UPPER BOUND OF FIBONACCI ENTRY POINTS

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Jon Maiga
sequencedb.net
jon@jonkagstrom.com

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ABSTRACT

The Fibonacci entry point for the integer m is the index of the earliest Fibonacci number where m appear as a factor. We prove that the upper bound of Fibonacci entry points is $2m$.

***K*eywords** Fibonacci number, Fibonacci entry point, Upper bound

1 Introduction

The Fibonacci numbers have been studied for centuries, dating back to 450 BC, India. In 1202 Leonardo of Pisa introduced the Fibonacci numbers to the western world in the voluminous book 'Liber Abacci'. The same book where he posed the famous rabbit problem [1].

A Fibonacci number is the sum of the two previous, forming the sequence A000045[1], 0,1,1,2,3,5,8,13,21,34,55,89... The Fibonacci numbers have many interesting properties of which we study the Fibonacci entry points[2].

An entry point is the earliest index where an integer enters the Fibonacci sequence. For example 3 enters as the 4-th Fibonacci number, hence 3 has the entry point 4. Also, 7 appears at earliest as a factor of the 8-th Fibonacci number, 21, therefore 7 has the entry point 8.

In 1961 Vorob'ev showed that the upper bound entry point of m is less than m^2 [1]. In section 3 we prove that the entry point of m is less or equal to $2m$ with equality if and only if $m = 6 \cdot 5^e$. To achieve this we create an upper bound function which allows us to narrow down candidates of m to integers with one or two prime factors. We also note that our upper bound can be expressed as the Dedekind psi function over the number of unitary divisors of m.

In section 4 we shortly present an alternative proof of a weaker upper bound.

Edit: Apparently, in 1975 J. Salle proved that the upper bound is less or equal to $2m$ [11]. This upper bound was further sharpened in 2013 by D. Marques [12]. The author was not aware of those results when writing this article. Our result is very similar to the results of Marques but hopefully we contribute with a slightly different perspective.

2 Definitions and lemmas

In this section we list some of our definitions and lemmas.

2.1 Definitions

Throughout the article m and e are arbitrary integers and p is prime.

Definition 1. Let F be the Fibonacci numbers such $\mathrm{F}_0 = 0$, $\mathrm{F}_1 = 1$ and $\mathrm{F}_m = \mathrm{F}_{m-1} + \mathrm{F}_{m-2}$.

Definition 2. Let $\mathrm{z}(m) = k$ denote the Fibonacci entry point, such that k is the smallest integer such that $m | \mathrm{F}_k$.

[1] All A-number are sequences to be found at https://oeis.org/AXXXXXX [3].
[2] Sometimes referred to as 'order of appearance' or 'rank of apparition'.

Definition 3. Let $\hat{\mathrm{z}}(m)$ denote our upper bound of $\mathrm{z}(m)$.

Definition 4. Let the upper bound ratio be $\mathrm{R}(m) = \frac{\hat{\mathrm{z}}(m)}{m}$.

2.2 Lemmas

Lemma 5. *If $p \neq 5$ then $p|\,\mathrm{F}_{p-1}$ or $p|\,\mathrm{F}_{p+1}$.*

Lemma 6. *If $p = 5$ then $p|\,\mathrm{F}_p$.*

Lemma 7. $\mathrm{z}(p_1^{e_1} \cdots p_n^{e_n}) = \mathrm{lcm}(\mathrm{z}(p_1^{e1}), \ldots, \mathrm{z}(p_n^{e_n}))$.

Lemma 8. *If $a|b$ then $\mathrm{z}(a)|\,\mathrm{z}(b)$.*

Lemma 9. $\mathrm{z}(p^e) = p^{e-1}\,\mathrm{z}(p)$ *if $p > 2$ and $\mathrm{z}(p) \neq \mathrm{z}(p^2)$.*

Lemma 10. $\mathrm{z}(2^e) = 3 \cdot 2^{e-2}$ *for $e > 2$.*

Lemma 11. $\mathrm{z}(5^e) = 5^e$.

Lemma 12. $\mathrm{z}(F_m) = m$ *for $m > 1$.*

Most of the lemmas are standard results, lemma 5 and lemma 6 can be found in Williams [4]. Robinson proved lemma 7 to lemma 10 in [5] and Marques provided lemma 11 in [6]. Since F_m is strictly increasing for $m > 1$ we can conclude that the entry point of F_k must be k which proves lemma 12.

2.3 The least common multiple

The least common multiple of a and b is the least integer that is divisible by both a and b. For example the least common multiple of 3 and 7 is 21 since it is the least integer that both a and b divides. Another example is $\mathrm{lcm}(4, 6) = 12$. The least common multiple is related to the greatest common divisor by the identity $\mathrm{lcm}(a, b) = \frac{|ab|}{\gcd(a,b)}$ [7].

The least common multiple is also well defined for multiple arguments, $\mathrm{lcm}(m_1, \ldots, m_n)$, and the idea is the same. One way to evaluate the least common multiple is to factorize all terms and take the product of prime factors with the highest powers [7].

Definition 13. $\mathrm{lcm}(m_1, \ldots, m_n) = \prod_{p|m} p^{max(e_p)}$ where $m = 2^{e_2}3^{e_3}5^{e_5} \cdots = \prod p^{e_p}$.

For example to evaluate $\mathrm{lcm}(6, 8, 45)$ we first factorize each term $6 = 2 \cdot 3$, $8 = 2^3$ and $45 = 3^2 \cdot 5$ then for each prime factor, 2, 3 and 5 we take the product of the factors with the highest powers $\mathrm{lcm}(6, 8, 45) = 2^3 \cdot 3^2 \cdot 5 = 360$.

From the definition of lcm, note that if all m_i are coprime, then the least common multiple is $\mathrm{lcm}(m_i) = \prod m_i$.

3 The upper bound of the Fibonacci entry points

In this section we prove our main result. However, first each subsection investigates the upper bound for a subset of m. In 3.1 we examine the highest observed upper bound. Because of lemma 6 we separately investigate 5 in 3.2. In 3.3 we consider single primes, then in 3.4 we find an upper bound for an arbitrary positive integer. In 3.5 we prove the main result. Finally in 3.6 we show a relationship between our upper bound and the Dedekind psi function.

3.1 The observed upper bound and ratios

The observed upper bound of $\mathrm{z}(m)$ is $2m$, for example the entry point of 6 is 12 since $\mathrm{F}_{12} = 144$ is the first Fibonacci number divisible by 12. We can therefore say $\mathrm{z}(6) = r \cdot 6 = 12$ where the ratio is $r = 2$. Another entry point with $r = 2$ is $\mathrm{z}(30) = 60$. An example of a ratio less than 2 is the ratio of 7, which is $\frac{\mathrm{z}(7)}{7} = \frac{8}{7}$.

The upper bound ratio R (definition 4) is central in the following subsections. Analogous to the ratio described in the previous paragraph, $\mathrm{R}(m)$ is the ratio between an upper bound and m.

3.2 The upper bound of 5

Let us examine the special prime 5. It's special because $p = 5$ is the only prime such $p|\,\mathrm{F}_p$.

Lemma 14. $\mathrm{z}(5^e m) \leq 5^e\,\mathrm{z}(m)$.

Proof. By lemma 11 we know that $\mathrm{z}(5^e) = 5^e$, then by lemma 7, $\mathrm{z}(5^e m) = \mathrm{lcm}(\mathrm{z}(5^e), \mathrm{z}(m)) = \mathrm{lcm}(5^e, \mathrm{z}(m))$. If $\gcd(5^e, \mathrm{z}(m)) = 1$ we get the maximum possible least common multiple and conclude that $\mathrm{z}(5^e m)$ has to be less or equal to $5^e\,\mathrm{z}(m)$. □

Corollary 15. $\mathrm{R}(5^e m) = \mathrm{R}(m)$.

Proof. By lemma 14 we have and the definition of R we get $\mathrm{R}(5^e m) = \frac{5^e\,\mathrm{z}(m)}{5^e m} = \mathrm{R}(m)$. □

Corollary 16. $\mathrm{R}(5^e) = 1$.

Proof. In corollary 15 we let $m = 1$ and then we get $\mathrm{R}(5^e) = \frac{5^e\,\mathrm{z}(1)}{5^e} = 1$. □

3.3 The upper bound of primes

Here we investigate the upper bound of the entry points for primes. Starting with the established upper bound for $\mathrm{z}(p)$

Lemma 17. $\mathrm{z}(p) \leq p + 1$ *for all primes* p.

Proof. If $p \neq 5$ then, by lemma 5, we have $p|\,\mathrm{F}_{p-1}$ or $p|\,\mathrm{F}_{p+1}$. By lemma 8 we apply z on both sides and get $\mathrm{z}(p)|\,\mathrm{z}(F_{p-1})$ and $\mathrm{z}(p)|\,\mathrm{z}(F_{p+1})$. By lemma 12 we get $\mathrm{z}(p)|p-1$ and $\mathrm{z}(p)|p+1$. As a consequence $\mathrm{z}(p) \leq p+1$. By inspection, $\mathrm{z}(5) = 5$ which is within the bound. □

Let's expand this by introducing an exponent to the prime $\mathrm{z}(p^e)$

Lemma 18. $\mathrm{z}(p^e) \leq p^{e-1}(p+1)$ *for all primes* p.

Proof. By lemma 9 we have $\mathrm{z}(p^e) = p^{e-1}\,\mathrm{z}(p)$ for all odd primes. Then by lemma 17 we can replace $\mathrm{z}(p)$ with $p+1$ and arrive at $\mathrm{z}(p^e) \leq p^{e-1}(p+1)$. For the even prime 2 we see by inspection that $\mathrm{z}(2) = 3$, $\mathrm{z}(2^2) = 6$, $\mathrm{z}(2^3) = 6$ and then by lemma 10 we have $\mathrm{z}(2^e) = 3 \cdot 2^{e-2}$. By lemma 11, $\mathrm{z}(5^e) = 5^e$. All of which are within the bound $p^{e-1}(p+1)$. □

Corollary 19. $\mathrm{R}(p^e) = \mathrm{R}(p) = 1 + \frac{1}{p}$ *for all primes* p.

Proof. By lemma 18 and the definition of R we get $\mathrm{R}(p^e) = \frac{p^{e-1}(p+1)}{p^e} = \frac{p+1}{p} = 1 + \frac{1}{p} = \mathrm{R}(p)$. □

Corollary 20. $\mathrm{R}(p^e) \leq \frac{3}{2}$ *for all primes* p.

Proof. Since $\lim_{p\to\infty} \mathrm{R}(p^e) = 1 + \frac{1}{p} = 1$ we see $p = 2$ maximizes $\mathrm{R}(p^e)$ and we get $\mathrm{R}(2) = \frac{3}{2}$. □

3.4 Upper bound of Fibonacci entry points

Let us leave the comfort zone where $m = p$ and consider the upper bound of composite numbers. To build up intuition we first examine $m = pq$ where p and q are distinct odd primes other than 5.

By lemma 6 $\mathrm{z}(pq) = \mathrm{lcm}(\mathrm{z}(p), \mathrm{z}(q))$. To get the upper bound we maximize this function by carefully selecting p and q such their entry points are at $p+1$ and $q+1$. Then $\mathrm{z}(pq) \leq \mathrm{lcm}(p+1, q+1)$, we make the observation that $p \pm 1$ is even for all odd primes. Hence both $p+1$ and $q+1$ have a factor of 2. Because both terms of the least common multiple contains a factor of 2 we can immediately conclude that $\mathrm{z}(pq) \leq \frac{(p+1)(q+1)}{2}$. Sometimes it's easier to see this by looking at the gcd equivalent, $\mathrm{lcm}(p+1, q+1) = \frac{(p+1)(q+1)}{\gcd(p+1,q+1)}$ where the denominator is always greater or equal to 2. Let us generalize this line of reasoning.

Definition 21. $\mathrm{c}(m) = \begin{cases} 2 & \text{if 2 is a square free factor of a composite } m \\ 1 & \text{otherwise} \end{cases}$

We now formulate our first theorem, an upper bound of the Fibonacci entry points, $\hat{\mathrm{z}}(m)$

Theorem 22. $\mathrm{z}(m) \leq \mathrm{c}(m)\dfrac{5^e \prod\limits^{n} p_i^{e_i-1}(p_i+1)}{2^{n-1}} = \hat{\mathrm{z}}(m)$ *where* $m = 5^e \cdot 2^{e_1}3^{e_2}7^{e_3}\cdots p_n^{e_n}$.

Note that we handle factors of 5 in m separately (due to corollary 15).

Proof. By lemma 7 we have $\mathrm{z}(m) = \mathrm{lcm}(\mathrm{z}(p_1^{e_1}), \ldots, \mathrm{z}(p_n^{e_n}))$.

To get an upper bound we maximize each term of lcm by invoking lemma 18. Then we have $\mathrm{z}(m) \leq \mathrm{lcm}(p_1^{e_1-1}(p_1 + 1), \ldots, p_n^{e_n-1}(p_n + 1))$. From here we get an upper bound of the least common multiple by assuming that each term is coprime and arrive at $\mathrm{z}(m) \leq \prod^n p_i^{e_i-1}(p + 1)$.

When every factor of m is odd ($p_i > 2$) then every maximization term, $p_i^{e-1}(p_i + 1)$, is even (because $p_i + 1$ is even). We recall from definition 13 that only the factor with the highest power is accounted in the least common multiple. This means that $n - 1$ factors of 2 are disregarded and we can state $\mathrm{z}(m) \leq \frac{\prod^n p_i^{e_i-1}(p_i+1)}{2^{n-1}}$. This completes the proof for all odd integers not divisible by 5.

Because 2 is the only even prime we get a special case for even composite numbers. The maximization term of 2 is odd, $2 + 1 = 3$. Because 3 is not necessarily a factor of the other even maximization terms, we can not remove it from the upper bound as we did with odd primes (the division by 2^{n-1}), to compensate we multiply the nominator by a factor of 2. The compensation is done via $\mathrm{c}(m)$.

Finally we see that maximization terms for 2^e when $e > 1$ contains a factor of 2. By inspection $\mathrm{z}(2^2) \leq 2 \cdot (2 + 1)$ and for $e > 2$ we have $\mathrm{z}(2^e) \leq 2^{e-2} \cdot 3$ (lemma 10). Therefore we only compensate whenever a square free 2 is a factor of composite m. This complets the proof for all integers not divisible by 5.

By lemma 14 we know that $\mathrm{z}(5^e m) = 5^e\, \mathrm{z}(m)$, which corresponds to the separately handled 5^e in theorem 22. This completes the proof for all integers. □

Corollary 23. *All* $\hat{\mathrm{z}}(m)$ *are integer values.*

Proof. From the proof of theorem 22 we know that $\hat{\mathrm{z}}(m)$ is always an integer, since there are always at least 2^{n-1} factors of 2 in the nominator. □

Corollary 24. $\hat{\mathrm{z}}(m)$ *is even for* $m \neq 2$.

Proof. From the proof of theorem 22 we get n factor of 2 in the nominator, except when $m = 2$, when we have $n - 1$ factors of 2. The latter is the only case when denominator of $\hat{\mathrm{z}}(m)$ cancel all factors of 2, making it the only odd $\hat{\mathrm{z}}(m)$. □

Theorem 25. $\mathrm{R}(m) = \mathrm{c}(m)\dfrac{\prod^n (1 + \frac{1}{p_i})}{2^{n-1}}$ *where* $m = 5^e \cdot 2^{e_1} 3^{e_2} 7^{e_3} \cdots p_n^{e_n}$.

Proof. By theorem 22 and the definition of R we get $\frac{\mathrm{R}(m)}{\mathrm{c}(m)} = \frac{5^e \prod^n p_i^{e_i-1}(p_i+1)}{2^{n-1} 5^e \prod^n p_i^{e_i}} = \frac{\prod^n (p_i+1)}{2^{n-1} \prod p_i} = \frac{\prod^n (1+\frac{1}{p_i})}{2^{n-1}}$. □

Corollary 26. $\mathrm{R}(m) \leq \frac{3^n}{4^{n-1}}$ *where* $m = 5^e \cdot 2^{e_1} 3^{e_2} 7^{e_3} \cdots p_n^{e_n}$.

Proof. Since the nominator of R is maximized by the lowest possible p_i and $\lim_{p\to\infty} 1 + \frac{1}{p_i} = 1$ we can assume, by setting all $p_i = 2$ that $\prod^n (1 + \frac{1}{p}) \leq \frac{3}{2}^n$. Further we assume that $\mathrm{c}(m) = 2$ for all m. Then we get that $\mathrm{R}(m)$ must be less or equal to $\mathrm{c}(m)\frac{\frac{3}{2}^n}{2^{n-1}} = 2 \cdot \frac{2 \cdot 3^n}{4^n} = \frac{3^n}{4^{n-1}}$. □

From corollary 26 we get a rough idea of how R behave, we assume that every factor has $\mathrm{c}(p_i) = 2$ and the same ratio as 2, which is the highest possible for single primes, $\mathrm{R}(p_i) = \frac{3}{2}$. Although this is an oversized upper estimation we can draw the following conclusion

Corollary 27. *The maximum* $\mathrm{R}(m)$ *can be found among* m *with either one or two factors, not counting* 5.

Proof. By inspection of the strictly decreasing function from corollary 26, $f(n) = \frac{3^n}{4^{n-1}}$, we get $f(1) = 3$, $f(2) = \frac{9}{4}$ and $f(3) = \frac{27}{16}$. Since $f(3)$ is less than 2 we can be sure that all m with three or more distinct factors have a ratio less than 2. We note that $f(n)$ is unaffected by factors of 5 (by corollary 26). Since the observed upper bound is 2 we do not need to consider composites with more than two distinct factors (not counting 5) to find the maximum R. □

3.5 The maximum upper bound

Next we state and prove our main result

Theorem 28. $\mathrm{z}(m) \leq 2m$ *for* $m \geq 1$.

Proof. We investigate all possible candidates of m that can maximize $\mathrm{R}(m)$.

For $m = 1$, by inspection we have $\mathrm{z}(1) = 1$, which is has a ratio of 1.

For $m = p^e$ we know by corollary 20 that $\mathrm{R}(p^e) \leq \frac{3}{2}$.

For $m = 5^e$ we know by corollary 15 that 5^e doesn't affect R so we do not consider it further. Next we consider all m with two distinct factors (except 5). In the nominator of R we have the product $\prod^n (1 + \frac{1}{p_i})$ to maximize it we select the two lowest possible p_i

For composites $m = 2^0 3^{e_2} 7^{e_3} \cdots$ we maximize R by $\mathrm{R}(3^{e_2} \cdot 7^{e_3}) = \frac{16}{21}$.

For composites $m = 2^{e_1} 3^{e_2} 7^{e_3} \cdots$ where $e_1 > 1$ we maximize R by $\mathrm{R}(2^{e_1} \cdot 3^{e_2}) = \frac{2}{3}$.

For composites $m = 2^1 3^{e_2} 7^{e_3} \cdots$ we maximize R by $\mathrm{R}(2 \cdot 3^{e_2}) = 2$.

We have examined all candidate m with one and two factors and found maximum upper bound at $\mathrm{R}(2 \cdot 3^e) = 2$. By $corollary$ 27 every m with three or more distinct factors have an upper bound ratio less than 2. Therefore we can be sure that $\mathrm{R}(m) \leq 2$ for all $m \geq 1$. Since $\frac{\mathrm{z}(m)}{m} \leq \mathrm{R}(m)$ the proof is complete. ☐

Corollary 29. $\mathrm{z}(m) = 2m$ *if and only if* $m = 6 \cdot 5^e$.

Proof. In the proof of theorem 28 we showed that $m = 2 \cdot 3^e$ has an upper bound ratio of 2.

Since $\mathrm{z}(2 \cdot 3^e) = \mathrm{lcm}(\mathrm{z}(2), \mathrm{z}(3^e) = \mathrm{lcm}(3, 3^{e-1}\,\mathrm{z}(3))$, we note that for all $e > 1$ both terms have a factor of 3. By definition 13 only the of highest factor of 3 is accounted, which allows us to adjust the upper bound by removing a factor of 3. When $e > 1$ we get an adjusted upper bound ratio of $\frac{\mathrm{R}(2 \cdot 3^e)}{3} = \frac{2}{3}$. Therefore $\mathrm{R}(2 \cdot 3^e) = 2$ if and only if $e = 1$.

Recall that $\mathrm{R}(m5^e) = \mathrm{R}(m)$ by corollary 15, therefore we conclude that all $\mathrm{R}(6 \cdot 5^e) = 2$. ☐

3.6 Relationship to the Dedekind psi function

Here we shortly remark that our upper bound can be expressed as a function of the ratio of the Dedekind psi function (A001615) over the number of unitary divisors (A034444) of m.

Definition 30. Let the Dedekind psi be denoted by $\psi(m) = m \prod^n 1 + \frac{1}{p_i} = \prod^n p^{e_i - 1}(p_i + 1)$.

Note that n is the number of distinct divisors, commonly denoted by omega, $\omega(m)$ (A001221). We then get the number of unitary divisors of m by $2^{\omega(m)}$. For simplicity we restrict our expression to odd m not divisible by 5,

Theorem 31. $\hat{\mathrm{z}}_\psi(m) = \dfrac{2\psi(m)}{2^{\omega(m)}}$ *for all odd* m *not divisible by* 5.

Proof. From theorem 22 we have $\hat{\mathrm{z}}(m) = \frac{\prod^n p^{e_i - 1}(p+1)}{2^{n-1}}$ for odd m not divisible by 5. We substitute the product in the nominator by the Dedekind psi function. Then we multiply the nominator by 2 to get 2^n in the denominator and replace n by $\omega(m)$. ☐

4 An alternative proof of a weaker upper bound

The period length of $\mathrm{F}_n \mod m$ is often referred to as the Pisano period, $\pi(m)$. For example $\mathrm{F}_n \mod 4 =$ A079343: 0,1,1,2,3,1,0,1,1,2,3,1,0,1,1,2,3,1,0,... for which the period length is 6. Hence $\pi(4) = 6$.

In 1960 Wall showed that $\pi(m) < m^2$ [8]. In 1992 Peter Freyd challenged the readers of the American Mathematical Monthly (E3410, March 92) to prove that $\pi(m) \leq 6m$ for all m. K.S. Brown found a proof and additional showed that $\pi(m) = 6m$ if and only if $m = 2 \cdot 5^e$ [9].

The number of zeros in a Pisano period is denoted $\mathrm{a}(m)$. The period above (A079343) has one zero per period, so $\mathrm{a}(4) = 1$. The only possible values of $\mathrm{a}(m)$ is 1, 2 or 4 [5].

Since $\mathrm{z}(m)$ and $\pi(m)$ are related by $\pi(m) = \mathrm{z}(m)\,\mathrm{a}(m)$ [5] we can apply the upper bound of $\pi(m)$ to $\mathrm{z}(m)$.

Theorem 32. $\mathrm{z}(m) \leq 6m$.

Proof. Since $\mathrm{z}(m) = \frac{\pi(m)}{\mathrm{a}(m)}$ and $\mathrm{a}(m) \geq 1$ it follows that $\mathrm{z}(m) \leq \pi(m)$. Since the upper bound of $\pi(m)$ is $6m$ we can conclude that $\mathrm{z}(m)$ also must be less or equal to $6m$. □

We encourage the reader to improve theorem 32 further by for example investigating the behaviour of $\mathrm{a}(m)$.

Acknowledgement

When I find interesting machine generated sequences in my hobby project, Sequence Database [10], they are often without context. However, thanks to automated mappings to sequences in the Online Encyclopedia of Integer Sequences [3], I can get context. The combination of machine generated and human audited integer sequences was the seed of this article. I am foremost grateful to all the volunteers and editors at the OEIS for all their hard work.

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Function table

m	$z(m)$	$\hat{z}(m)$	$\frac{z(m)}{m}$	$R(m) = \frac{\hat{z}(m)}{m}$
1	1	1	1.	1.
2	3	3	1.5	1.5
3	4	4	1.33333	1.33333
4	6	6	1.5	1.5
5	5	5	1.	1.
6	12	12	2.	2.
7	8	8	1.14286	1.14286
8	6	12	0.75	1.5
9	12	12	1.33333	1.33333
10	15	15	1.5	1.5
11	10	12	0.909091	1.09091
12	12	12	1.	1.
13	7	14	0.538462	1.07692
14	24	24	1.71429	1.71429
15	20	20	1.33333	1.33333
16	12	24	0.75	1.5
17	9	18	0.529412	1.05882
18	12	36	0.666667	2.
19	18	20	0.947368	1.05263
20	30	30	1.5	1.5
21	8	16	0.380952	0.761905
22	30	36	1.36364	1.63636
23	24	24	1.04348	1.04348
24	12	24	0.5	1.
25	25	25	1.	1.
26	21	42	0.807692	1.61538
27	36	36	1.33333	1.33333
28	24	24	0.857143	0.857143
29	14	30	0.482759	1.03448
30	60	60	2.	2.
31	30	32	0.967742	1.03226
32	24	48	0.75	1.5
33	20	24	0.606061	0.727273
34	9	54	0.264706	1.58824
35	40	40	1.14286	1.14286
36	12	36	0.333333	1.
37	19	38	0.513514	1.02703
38	18	60	0.473684	1.57895
39	28	28	0.717949	0.717949
40	30	60	0.75	1.5
41	20	42	0.487805	1.02439
42	24	48	0.571429	1.14286
43	44	44	1.02326	1.02326
44	30	36	0.681818	0.818182
45	60	60	1.33333	1.33333
46	24	72	0.521739	1.56522
47	16	48	0.340426	1.02128
48	12	48	0.25	1.
49	56	56	1.14286	1.14286
50	75	75	1.5	1.5