 2^\Theta \rightarrow [0,1]$ such that $Pos(\emptyset) = 0$, $Pos(\Theta) = 1$ and

$$Pos(\bigcup_i A_i) = \sup_i Pos(A_i)$$

for any family of subsets $\{A_i | A_i \in 2^\Theta, i \in I\}$, where I is an arbitrary set index. Each possibility measure Pos is uniquely characterized by a *membership function* or *possibility distribution* $\pi : \Theta \rightarrow [0,1]$, $\pi(x) \doteq Pos(\{x\})$, via the formula

$$Pos(A) = \sup_{x \in A} \pi(x).$$

2.3 A bridge between belief and possibility

Many authors, like Yager (Yager 1999) and Romer (Roemer & Kandel 1995) among the others, have studied the connection between fuzzy theory and ToE (Caro & Nadjar 1999). Klir et al. (Klir, Zhenyuan, & Harmanec 1997) and Heilpern (Heilpern 1997), for instance, discussed the relations among fuzzy and belief measures and possibility theory. The points of contact between evidential formalism and possibility theory have been briefly investigated in (Smets 1990).

Many of the studies cited above have pointed out that possibility measures coincide in the theory of evidence with the class of consonant belief functions. Let us call *plausibility assignment* (pl.ass.) $\bar{pl}_b$ (Joslyn 1991) the restriction of the plausibility function to singletons

$$\bar{pl}_b(x) = pl_b(\{x\}).$$

From Proposition 1 it follows immediately that

Proposition 2. *The plausibility function pl_b associated with a belief function b on a domain Θ is a possibility measure iff b is consonant, with the pl.ass. playing the role of the membership function: $\pi = \bar{pl}_b$.*

2.4 Cs.b.f.s and possibility distributions

However, it is not necessary for a belief function to be consonant in order for its plausibility assignment to be an admissible possibility *distribution* (Joslyn 1991).

Lemma 1. *b is consistent iff $\exists\, x \in \Theta$ s.t. $\bar{pl}_b(x) = 1$.*

$\bar{pl}_b(x) = 1$ for some $x \in \Theta$ is equivalent to $\sum_{A \ni x} m_b(A) = 1$. This is true iff

$$\bigcap_{m_b(A) \neq 0} A \ni x \neq \emptyset.$$

Theorem 1. *The plausibility assignment $\bar{pl}_b$ associated with a b.f. b is a possibility distribution iff the b.f. b is consistent.*

Proof. Given Lemma 1 this is equivalent to say that $\bar{pl}_b$ is a possibility distribution iff $\bar{pl}_b(x) = 1$ for some $x \in \Theta$. But by definition of possibility measures $Pos(\cup_i A_i) = \sup_i Pos(A_i)$ and $Pos(\Theta) = 1$ so that

$$Pos(\Theta) = 1 = Pos(\cup_{x \in \Theta} x) = \sup_x Pos(x) = \sup_x \pi(x)$$

for all membership functions: $\pi(x) = 1$ for some $x \in \Theta$. □

2.5 A unified description in terms of complexes

Possibility theory (in the finite case) is then embedded in the ToE. Two are the elements of this relationship: consonant b.f.s as representatives of possibility measures, and consistent b.f.s as counterparts of membership functions. As we will see in Section 5 the notion of consistency is also related to that of *combinability* in Dempster's framework, as the condition under which belief measures can be merged is expressed in terms of possibility distributions.

Both semantics of consistent b.f.s can be seen in an unified fashion by recurring to the language of convex geometry, and in particular the notion of *simplicial complex* (Dubrovin, Novikov, & Fomenko 1986). The formalism of simplicial complexes is powerful enough to describe both the nexus between consistency and combinability, and the link between possibilistic and probabilistic approximation. We first recall the bases of the geometric approach to uncertainty theory.

3 A geometric approach

3.1 The space of belief functions

Given a frame of discernment Θ, a b.f. $b : 2^\Theta \to [0,1]$ is completely specified by its $N-1$ belief values

$$\{b(A), A \subseteq \Theta, A \neq \emptyset\},$$

$N \doteq 2^{|\Theta|}$, and can then be represented as a point of $\mathbb{R}^{N-1}$. The *belief space* associated with Θ is the set of points $\mathcal{B}$ of $\mathbb{R}^{N-1}$ which correspond to b.f.s. Let us call

$$b_A \doteq b \in \mathcal{B} \; s.t. \; m_b(A) = 1, \; m_b(B) = 0 \; \forall B \neq A \quad (2)$$

the unique b.f. assigning all the mass to a *single* subset A of Θ (A-th *basis* belief function).
We proved that (Cuzzolin 2007b), denoting by $\mathcal{E}_b$ the list of focal elements of b,

Proposition 3. *The set of all the belief functions with focal elements in a given collection L is closed and convex in $\mathcal{B}$:*

$$\{b : \mathcal{E}_b \subseteq L\} = Cl(b_A : A \in L),$$

where Cl denotes the convex closure operator:

$$Cl(b_1, ..., b_k) = \Big\{ b \in \mathcal{B} : b = \alpha_1 b_1 + \cdots + \alpha_k b_k, \sum_i \alpha_i = 1, \alpha_i \geq 0 \; \forall i \Big\}.$$

As a consequence, the belief space $\mathcal{B}$ is the convex closure of all the basis belief functions b_A,

$$\mathcal{B} = Cl(b_A, \; \emptyset \subsetneq A \subseteq \Theta).$$

More precisely $\mathcal{B}$ is an $N-2$-dimensional *simplex*, i.e. the convex closure of $N-1$ (affinely independent (Dubrovin, Novikov, & Fomenko 1986)) points of the Euclidean space $\mathbb{R}^{N-1}$. The *faces* of a simplex are all the simplices generated by a subset of its vertices. Each belief function $b \in \mathcal{B}$ can be written as a convex sum as

$$b = \sum_{\emptyset \subsetneq A \subseteq \Theta} m_b(A) b_A. \quad (3)$$

Since a probability is a b.f. assigning non zero masses to singletons only, Proposition 3 implies that the set $\mathcal{P}$ of all Bayesian b.f.s is the simplex

$$\mathcal{P} = Cl(b_x, x \in \Theta).$$

3.2 Binary case

As an example let us consider a frame of discernment containing only two elements, $\Theta_2 = \{x, y\}$. In this very simple case each b.f. $b : 2^{\Theta_2} \to [0,1]$ is completely determined by its belief values $b(x), b(y)$ as it is always true that $b(\Theta) = 1, b(\emptyset) = 0 \; \forall b \in \mathcal{B}$. We can then represent b as the vector

$$[b(x) = m_b(x), b(y) = m_b(y)]'$$

of $\mathbb{R}^{N-2} = \mathbb{R}^2$ (since $N = 2^2 = 4$). Since $m_b(x) \geq 0$, $m_b(y) \geq 0$, and $m_b(x) + m_b(y) \leq 1$ the set $\mathcal{B}_2$ of all the possible belief functions on Θ_2 is the triangle of Figure 1, whose vertices are the points $b_\Theta = [0,0]'$, $b_x = [1,0]'$, $b_y = [0,1]'$ which correspond respectively to the vacuous belief function b_Θ ($m_{b_\Theta}(\Theta) = 1$), the Bayesian b.f. b_x with $m_{b_x}(x) = 1$, and the Bayesian b.f. b_y with $m_{b_y}(y) = 1$. The region $\mathcal{P}_2$ of all Bayesian b.f.s on Θ_2 is the segment

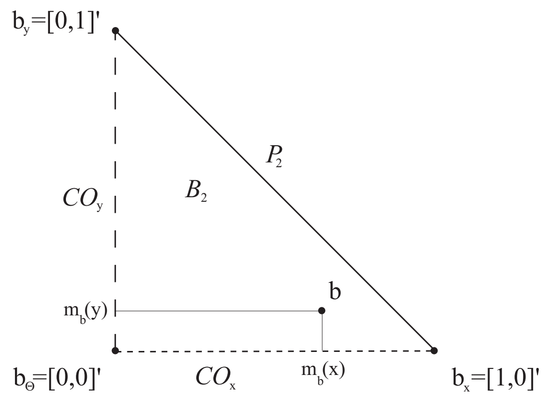

Figure 1: The belief space $\mathcal{B}$ for a binary frame is a triangle in $\mathbb{R}^2$ whose vertices are the basis b.f.s focused on $\{x\}, \{y\}$ and Θ, (b_x, b_y, b_Θ respectively). The probability region is the segment $Cl(b_x, b_y)$, while consonant and consistent b.f.s live in the union of two segments $\mathcal{CS}_x = \mathcal{CO}_x = Cl(b_\Theta, b_x)$ and $\mathcal{CS}_y = \mathcal{CO}_y = Cl(b_\Theta, b_y)$.

$Cl(b_x, b_y)$. In the binary case consonant belief functions can have as sets of focal elements one between $\{\{x\}, \Theta_2\}$ and $\{\{y\}, \Theta_2\}$. Therefore the space of co.b.f.s $\mathcal{CO}_2$ is the union of two convex components

$$\mathcal{CO}_2 = \mathcal{CO}_x \cup \mathcal{CO}_y = Cl(b_\Theta, b_x) \cup Cl(b_\Theta, b_y)$$

and coincides with the region $\mathcal{CS}_2$ of consistent b.f.s, as the latter cannot have both $\{x\}$ and $\{y\}$ as focal elements.

3.3 The consonant complex

In the general case (Cuzzolin 2004b) the geometry of co.b.f.s can be described by means of the notion of simplicial complex (Dubrovin, Novikov, & Fomenko 1986).

Definition 1. *A* simplicial complex *is a collection Σ of simplices which satisfies the following properties:*
1. if a simplex belongs to Σ, then all its faces of any dimension belong to Σ;
2. the intersection of any two simplices is a face of both the intersecting simplices.

Let us consider for instance two triangles (2-dimensional simplices) in $\mathbb{R}^2$. Roughly speaking, the second condition says that their intersection cannot contain points of their interiors (Figure 2-left) or be an arbitrary subset of their borders (middle), but has to be a *face* (right, in this case a single vertex). It can be proven that (Cuzzolin 2004b)

Proposition 4. *The region $\mathcal{CO}$ of consonant belief functions in the belief space is a simplicial complex.*

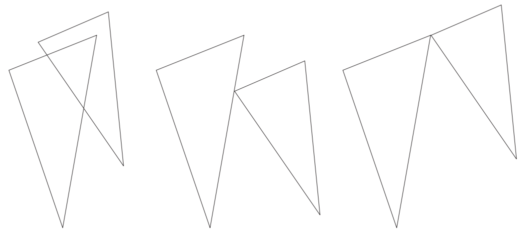

Figure 2: Constraints on the intersection of simplices in a complex. Only the right-hand pair meets condition 2. of the definition of simplicial complex.

More precisely, $\mathcal{CO}$ is a collection of maximal simplices $Cl(b_{A_1}, ..., b_{A_n})$, each of them associated with a maximal chain of subsets in 2^Θ: $A_1 \subset \cdots \subset A_n$, $|A_i| = i$. In the binary example $\mathcal{CO}_x$ and $\mathcal{CO}_y$ are the two maximal simplices forming a simplicial complex (as they intersect in a vertex).

4 Geometry of the consistent subspace

As co.b.f.s and cs.b.f.s are associated with possibility measures and distributions respectively, it is natural to conjecture that consistent belief functions may have a similar geometric behavior. All possible lists of f.e.s associated with consistent b.f.s obviously correspond to all possible collections of intersecting events:

$$\{A_1, ..., A_m \subseteq \Theta : \bigcap_{i=1}^m A_i \neq \emptyset\}.$$

Geometrically, Proposition 3 implies that all the b.f.s whose focal elements belong to such a collection form the simplex $Cl(b_{A_1}, ..., b_{A_m})$. This collection is "maximal" when it is not possible to add another event A_{m+1} such that $\cap_{i=1}^{m+1} A_i \neq \emptyset$. Collections of events with non-empty intersection are maximal iff they have the form

$$\{A \subseteq \Theta : A \ni x\} \quad (4)$$

for some singleton $x \in \Theta$. By Proposition 3 the region of cs.b.f.s is the union of a number of simplices, each associated with a maximal collection of the form (4):

$$\mathcal{CS} = \bigcup_{x \in \Theta} Cl(b_A, A \ni x).$$

The number of such maximal simplices of $\mathcal{CS}$ is then obviously the number of singletons, i.e. the cardinality $n \doteq |\Theta|$ of Θ. Each of them has

$$|\{A : A \ni x\}| = |\{A \subseteq \Theta : A = \{x\} \cup B, B \subset \{x\}^c\}| =$$

$= 2^{|\{x\}^c|} = 2^{n-1}$ vertices, so that their dimension as simplices of $\mathcal{B}$ is $2^{n-1} - 1 = \frac{\dim \mathcal{B}}{2}$ (as the dimension of the whole belief space is $\dim \mathcal{B} = 2^n - 2$).
As b_Θ belongs to all maximal simplices $\mathcal{CS}$ is connected.

4.1 A ternary example

In the case of a frame of size 3 $\Theta = \{x, y, z\}$ all b.f.s $b \in \mathcal{B}_3$ are 6-dimensional vectors:

$$[b(x), b(y), b(z), b(\{x, y\}), b(\{x, z\}), b(\{y, z\})]'.$$

Let us pick for instance two possible cores $\mathcal{C}_1 = \{x, y\}$ and $\mathcal{C}_2 = \{x\}$. The lists of focal elements associated with cs.b.f.s with cores $\mathcal{C}_1$ and $\mathcal{C}_2$ are respectively

$$\mathcal{E}_b^x = \{A \ni x\} = \{\{x\}, \{x, y\}, \{x, z\}, \Theta\}$$
$$\mathcal{E}_b^{x,y} = \{A \supseteq \{x, y\}\} = \{\{x, y\}, \Theta\} \subsetneq \mathcal{E}_b^x$$

which confirms that all maximal lists of f.e.s for consistent b.f.s are associated with singletons of Θ (x in this case). In the ternary case the maximal collections (4) of consistent f.e.s are then $\{A \ni x\}$, $\{A \ni y\}$, and $\{A \ni z\}$. The number of simplicial components is 3, and their dimension $|\{A \ni x\}| - 1 = 3$:

$$Cl(b_A : A \ni x) = Cl(b_x, b_{\{x,y\}}, b_{\{x,z\}}, b_\Theta),$$
$$Cl(b_A : A \ni y) = Cl(b_y, b_{\{x,y\}}, b_{\{y,z\}}, b_\Theta),$$
$$Cl(b_A : A \ni z) = Cl(b_z, b_{\{x,z\}}, b_{\{y,z\}}, b_\Theta).$$

The geometry of consistent belief functions in the ternary frame can then be represented as in Figure 3.
The consonant subspace $\mathcal{CO}_3$, for comparison, is the union

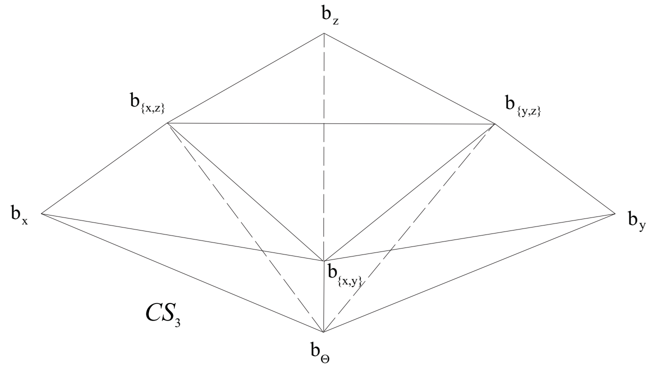

Figure 3: The consistent $\mathcal{CS}_3$ subspace for $\Theta = \{x, y, z\}$.

of the six simplices $Cl(b_x, b_{\{x,z\}}, b_\Theta)$, $Cl(b_x, b_{\{x,y\}}, b_\Theta)$, $Cl(b_y, b_{\{x,y\}}, b_\Theta)$, $Cl(b_y, b_{\{y,z\}}, b_\Theta)$, $Cl(b_z, b_{\{y,z\}}, b_\Theta)$, and $Cl(b_z, b_{\{x,z\}}, b_\Theta)$ which are also faces of $\mathcal{CS}_3$.

4.2 The consistent complex

The region of consistent b.f.s is indeed also a simplicial complex, i.e. a collection of simplices satisfying Definition 1.

Theorem 2. *$\mathcal{CS}$ is a simplicial complex.*

Proof. Property 1. of Definition 1 is trivially satisfied. As a matter of fact, if a simplex $Cl(b_{A_1}, ..., b_{A_n})$ corresponds to focal elements with non-empty intersection, clearly points of any face of this simplex (obtained by selecting a subset of vertices) will be b.f.s with non-empty core, and will then correspond to cs.b.f.s. About property 2., consider the intersection of two maximal simplices of $\mathcal{CS}$ associated with two distinct cores $\mathcal{C}_1, \mathcal{C}_2 \subset \Theta$:

$$Cl(b_A : A \supseteq \mathcal{C}_1) \cap Cl(b_A : A \supseteq \mathcal{C}_2).$$

Now, each convex closure of points $b_1, ..., b_m$ in a Cartesian space is included in the *affine* space they generate:

$$Cl(b_1, ..., b_m) \subsetneq a(b_1, ..., b_m) \doteq$$
$$\doteq \left\{b : b = \alpha_1 b_1 + \cdots + \alpha_m b_m, \sum_i \alpha_i = 1\right\}$$

(since this just means that we relax the positivity constraint on the coefficients α_i). But the basis b.f.s $\{b_A : \emptyset \subsetneq A \subsetneq \Theta\}$ are linearly independent (as it is straightforward to check), so that

$$a(b_A, A \in L_1) \cap a(b_A, A \in L_2) \neq \emptyset \Leftrightarrow L_1 \cap L_2 \neq \emptyset$$

where L_1, L_2 are lists of subsets of Θ. Here $L_1 = \{A \subseteq \Theta : A \supseteq \mathcal{C}_1\}$, $L_2 = \{A \subseteq \Theta : A \supseteq \mathcal{C}_2\}$, so that the condition is

$$\begin{array}{c} \{A \subseteq \Theta : A \supseteq \mathcal{C}_1\} \cap \{A \subseteq \Theta : A \supseteq \mathcal{C}_2\} = \\ = \{A \subseteq \Theta : A \supseteq \mathcal{C}_1 \cup \mathcal{C}_2\} \neq \emptyset. \end{array}$$

As $\mathcal{C}_1 \cup \mathcal{C}_2 \supseteq \mathcal{C}_1, \mathcal{C}_2$ we have that $Cl(b_A, A \supseteq \mathcal{C}_1 \cup \mathcal{C}_2)$ is a face of both simplices. □

5 The twin geometry of consistency and combinability

The geometric approach to the theory of evidence can be applied in particular to possibility theory by analyzing the geometry of consonant and consistent belief functions. Some sort of duality seems to appear, as the geometric counterparts of belief measures are simplices, while the geometric loci of possibility measures and assignments are simplicial complexes (see Figure 4-left).

A similar duality appears when considering the relationship

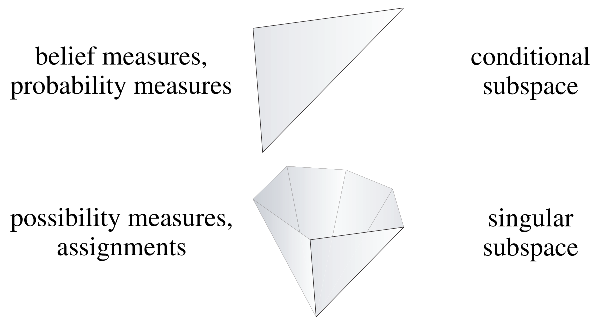

Figure 4: A pictorial representation of geometric dualities between notions of uncertainty theory.

between the notion of consistency and that of *combinability* in Dempster's theory.

Definition 2. *The* orthogonal sum *or* Dempster's sum *of two b.f.s b_1, b_2 on Θ is a new belief function $b_1 \oplus b_2$ on Θ with b.p.a.*

$$m_{b_1 \oplus b_2}(A) = \frac{\sum_{B \cap C = A} m_{b_1}(B)\, m_{b_2}(C)}{\sum_{B \cap C \neq \emptyset} m_{b_1}(B)\, m_{b_2}(C)}$$

where m_{b_i} denotes the b.p.a. associated with b_i.

When the denominator of the above equation is nil the two functions are said to be *non-combinable*. Now, we have seen (Cuzzolin 2004a) that the *conditional subspace*

$$\langle b \rangle \doteq \{b \oplus b', \forall b' \in \mathcal{B} : \exists b \oplus b'\}$$

obtained by combining through Dempster's rule a given b.f. b with all other b.f.s on the same frame (if such a combination exists) is a simplex. Let us focus here on *non-combinable* belief functions, and call

$$Sing \doteq \{b \in \mathcal{B} : \exists b' \in \mathcal{B} : \nexists b \oplus b'\}$$

the class of belief functions on Θ which are not combinable with each and every other b.f. (*singular subspace*).
The singular subspace is itself a simplicial complex: The duality between combinable/non-combinable b.f.s is again reflected in the dichotomy simplex-complex (Figure 4-right). This is related to the fact that cs.b.f.s can be constructed from non-combinable b.f.s, and vice-versa.

B.F.s in $Sing$ are characterized by the property that the union of their focal elements is a *proper* subset of Θ:

$$b \in Sing \Leftrightarrow \bigcup_{A_i \in \mathcal{E}_b} A_i \subsetneq \Theta,$$

where $\mathcal{E}_b$ denotes again the list of focal elements of b. Equivalently, there exists a non-empty subset of Θ which has empty intersections with each f.e. of b. Any b.f. b' with focal elements in this subset will not be combinable with b. We can then write

$$b \in Sing \Leftrightarrow \bigcup_{A_i \in \mathcal{E}_b} A_i \subseteq \{x\}^c$$

for some element $x \in \Theta$. By Proposition 3, b.f.s with focal elements in the list $L = \{A \subseteq \{x\}^c\}$ form the simplex $Cl(b_A : A \subseteq \{x\}^c)$. As there are n of such subsets (one for each singleton) the region of "singular" b.f.s is

$$Sing = \bigcup_{x \in \Theta} Cl(b_A : A \subseteq \{x\}^c). \tag{5}$$

Theorem 3. *$Sing$ (5) is a simplicial complex.*

Proof. As a matter of fact, following the same line of the proof of Theorem 2, each pair of simplices in the collection (5) has a common intersection

$$\begin{array}{c} Cl(b_A : A \subseteq \{x\}^c) \cap Cl(b_A : A \subseteq \{y\}^c) = \\ = Cl(b_A : A \subseteq \{x, y\}^c) \end{array}$$

which is a face of both (Property 2 of Definition 1). Besides, their faces correspond to b.f.s whose union of focal elements is obviously a proper subset of Θ (having less focal elements), and then belong to $Sing$ (Property 1). □

Figure 5 shows the singular complex for a ternary frame, and its relationship with $\mathcal{CO}_3$ ($\mathcal{CS}_3$ is not shown for sake of simplicity). Examining Figures 5 and 3 we can see that each maximal component $Sing_x \doteq Cl(b_A : A \subseteq \{x\}^c, A \neq \emptyset)$ of $Sing$ corresponds to a component $\mathcal{CS}_x$ of $\mathcal{CS}$:

$$\begin{array}{c} Sing_x = Cl(b_A : \emptyset \subsetneq A \subset \{x\}^c) \\ \updownarrow \\ \mathcal{CS}_x = Cl(b_A : A \ni x) = Cl(b_x, b_A : A \supsetneq \{x\}) = \\ = Cl(b_x, Cl(b_A : A = B \cup \{x\}, \emptyset \subsetneq B \subset \{x\}^c)). \end{array} \tag{6}$$

The interpretation is straightforward: each consistent b.f. is obtained by a singular b.f. by adding to each of its f.e.s a subset of $\Theta \setminus \cup_i A_i$, $A_i \in \mathcal{E}_b$. In fact, each maximal simplex $Sing_x$ of the singular complex is nothing but a replica of the belief space $\mathcal{B}_{\{x\}^c}$ for the frame $\{x\}^c$: for instance, in the above ternary example the triangle $Cl(b_x, b_y, b_{\{x,y\}})$ is isomorphic to the binary belief space $\mathcal{B}_2$ (see Figure 1).

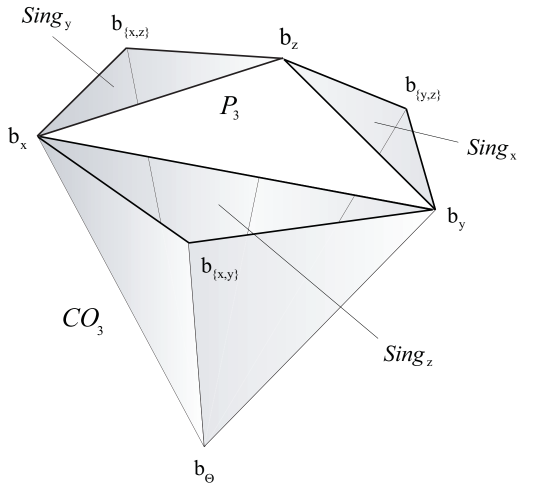

Figure 5: The singular subspace for a ternary frame, and the related consonant subspace. Each maximal component $Sing_x$ of $Sing$ is isomorphic to the belief space $\mathcal{B}_2 = Cl(b_y, b_z, b_{\{y,z\}})$ for the binary frame ($\{x\}^c = \{y, z\}$).

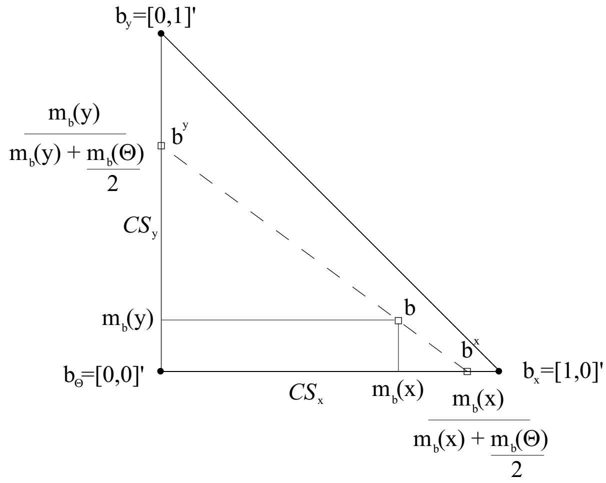

Figure 6: A belief function b as convex combination of its consistent coordinates in the binary belief space.

6 A decomposition in consistent components

A natural application of the geometric approach is the problem of finding approximations of belief functions belonging a given class of measures. We then close this paper by pointing out an interesting decomposition (closely related to the pignistic transformation (Smets & Kennes 1994)) of any belief function b into consistent components, which be interpreted as the natural projections of b on the maximal components of the consistent simplicial complex.
Let us then consider again the binary case. As a matter of fact, any b.f. $b \in \mathcal{B}_2$ $b = m_b(x)b_x + m_b(y)b_y + m_b(\Theta)b_\Theta$ can be written as the following combination (Figure 6) $b = \Big(m(x) + \frac{m(\Theta)}{2}\Big)\left(\frac{m(x)}{m(x)+\frac{m(\Theta)}{2}} b_x + \frac{\frac{m(\Theta)}{2}}{m(x)+\frac{m(\Theta)}{2}} b_\Theta\right) +$ $+\Big(m(y)+\frac{m(\Theta)}{2}\Big)\left(\frac{m(y)}{m(y)+\frac{m(\Theta)}{2}})b_y + \frac{\frac{m(\Theta)}{2}}{m(y)+\frac{m(\Theta)}{2}} b_\Theta\right)$, which is convex, as $(m_b(x) + m_b(\Theta)/2) + (m_b(y) + m_b(\Theta)/2) = m_b(x) + m_b(y) + m_b(\Theta) = 1$ and $m_b(x) + m_b(\Theta)/2 \geq 0$, $m_b(y) + m_b(\Theta)/2 \geq 0$.
In fact, this is the only way each belief function $b \in \mathcal{B}_2$ can be consistently decomposed as a convex combination of two points of $\mathcal{CS}_x, \mathcal{CS}_y$:

$$b^x = \frac{m(x)}{m(x) + \frac{m(\Theta)}{2}} b_x + \frac{\frac{m(\Theta)}{2}}{m(x) + \frac{m(\Theta)}{2}} b_\Theta,$$
$$b^y = \frac{m(y)}{m(y) + \frac{m(\Theta)}{2}})b_y + \frac{\frac{m(\Theta)}{2}}{m(y) + \frac{m(\Theta)}{2}} b_\Theta.$$

Now, we can notice that $m(x) + \frac{m(\Theta)}{2} = BetP[b](x)$ and $m(y) + \frac{m(\Theta)}{2} = BetP[b](y)$, where $BetP[b]$ denotes the *pignistic probability* associated with b:

$$BetP[b](x) \doteq \sum_{A \ni x} \frac{m(A)}{|A|}.$$

In other words, any b.f. $b \in \mathcal{B}_2$ can be written as a convex combination of two consistent belief functions

$$b = BetP[b](x)b^x + BetP[b](y)b^y$$

$b^x \in \mathcal{CS}_x$ and $b^y \in \mathcal{CS}_y$, whose coefficients are the values of the pignistic function.
In the general case, the consistent belief functions

$$b^x \doteq \frac{1}{BetP[b](x)} \sum_{A \ni x} \frac{m(A)}{|A|} b_A, \qquad x \in \Theta$$

can be considered as "consistent projections" of b onto the maximal components $\mathcal{CS}_x$, $x \in \Theta$ of the consistent subspace. As a matter of fact we can write

$$b = \sum_{A \subseteq \Theta} m(A) b_A = \sum_{x \in \Theta} \sum_{A \ni x} \frac{m(A)}{|A|} b_A =$$
$$= \sum_{x \in \Theta} BetP[b](x) \frac{\sum_{A \ni x} \frac{m(A)}{|A|} b_A}{BetP[b](x)} = \sum_{x \in \Theta} BetP[b](x) b^x. \tag{7}$$

According to Equation (7), each b.f. b lives in the $n-1$ dimensional simplex $\mathcal{P}^b \doteq Cl(b^x, x \in \Theta)$ (see Figure 7) and its convex coordinates in $\mathcal{P}^b$ ("consistent" coordinates) coincide with the coordinates of the pignistic probability in the probability simplex $\mathcal{P}$.

This argument in a sense mirrors another well known result which states that a belief function is a convex sum of a probability measure and a possibility measure. This is clear from Figure 1, where the reader can easily appreciate that b lies on a manifold of segments joining $\mathcal{CO}_x$ ($\mathcal{CO}_y$) and $\mathcal{P}$.

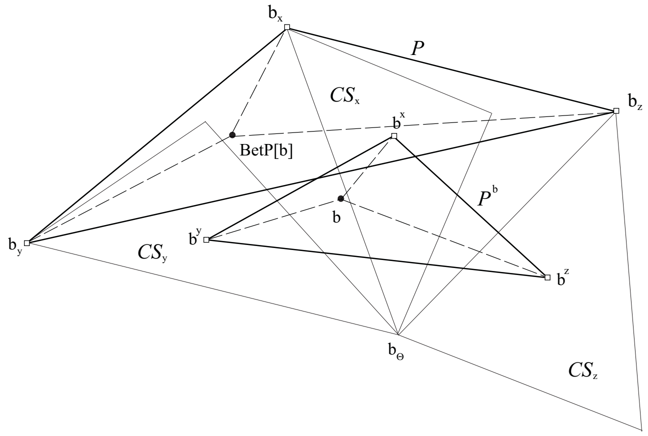

Figure 7: Pictorial representation of the role of the pignistic values $BetP[b](x)$ for a belief function and the related pignistic function. Both b and $BetP[b]$ live in a simplex (respectively $\mathcal{P} = Cl(b_x, x \in \Theta)$ and $\mathcal{P}^b = Cl(b^x, x \in \Theta)$) on which they possess the same convex coordinates. The vertices b^x, $x \in \Theta$ of the simplex $\mathcal{P}^b$ can be interpreted as consistent projections of the belief function b on the simplicial complex of consistent belief functions $\mathcal{CS}$.

6.1 Consistent coordinates and inner approximations

Equation (7) draws a connection between the notions of belief, probability, and possibility as it relates each belief function to its "natural" probabilistic (the pignistic function) and consistent (the quantities b^x) proxy. It remains to understand whether those functions b^x can be interpreted as some sort of consistent approximations of b, i.e. the cs.b.f.s which minimize some sort of distance between b and the consistent subspace.

As a matter of fact, consistent belief functions can be easily approximated in terms of possibility measures or consonant belief functions. *Inner consonant approximations* (Dubois & Prade 1990) of a b.f. b are those co.b.f.s such that

$$c(A) \geq b(A) \quad \forall A \subseteq \Theta$$

(or equivalently $pl_c(A) \leq pl_b(A)\ \forall A$). Such an approximation exists indeed iff b is consistent. In the binary case this means that inner approximations of b exist iff b is already consonant: $b \in \mathcal{CO}_x$ or $b \in \mathcal{CO}_y$. The optimal inner approximation is the co.b.f. $\hat{c}$ such that $pl_{\hat{c}}(x) = pl_b(x)$ $\forall xin\Theta$. It is rather interesting to wonder what are the relationships between the consistent coordinates obtained above and Dubois and Prade's inner approximations: We are going to investigate this issue in the near future (Cuzzolin 2007c).

7 Comments

In this paper we completed the analysis of the geometry of finite possibility measures by focusing on consistent belief functions, in virtue of their relationships with possibility assignments, on one side, and singular belief functions on the other. On a wider perspective, this study places a new element in the geometric semantics of the theory of evidence. As belief functions are points of a simplex, possibility measures form a simplicial complex, and Dempster's rule itself is nothing but an intersection of linear spaces (Cuzzolin 2004a), the Dempster-Shafer formalism can be in fact seen as some form of geometric calculus.
The full potential of the geometric approach can be appreciated though in the approximation problem: in the near future we will develop the preliminary results of (Cuzzolin 2004b) and Section 6 into a complete description of the consonant and consistent approximation problems by geometric methods, and relate them to inner and outer approximations (Dubois & Prade 1990).

The natural evolution of the belief space formalism is possibly the confluence with the field of geometric probability or continuous combinatorics (Klain & Rota 1997), which studies invariant measures on sets of geometric objects and relates them to additive probability measures. Belief functions can be indeed seen as iso-volumes of a convex body in $\mathbb{R}^{2^n-1}$, i.e. the vector of the volumes of all its orthogonal projections onto the space spanned by a subset of the reference axes.

References

Baroni, P. 2004. Extending consonant approximations to capacities. In *Proceedings of IPMU*, 1127–1134.

Barthelemy, J. 2000. Monotone functions on finite lattices: an ordinal approach to capacities, belief and necessity functions. In Fodor, J.; Baets, B. D.; and Perny, P., eds., *Preferences and Decisions under Incomplete Knowledge*. 195–208.

Berger. 1990. Robust Bayesian analysis: Sensitivity to the prior. *Journal of Statistical Planning and Inference* 25:303–328.

Caro, L., and Nadjar, A. B. 1999. Generalization of the Dempster-Shafer theory: A fuzzy-valued measure. *IEEE Transactions on Fuzzy Systems* 7:255–270.

Cozman, F. G. 1999. Calculation of posterior bounds given convex sets of prior probability measures and likelihood functions. *Journal of Computational and Graphical Statistics* 8(4):824–838.

Cuzzolin, F. 2004a. Geometry of Dempster's rule of combination. *IEEE Transactions on Systems, Man and Cybernetics part B* 34:2:961–977.

Cuzzolin, F. 2004b. Simplicial complexes of finite fuzzy sets. In *Proceedings of the* 10^{th} *International Conference on Information Processing and Management of Uncertainty IPMU'04, Perugia, Italy*, 1733–1740.

Cuzzolin, F. 2007a. Dual properties of relative belief of singletons. *submitted to the IEEE Transactions on Fuzzy Systems*.

Cuzzolin, F. 2007b. A geometric approach to the theory of evidence. *IEEE Transactions on Systems, Man and Cybernetics part C, to appear*.

Cuzzolin, F. 2007c. Metric consistent approximations of belief functions. *(in preparation)*.

Cuzzolin, F. 2007d. Moebius inverses of plausibility and commonality functions and their geometric interpretation. *submitted to the International Journal of Uncertainty, Fuzziness, and Knowledge-Based Systems*.

Cuzzolin, F. 2007e. Two new Bayesian approximations of belief functions based on convex geometry. *IEEE Transactions on Systems, Man and Cybernetics part B* 37(4).

Dempster, A. P. 1968. Upper and lower probabilities generated by a random closed interval. *Annals of Mathematical Statistics* 39:957–966.

Dubois, D., and Prade, H. 1988. *Possibility theory*. New York: Plenum Press.

Dubois, D., and Prade, H. 1990. Consonant approximations of belief functions. *International Journal of Approximate Reasoning* 4:419–449.

Dubrovin, B. A.; Novikov, S. P.; and Fomenko, A. T. 1986. *Sovremennaja geometrija. Metody i prilozenija*. Moscow: Nauka.

Fagin, R., and Halpern, J. 1988. Uncertainty, belief and probability. In *Proc. Intl. Joint Conf. in AI (IJCAI-89)*, 1161–1167.

Heilpern, S. 1997. Representation and application of fuzzy numbers. *Fuzzy Sets and Systems* 91:259–268.

Joslyn, C., and Klir, G. 1992. Minimal information loss possibilistic approximations of random sets. In Bezdek, J., ed., *Proc. 1992 FUZZ-IEEE Conference*, 1081–1088.

Joslyn, C. 1991. Towards an empirical semantics of possibility through maximum uncertainty. In Lowen, R., and Roubens, M., eds., *Proc. IFSA 1991*, volume A, 86–89.

Klain, D. A., and Rota, G.-C. 1997. *Introduction to Geometric Probability*. Cambridge University Press.

Klir, G. J.; Zhenyuan, W.; and Harmanec, D. 1997. Constructing fuzzy measures in expert systems. *Fuzzy Sets and Systems* 92:251–264.

Melkonyan, T., and Chambers, R. 2006. Degree of imprecision: Geometric and algebraic approaches. *forthcoming in the International Journal of Approximate Reasoning*.

Nguyen, H. T., and Wang, T. 1997. Belief functions and random sets. In *Applications and Theory of Random Sets, The IMA Volumes in Mathematics and its Applications, Vol. 97*. Springer. 243–255.

Roemer, C., and Kandel, A. 1995. Applicability analysis of fuzzy inference by means of generalized Dempster-Shafer theory. *IEEE Transactions on Fuzzy Systems* 3:4:448–453.

Seidenfeld, T., and Wasserman, L. 1993. Dilation for convex sets of probabilities. *Annals of Statistics* 21:1139–1154.

Shafer, G. 1976. *A Mathematical Theory of Evidence*. Princeton University Press.

Smets, P., and Kennes, R. 1994. The Transferable Belief Model. *Artificial Intelligence* 66:191–234.

Smets, P. 1990. The transferable belief model and possibility theory. In Y., K., ed., *Proceedings of NAFIPS-90*, 215–218.

Yager, R. R. 1999. Class of fuzzy measures generated from a Dempster-Shafer belief structure. *International Journal of Intelligent Systems* 14:1239–1247.

Yao, Y. Y., and Lingras, P. J. 1998. Interpretations of belief functions in the theory of rough sets. *Information Sciences* 104(1-2):81–106.