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A Survey of Attitude Representations

Malcolm D. Shuster[1]

Abstract

A survey of the attitude representations is given in a single consistent notation and set of conventions. The relations between the various representations of the attitude, and the kinematic equations are given completely. The transformations connecting different attitude covariance representations are presented for the case where the errors in the attitude are sufficiently small that they can be represented by an infinitesimal rotation. Examples of the use of each representation are presented and some historical notes provided.

Contents

Introduction

In comparison with orbit studies, where the position of an object is represented as one of a small number of known representations, which are almost always the familiar Cartesian, cylindrical, and spherical coordinates of vectors, the representation of attitude, i.e., orientation, is exceedingly more diverse. The attitude rep-

[1]Senior Professional Staff, Space Department, The Johns Hopkins University Applied Physics Laboratory, Laurel, MD 20723-6099.

resentations in current use comprise vectors of three and four components, as well as 2×2, 3×3 and 4×4 matrices. For each of these representations there is some application for which its use is advantageous.

The development of the attitude representations can be found in many books on Classical Mechanics and on attitude control, in particular, in the books of Goldstein [1], Hughes [2], Junkins and Turner [3], the short section by Markley [4] in Wertz [5], and in the book of Kane, Likins, and Levinson [6], to cite the most recent and most useful. The present treatment has much in common with that found in each of these excellent texts and with the forthcoming book by the present author [7]. However, an attempt has been made to be much more complete and to make a strong distinction between abstract and numerical quantities.

The conventions adopted here are consistent with those of Goldstein [1], Hughes [2], and Markley [4] and are mostly consistent with those of Junkins and Turner [3] (who label the scalar component of the quaternion differently). The differences in the sign conventions which distinguish many expressions in this work from those of Kane, Likins and Levinson [6] result from the fact that the present work has adopted the passive description of rotations as have Goldstein, Hughes, Junkins and Turner, and Markley, while Kane, Likins and Levinson and several of the references cited in this survey have adopted the active description. Our conventions seem to be largely consistent with those adopted in Inertial Navigation [8, 9].

An attempt has been made to include all of the useful results on attitude representations as well as many results whose utility may be seriously open to question. Therefore, in order to keep the length of this article almost within manageable proportions, only the slimmest skeleton of a derivation, if any, has been presented, very little discussion, and hardly any figures. Despite its breadth, this survey may be justly accused of superficiality. This work makes no claim to be didactic, although experienced readers may find some didactic value in it.

On the other hand, a great deal of information, amounting to more than 600 equations, is presented in one place, in a single set of conventions, from a single point of view, and explicitly stating what analytically each item means. The growing sophistication of attitude dynamics, control and determination activities may be judged from a comparison of this survey and the review by Mayer [10], which appeared in 1960.

Readers should take note that while this material has been presented largely from the standpoint of spacecraft attitude, the material applies equally well to aircraft, seacraft, landcraft, and missiles. For this reason the word vehicle has been used instead of spacecraft wherever possible.

Vector Spaces

Physical vectors, the basic quantities in the study of attitude, do not have a numerical value by themselves but only in conjunction with other vectors (generally an orthonormal basis as discussed in the following section). However, the most basic properties of vectors, i.e., the algebraic properties which define a vector space, do not depend on the specific numerical values of the vectors.

A *vector space* [11, 12] consists of a set of vectors, $\mathcal{V} = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \ldots\}$, a set of scalars, $\mathcal{F} = \{a, b, c, \ldots\}$, and two operations: vector addition and multiplication of a vector by a scalar. The set of scalars is a field, i.e., a set whose elements have the same algebraic properties as the real (or complex) numbers. In real-world applications the field is almost always the field of real numbers, though in rare cases the complex numbers are also used, as in the exploration of the abstract properties of quaternions.

The set of vectors forms a group under vector addition satisfying necessarily:

(a) if $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors, then so is $\boldsymbol{u} + \boldsymbol{v}$,

(b) vector addition is associative and commutative:

$$(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}), \tag{1}$$

and

$$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}, \tag{2}$$

(c) there exists a vector $\boldsymbol{0}$ such that

$$\boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v} \tag{3}$$

for every vector $\boldsymbol{v}$, and

(d) for every vector $\boldsymbol{v}$ there exists a negative vector $(-\boldsymbol{v})$ such that

$$\boldsymbol{v} + (-\boldsymbol{v}) = \boldsymbol{0}. \tag{4}$$

By convention we write

$$\boldsymbol{u} + (-\boldsymbol{v}) \equiv \boldsymbol{u} - \boldsymbol{v} \tag{5}$$

and speak of vector subtraction.

Multiplication of a vector by a scalar is defined and satisfies

$$a(b\boldsymbol{v}) = (ab)\boldsymbol{v}, \tag{6}$$

$$(a + b)\boldsymbol{v} = a\boldsymbol{v} + b\boldsymbol{v}, \tag{7}$$

$$a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}, \tag{8}$$

as well as

$$1\boldsymbol{v} = \boldsymbol{v}. \tag{9}$$

It follows that

$$0\boldsymbol{v} = \boldsymbol{0}, \quad \text{and} \quad (-a)\boldsymbol{v} = -(a\boldsymbol{v}). \tag{10}$$

If further the vector space is endowed with a scalar product $(\cdot)$ whose values are in $\mathcal{F}$ and which satisfies

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}, \tag{11}$$

$$\boldsymbol{v} \cdot \boldsymbol{v} \geq 0, \tag{12}$$

$$\boldsymbol{v} \cdot \boldsymbol{v} = 0 \quad \text{if and only if} \quad \boldsymbol{v} = \boldsymbol{0}, \tag{13}$$

$$\boldsymbol{u} \cdot (a\boldsymbol{v}) = a(\boldsymbol{u} \cdot \boldsymbol{v}), \tag{14}$$

$$(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}, \tag{15}$$

$\mathcal{V}$ is said to be an *inner-product space.* (Note that if $\mathcal{F}$ were the field of complex numbers, the right member of equation (11) would be replaced by its complex conjugate.) The length of a vector, written $|v|$, is defined as

$$|v| = (v \cdot v)^{1/2}. \tag{16}$$

In three-dimensional space a non-trivial vector product ($\times$), with values in $\mathcal{V}$, can be constructed which satisfies

$$u \times v = -v \times u, \tag{17}$$

$$(au) \times v = a(u \times v), \tag{18}$$

$$(u + v) \times w = u \times w + v \times w. \tag{19}$$

There is no commonly accepted name for a vector space with both a scalar and a vector product, and the concept seems to be useful only in three dimensions.[2] These relations define all the abstract properties of vectors without ever saying what a vector is. Henceforth, our presentation will be specialized to a three-dimensional space.

Right-Handed Orthonormal Bases

Any vector v in three-dimensional space can be written in the form

$$v = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}, \tag{20}$$

where $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ are the *basis vectors,* which are necessarily linearly independent. The numbers a, b and c are the *coordinates* (or *components*) of v with respect to this basis. As long as the basis vectors are linearly independent, the coordinates are well defined.

If the basis vectors satisfy

$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = 0, \tag{21}$$

the basis is said to be *orthogonal.* If, in addition, it is true that

$$\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1, \tag{22}$$

the basis is said to be *orthonormal,* and the basis vectors are then usually indicated by a caret, $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$. An orthonormal basis which satisfies

$$\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}}, \qquad \hat{\boldsymbol{j}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{i}}, \qquad \hat{\boldsymbol{k}} \times \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}}, \tag{23}$$

is further said to be *right-handed* or *dextral.* (The basis is said to be *left-handed* if the sign of the right-member of each of the above equations is reversed.) It is the scalar product which permits the components of a vector to be computed. For an orthonormal basis,

$$a = \hat{\boldsymbol{i}} \cdot v, \qquad b = \hat{\boldsymbol{j}} \cdot v, \qquad c = \hat{\boldsymbol{k}} \cdot v. \tag{24}$$

The notation for the scalar and vector products can be made more compact if instead of $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$, we write $\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3$. Then the orthonormality conditions can be

[2] In higher-dimensional vector spaces the equivalent construction is accomplished as an exterior product. For more information on exterior algebras see the book by Flanders [13].

written as

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}, \qquad i, j = 1, 2, 3, \tag{25}$$

where δ_{ij} is the *Kronecker symbol*, defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{26}$$

The condition for right-handedness becomes likewise

$$\hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k) = \epsilon_{ijk}, \tag{27}$$

where ϵ_{ijk} is the *Levi-Civita symbol*, which is defined by

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \qquad \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1, \tag{28}$$

and all other elements vanish. Thus, ϵ_{ijk} is antisymmetric with respect to the interchange of any two indices.

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}. \tag{29}$$

Equation (29) and the value of ϵ_{123} are sufficient to completely specify ϵ_{ijk}. The Levi-Civita symbol satisfies

$$\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}, \quad \text{and} \quad \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{mjk} = 2\delta_{im}. \tag{30}$$

Equation (27) is equivalent to

$$\hat{e}_i \times \hat{e}_j = \sum_{k=1}^{3} \epsilon_{ijk}\hat{e}_k. \tag{31}$$

As a result of these equations, if

$$u = u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3, \quad \text{and} \quad v = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3, \tag{32}$$

are any two vectors, then

$$u \cdot v = \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}u_iv_j = \sum_{k=1}^{3} u_kv_k, \tag{33}$$

and

$$u \times v = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}u_iv_j\hat{e}_k. \tag{34}$$

Equation (34) can be rewritten in more familiar form in terms of the determinant

$$u \times v = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{35}$$

The Matrix Representation of Vectors

The vectors u and v designate abstract quantities such as the position or the velocity of the vehicle. Likewise $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$ designate the abstract directions of

the three coordinate axes. By *abstract* we mean that we can attach a meaning to these quantities without having to attach a numerical (concrete) value to them. These *abstract* vectors satisfy all the above relations satisfied by elements of a vector space and the scalar and vector product relations. The numerical values *associated with* abstract vectors, are the coordinates, which are arranged here as column vectors, i.e., as 3×1 matrices. These are not unique but depend on the choice of the coordinate system, i.e., the basis.

Consider again $\boldsymbol{v} = v_1\hat{\boldsymbol{e}}_1 + v_2\hat{\boldsymbol{e}}_2 + v_3\hat{\boldsymbol{e}}_3$, and denote by $\mathcal{E}$ the basis $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$. The components of $\boldsymbol{v}$ with respect to $\mathcal{E}$ are called the *representation* of $\boldsymbol{v}$ with respect to $\mathcal{E}$, which is written as a *column matrix* (or *column vector*)

$$\mathbf{v}_{\mathcal{E}} \equiv \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \tag{36}$$

where for an orthonormal basis

$$v_i = \hat{\boldsymbol{e}}_i \cdot \boldsymbol{v}. \tag{37}$$

To avoid ambiguity, it is generally necessary to indicate the basis as well as to identify the abstract vector whose components are being considered. When there is no confusion, the basis designation will be dropped. Unless otherwise noted, in order to distinguish between abstract vectors and vector representations, the former are indicated by bold italic letters and the latter by bold unslanted *sans-serif* letters. Often to save space equation (36) is written in terms of a *row matrix* or *row vector* as

$$\mathbf{v}_{\mathcal{E}} \equiv [v_1 \quad v_2 \quad v_3]^T, \quad \text{or} \quad \mathbf{v}_{\mathcal{E}}^T \equiv [v_1 \quad v_2 \quad v_3], \tag{38}$$

where the superscript T denotes the matrix transpose.

Of particular interest is the representation of a basis with respect to itself,

$$(\hat{\mathbf{e}}_1)_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \equiv \hat{\mathbf{1}}, \qquad (\hat{\mathbf{e}}_2)_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \equiv \hat{\mathbf{2}}, \qquad (\hat{\mathbf{e}}_3)_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \equiv \hat{\mathbf{3}}. \tag{39}$$

Equations (39) are true for every basis. Thus, the numerical values, $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$, and $\hat{\mathbf{3}}$ are independent of the basis (as are the values of δ_{ij} and ε_{ijk}). Their physical significance, however, depends on the specific identification of the basis $\mathcal{E}$.

The scalar and vector products of abstract vectors can now be extended to representations. If $\mathbf{u}$ and $\mathbf{v}$ are the respective representations of the abstract vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ with respect to a common basis, then the scalar product for representations is defined to have the same value as the scalar product for the corresponding abstract vectors. Thus, from equation (33),

$$\mathbf{u} \cdot \mathbf{v} \equiv \sum_{k=1}^{3} u_k v_k. \tag{40}$$

Likewise, the vector product of two representations with respect to a common

basis is defined to be equal to the representation with respect to that basis of the vector product of the two corresponding abstract vectors. Thus, from equation (34),

$$\mathbf{u} \times \mathbf{v} \equiv \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}. \tag{41}$$

The scalar product of two vector representations can likewise be written in terms of matrix operations as

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}. \tag{42}$$

Likewise, for the vector product we define the antisymmetric matrix $[[\mathbf{u}]]$ (read "*u antisymmetric*") according to

$$[[\mathbf{u}]]_{ij} \equiv \sum_{k=1}^{3} \varepsilon_{ijk}u_k, \tag{43}$$

or equivalently,

$$[[\mathbf{u}]] = \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{bmatrix}. \tag{44}$$

It follows then from equation (41) that

$$\mathbf{u} \times \mathbf{v} = -[[\mathbf{u}]]\mathbf{v}. \tag{45}$$

Some authors define instead

$$[\mathbf{u}\times] \equiv -[[\mathbf{u}]], \tag{46}$$

in order that the sign in equation (45) be reversed. In practice, however, one calculates vector products infrequently, and the matrix defined by equation (44) turns out to be more convenient overall. It could be said with no little justification that the theoretical study of attitude is the study of the matrix $[[\mathbf{u}]]$.

While abstract vectors have the advantage of being "coordinate-free" and expressions in terms of abstract vectors are generally physically meaningful, they have the disadvantage of being subject to two different multiplication operations. For vector representations, on the other hand, these two products become simple matrix multiplication, which can be manipulated much more readily. The matrix definitions of the scalar and vector products as given by equations (42) and (45) can be extended to the computation of scalar and vector products of arbitrary 3×1 matrices, which need not correspond to vector representations with respect to a common basis. This turns out to be very useful in practice. Also, vector *data* consists of coordinates, not abstract quantities. Hence, it is always representations of a vector which figure in real-world applications.

The matrix $[[\mathbf{u}]]$ satisfies

$$[[\mathbf{u}]]^T = -[[\mathbf{u}]], \tag{47}$$

$$[[\mathbf{u}]]\mathbf{v} = -[[\mathbf{v}]]\mathbf{u}, \tag{48}$$

$$[[\mathbf{u}]]\mathbf{u} = \mathbf{0}, \tag{49}$$

$$[[\mathbf{u}]]\,[[\mathbf{v}]] = -(\mathbf{u}\cdot\mathbf{v})I + \mathbf{v}\mathbf{u}^T, \tag{50}$$

$$[[\mathbf{u}]]^3 = -|\mathbf{u}|^2[[\mathbf{u}]], \tag{51}$$

$$[[\mathbf{u}]]\,[[\mathbf{v}]] - [[\mathbf{v}]]\,[[\mathbf{u}]] = \mathbf{v}\mathbf{u}^T - \mathbf{u}\mathbf{v}^T = -[[\mathbf{u}\times\mathbf{v}]], \tag{52}$$

$$\mathbf{u}\mathbf{v}^T[[\mathbf{w}]] + [[\mathbf{w}]]\mathbf{v}\mathbf{u}^T = -[[\mathbf{u}\times(\mathbf{v}\times\mathbf{w})]]. \tag{53}$$

These results follow directly from the definition of $[[\mathbf{u}]]$ or from equation (50) which is easily verified by direct substitution. Equation (50) is equivalent to the Grassman identity:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}. \tag{54}$$

We note also the Jacobi identity:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{0}. \tag{55}$$

If M is an arbitrary square matrix, it is also true that

$$M[[\mathbf{u}]] + [[\mathbf{u}]]M^T + [[M^T\mathbf{u}]] = (\operatorname{tr} M)\,[[\mathbf{u}]], \tag{56}$$

and

$$M[[\mathbf{u}]]M^T = [[\operatorname{adj}(M^T)\mathbf{u}]], \tag{57}$$

where $\operatorname{adj}(M)$ denotes the adjoint matrix [14] of M. From this we may derive

$$M[[\mathbf{u}]]M^T = (\det M)\,[[(M^T)^{-1}\mathbf{u}]], \tag{58}$$

where M is any nonsingular matrix. It follows from equation (52) for any 3×3 matrix that

$$[[(M\mathbf{u})\times(M\mathbf{v})]] = M[[\mathbf{u}\times\mathbf{v}]]M^T. \tag{59}$$

If we write M in terms of its columns

$$M = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3], \tag{60}$$

then clearly

$$\det M = \mathbf{u}_1\cdot(\mathbf{u}_2\times\mathbf{u}_3). \tag{61}$$

From equations (60) and (61) it follows for the special case

$$M_{ij} = \mathbf{u}_i\cdot\mathbf{v}_j, \tag{62}$$

that

$$\det M = [\mathbf{u}_1\cdot(\mathbf{u}_2\times\mathbf{u}_3)]\,[\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)]. \tag{63}$$

The representation of vectors by column matrices requires only the existence of a basis. However, in order to compute the components of the vectors with respect to that basis, the vector space must possess a scalar product, whose definition must precede that of components. The ultimate definition of the scalar product then is not equation (33) or (40), which are only the representation of a scalar product, but some geometrical construction, such as the construction of a perpendicular projection. Once a scalar product exists, however, it is possible to define the vector product in terms of representations. Note also that there is no fundamental connection between right-handedness and the physical orientation of the axes. Equation (23) makes no statement about right-handed screws, which are simply a convention, only about the algebraic nature of the operation. The special status accorded right-handed coordinate frames is therefore artificial to some extent. What is important, however, is that physical rotations not transform right-handed coordinate systems into left-handed coordinate systems.

Orthogonal Transformations

Let $\mathcal{E} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ and $\mathcal{E}' = \{\hat{e}'_1, \hat{e}'_2, \hat{e}'_3\}$ denote two orthonormal bases. Then an arbitrary abstract vector x can be represented either as

$$x = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3, \quad \text{or as} \quad x = x'_1\hat{e}'_1 + x'_2\hat{e}'_2 + x'_3\hat{e}'_3 . \tag{64}$$

Thus, the representation vectors are

$$\mathbf{x} \equiv \mathbf{x}_{\mathcal{E}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}' \equiv \mathbf{x}_{\mathcal{E}'} = \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}. \tag{65}$$

Now, $\mathcal{E}$ and $\mathcal{E}'$ can each be expanded in terms of the other basis as

$$\hat{e}'_i = \sum_{j=1}^{3} C_{ij}\hat{e}_j, \quad \text{and} \quad \hat{e}_i = \sum_{j=1}^{3} C'_{ij}\hat{e}'_j , \tag{66}$$

with

$$C_{ij} = \hat{e}'_i \cdot \hat{e}_j, \quad \text{and} \quad C'_{ij} = \hat{e}_i \cdot \hat{e}'_j . \tag{67}$$

Thus,

$$C'_{ij} = C_{ji} , \tag{68}$$

or as matrices

$$C' = C^T. \tag{69}$$

The coefficients C_{ij} and C'_{ij} are the cosines of the angles between the two sets of axes, or *direction cosines.* The matrix C, therefore, is called the *direction-cosine matrix.*

From equation (67) the columns of C_{ij} are seen to be the representations of the $\hat{e}_i$, $i = 1, 2, 3$, with respect to the basis $\mathcal{E}'$ while the rows of C_{ij} are the representations of the $\hat{e}'_i$, $i = 1, 2, 3$, with respect to the basis $\mathcal{E}$. Thus, in terms of columns,

$$C = [(\hat{\mathbf{e}}_1)_{\mathcal{E}'} \quad (\hat{\mathbf{e}}_2)_{\mathcal{E}'} \quad (\hat{\mathbf{e}}_3)_{\mathcal{E}'}] = [(\hat{\mathbf{e}}'_1)_{\mathcal{E}} \quad (\hat{\mathbf{e}}'_2)_{\mathcal{E}} \quad (\hat{\mathbf{e}}'_3)_{\mathcal{E}}]^T. \tag{70}$$

From

$$x_i = \hat{\boldsymbol{e}}_i \cdot \boldsymbol{x}, \quad \text{and} \quad x_i' = \hat{\boldsymbol{e}}_i' \cdot \boldsymbol{x}, \tag{71}$$

if follows that

$$x_i' = \sum_{j=1}^{3} C_{ij} x_j, \quad \text{and} \quad x_i = \sum_{j=1}^{3} C_{ji} x_j', \tag{72}$$

or, in matrix notation,

$$\mathbf{x}' = C\mathbf{x}, \quad \text{and} \quad \mathbf{x} = C^T\mathbf{x}', \tag{73}$$

where the indicated operation is matrix multiplication. Substituting these two equations into one another, it follows immediately from equations (73) that

$$C^TC = CC^T = I, \quad \text{or} \quad C^T = C^{-1}. \tag{74}$$

This follows also from equation (70). Such a matrix is said to be *orthogonal*. From equation (74)

$$1 = \det(CC^T) = (\det C)(\det C^T) = (\det C)^2, \tag{75}$$

from which it follows that

$$\det C = \pm 1. \tag{76}$$

When $\det C = 1$, the orthogonal matrix is said to be proper or special. Otherwise, it is *improper*. Proper orthogonal matrices represent rotations, while an improper orthogonal matrix may be factored as the product of a proper orthogonal matrix and an inversion. An inversion is simply the matrix $-I$, and changes the sign of every component.

If $\mathcal{E}$, $\mathcal{E}'$, and $\mathcal{E}''$ are three orthonormal bases connected by two orthogonal transformations described by the orthogonal matrices C' and C''

$$\mathcal{E}'' \xleftarrow{C''} \mathcal{E}' \xleftarrow{C'} \mathcal{E},$$

then the combined orthogonal transformation is described by the orthogonal matrix C according to

$$\mathcal{E}'' \xleftarrow{C} \mathcal{E},$$

and

$$C = C''C'. \tag{77}$$

For any orthogonal matrix

$$(C\mathbf{x}) \cdot (C\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}, \tag{78}$$

and

$$(C\mathbf{x}) \times (C\mathbf{y}) = (\det C)C(\mathbf{x} \times \mathbf{y}). \tag{79}$$

Equation (79) is equivalent to

$$C[[\mathbf{u}]]C^T = (\det C)\,[[C\mathbf{u}]]. \tag{80}$$

Equation (80) is a special case of equation (58). Thus, every orthogonal transformation preserves scalar products, but only proper orthogonal transformations preserve vector products. Rotations are proper orthogonal transformations. Note that (det C) C in equations (79) and (80) is a proper orthogonal matrix even if C is improper orthogonal.

In terms of components, equation (80) is equivalent to

$$\sum_{m=1}^{3}\sum_{n=1}^{3}\varepsilon_{lmn}C_{mi}C_{nj} = (\det C)\sum_{k=1}^{3}\varepsilon_{ijk}C_{kl}\,. \tag{81}$$

Applying equation (30) leads to [15–17]

$$(\operatorname{tr} C)C - C^2 = (\det C)[(\operatorname{tr} C)I - C^T], \tag{82}$$

and

$$(\operatorname{tr} C)^2 - \operatorname{tr}(C^2) = 2(\det C)(\operatorname{tr} C)\,. \tag{83}$$

It is important to distinguish the algebraic properties of vectors (what, in fact, makes vectors *vectors*) from the tensorial properties of vectors, i.e., how they transform under changes of bases, which is not so much a property of vectors as of their representations.

The Rotation Matrix and Related Quantities

A rotation about the z-axis, or rather the $\hat{e}_3$-axis, through an angle θ is represented pictorially in Fig. 1. Mathematically, this rotation is described by

$$\hat{e}_1' = \cos\theta\hat{e}_1 + \sin\theta\hat{e}_2\,, \tag{84a}$$

$$\hat{e}_2' = -\sin\theta\hat{e}_1 + \cos\theta\hat{e}_2\,, \tag{84b}$$

$$\hat{e}_3' = \hat{e}_3\,, \tag{84c}$$

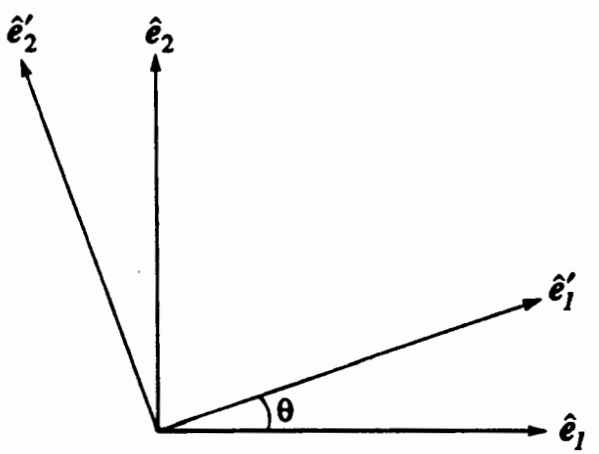

FIG. 1. Rotation about the $\hat{e}_3$-Axis.

or

$$\hat{\mathbf{e}}_i' = \sum_{j=1}^{3} R_{ij}(\hat{\mathbf{3}}, \theta)\hat{\mathbf{e}}_j \,. \tag{85}$$

Thus,

$$R(\hat{\mathbf{3}}, \theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{86}$$

and

$$\mathbf{x}' = R(\hat{\mathbf{3}}, \theta)\mathbf{x} \,. \tag{87}$$

In this context the direction-cosine matrix R is called the *rotation matrix*. Necessarily, R is proper orthogonal. Note that the rotation matrix is a function of the representation of the axis of rotation and not of the abstract vector. (The value of the matrix changes if the indices of the coordinate axes are altered, even if the physical rotation axis remains the same.) Since the representation of the axis of rotation is necessarily the same with respect to the primed and unprimed bases, no confusion can result as to which value to choose. For rotations about the other two axes

$$R(\hat{\mathbf{1}}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \qquad R(\hat{\mathbf{2}}, \theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{88}$$

obtained by cyclic permutation of equation (84)

A general expression for the rotation matrix for an arbitrary axis of rotation may be obtained by noting that

$$R(\hat{\mathbf{3}}, \theta)\hat{\mathbf{1}} = \cos\theta\,\hat{\mathbf{1}} - \sin\theta\,\hat{\mathbf{2}} = \cos\theta\,\hat{\mathbf{1}} - \sin\theta\,\hat{\mathbf{3}} \times \hat{\mathbf{1}}, \tag{89a}$$

$$R(\hat{\mathbf{3}}, \theta)\hat{\mathbf{2}} = \cos\theta\,\hat{\mathbf{2}} + \sin\theta\,\hat{\mathbf{1}} = \cos\theta\,\hat{\mathbf{2}} - \sin\theta\,\hat{\mathbf{3}} \times \hat{\mathbf{2}}, \tag{89b}$$

$$R(\hat{\mathbf{3}}, \theta)\hat{\mathbf{3}} = \hat{\mathbf{3}} \,. \tag{89c}$$

Hence, by analogy, if $\hat{\mathbf{n}}$ is an arbitrary unit column vector, and $\mathbf{v}_\perp$ is the projection of the column vector $\mathbf{v}$, onto the plane perpendicular to $\hat{\mathbf{n}}$ (this is also a three-dimensional column vector), then

$$R(\hat{\mathbf{n}}, \theta)\mathbf{v}_\perp = \cos\theta\,\mathbf{v}_\perp - \sin\theta\,\hat{\mathbf{n}} \times \mathbf{v}_\perp \tag{90a}$$

$$= \cos\theta\,\mathbf{v}_\perp + \sin\theta\,[[\hat{\mathbf{n}}]]\mathbf{v}_\perp \,. \tag{90b}$$

Also, $\mathbf{v}_\parallel$, the projection of $\mathbf{v}$ along $\hat{\mathbf{n}}$, is not changed by the rotation, since by analogy with equation (89c)

$$R(\hat{\mathbf{n}}, \theta)\hat{\mathbf{n}} = \hat{\mathbf{n}} \,. \tag{91}$$

Thus, in general,

$$R(\hat{\mathbf{n}}, \theta)\mathbf{v} = \mathbf{v}_\parallel + \cos\theta\,\mathbf{v}_\perp + \sin\theta\,[[\hat{\mathbf{n}}]]\mathbf{v}_\perp \,, \tag{92}$$

From equation (54) it follows that

$$\mathbf{v} = (\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}} - \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{v}) \tag{93}$$

$$= \hat{\mathbf{n}}\hat{\mathbf{n}}^T\mathbf{v} - [[\hat{\mathbf{n}}]]^2\mathbf{v} \tag{94}$$

$$\equiv \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}, \tag{95}$$

which gives the general decomposition of $\mathbf{v}$ into components which are parallel and perpendicular to $\hat{\mathbf{n}}$. Thus, for an arbitrary column vector $\mathbf{v}$ and arbitrary unit column vector $\hat{\mathbf{n}}$,

$$R(\hat{\mathbf{n}}, \theta)\mathbf{v} = \hat{\mathbf{n}}\hat{\mathbf{n}}^T\mathbf{v} - \cos\theta\,[[\hat{\mathbf{n}}]]^2\mathbf{v} + \sin\theta\,[[\hat{\mathbf{n}}]]\mathbf{v}, \tag{96}$$

where equation (51) has been used to remove the term in $[[\hat{\mathbf{n}}]]^3$. Since $\mathbf{v}$ is an arbitrary column vector, it follows that

$$R(\hat{\mathbf{n}}, \theta) = \cos\theta I + (1 - \cos\theta)\hat{\mathbf{n}}\hat{\mathbf{n}}^T + \sin\theta\,[[\hat{\mathbf{n}}]], \tag{97}$$

or in terms of individual elements,

$$R(\hat{\mathbf{n}}, \theta) = \begin{bmatrix} c + n_1^2(1-c) & n_1n_2(1-c) + n_3s & n_1n_3(1-c) - n_2s \\ n_2n_1(1-c) - n_3s & c + n_2^2(1-c) & n_2n_3(1-c) + n_1s \\ n_3n_1(1-c) + n_2s & n_3n_2(1-c) - n_1s & c + n_3^2(1-c) \end{bmatrix}, \tag{98}$$

where $c = \cos\theta$ and $s = \sin\theta$. Equivalently,

$$R(\hat{\mathbf{n}}, \theta) = I + \sin\theta\,[[\hat{\mathbf{n}}]] + (1 - \cos\theta)\,[[\hat{\mathbf{n}}]]^2. \tag{99}$$

Any of equations (96) through (99) is generally known as Euler's formula [18].[3] The derivation of Euler's formula has been subject to constant repetition in the literature. Beatty [19] cites more than a dozen such works which appeared between 1962 and 1976.

If we write the abstract axis of rotation vector as

$$\hat{\boldsymbol{n}} \equiv \sum_{i=1}^{3} n_i\hat{\boldsymbol{e}}_i = \sum_{i=1}^{3} n_i\hat{\boldsymbol{e}}_i', \tag{100}$$

then equation (85) can be written equivalently as

$$\hat{\boldsymbol{e}}_i' = \cos\theta\hat{\boldsymbol{e}}_i + (1 - \cos\theta)(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{e}}_i)\hat{\boldsymbol{n}} + \sin\theta\hat{\boldsymbol{n}} \times \hat{\boldsymbol{e}}_i \tag{101a}$$

$$= \hat{\boldsymbol{e}}_i + \sin\theta\hat{\boldsymbol{n}} \times \hat{\boldsymbol{e}}_i + (1 - \cos\theta)\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{e}}_i). \tag{101b}$$

These should be compared with equations (97) and (99). Note in particular the signs multiplying $\sin\theta$. Equation (100) is the abstract counterpart of equation (91).

The angle and axis of rotation are recoverable from R according to

$$\cos\theta = \frac{1}{2}(\mathrm{tr}\,R - 1), \tag{102}$$

[3] In its original presentation by Euler the rotation matrix was written as a function of three angles: the two spherical angles describing the axis of rotation and the angle of rotation.

and for $\sin\theta \neq 0$,

$$\hat{\mathbf{n}} = \frac{1}{2\sin\theta}\begin{bmatrix} R_{23} - R_{32} \\ R_{31} - R_{13} \\ R_{12} - R_{21} \end{bmatrix}, \tag{103a}$$

or in component form,

$$n_i = \frac{1}{2\sin\theta}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}R_{jk}\,. \tag{103b}$$

For $\theta = 0, \hat{\mathbf{n}}$ is undefined but not physically significant. For $\theta = \pi, R$ has the form

$$R(\hat{\mathbf{n}}, \pi) = -I + 2\hat{\mathbf{n}}\hat{\mathbf{n}}^T, \tag{104}$$

so that any column of $I + R(\hat{\mathbf{n}}, \pi)$ is proportional to $\hat{\mathbf{n}}$. The sign of $\hat{\mathbf{n}}$ is not significant in this case.

An important result is Euler's theorem [20, 21], which states effectively that every proper orthogonal matrix is a rotation matrix, i.e., it can be described by Euler's formula. Thus, the expressions rotation matrix and proper orthogonal matrix may be used interchangeably.

The rotation matrix, because it is orthogonal, satisfies

$$R^TR = I. \tag{105}$$

Since R^TR is symmetric, equation (105) implies only six constraints. Thus, the fact that rotations can have at most three degrees of freedom and can be described by at most three independent parameters follows directly from the orthogonality property.

As a consequence of equations (80) and (99) the rotation matrix satisfies

$$R'R(\hat{\mathbf{n}}, \theta)R'^T = R(\hat{\mathbf{n}}', \theta)\,, \tag{106}$$

where R' is an arbitrary rotation matrix and

$$\hat{\mathbf{n}}' = R'\hat{\mathbf{n}}\,. \tag{107}$$

Like the rotation matrix R, the axis and angle of rotation $(\hat{\mathbf{n}}, \theta)$, also constitute a representation of the rotation. Note, however, that the description of a rotation in terms of the axis and angle of rotation is not unique, since $(\hat{\mathbf{n}}, \theta)$ and $(-\hat{\mathbf{n}}, -\theta)$ (or, equivalently $(-\hat{\mathbf{n}}, 2\pi - \theta)$) correspond to the same rotation. An important related representation is the *rotation vector* $\boldsymbol{\theta}$, defined by

$$\boldsymbol{\theta} \equiv \theta\hat{\mathbf{n}}\,. \tag{108}$$

Of particular importance is the case when the angle of rotation, θ, is infinitesimally small. In that case one usually writes the angle of rotation as $\Delta\theta$, and notes that $\sin\Delta\theta \approx \Delta\theta$, and $\cos\Delta\theta \approx 1$. Euler's formula becomes then

$$R = I + [[\Delta\boldsymbol{\theta}]] + O(|\Delta\boldsymbol{\theta}|^2)\,, \tag{109}$$

where

$$\Delta\boldsymbol{\theta} \equiv \Delta\theta\hat{\mathbf{n}}, \tag{110}$$

is the *infinitesimal rotation vector*, and $O(x^c)$ denotes a quantity which becomes proportional to x^c as x tends to zero. The components of $\Delta\boldsymbol{\theta}$ are termed the *infinitesimal angles*. In contradistinction to finite rotations, infinitesimal rotations commute when quadratic and higher-order terms can be ignored.

From

$$R(\hat{\mathbf{n}}, \theta_2)R(\hat{\mathbf{n}}, \theta_1) = R(\hat{\mathbf{n}}, \theta_1 + \theta_2), \tag{111}$$

it follows by induction and from equation (109) that

$$R(\hat{\mathbf{n}}, \theta) = \lim_{N\to\infty}[R(\hat{\mathbf{n}}, \theta/N)]^N = \lim_{N\to\infty}\left[I + \frac{\theta}{N}[[\hat{\mathbf{n}}]]\right]^N = \exp\{[[\boldsymbol{\theta}]]\}, \tag{112}$$

where $\exp\{\cdot\}$ denotes the matrix exponential function. Explicit evaluation of the matrix exponential function via the Taylor series leads to equation (97) or (99).

The composition rule for the axis and angle of rotation or, equivalently, for the rotation vector are derived most easily from the composition rule for the Euler-Rodrigues symmetric parameters, which are treated in a later section of this survey. If two successive rotations are represented in terms of the axis and angle of rotation,

$$R(\hat{\mathbf{n}}_3, \theta_3) = R(\hat{\mathbf{n}}_2, \theta_2)R(\hat{\mathbf{n}}_1, \theta_1), \tag{113}$$

then the relevant composition rules are

$$\cos(\theta_3/2) = \cos(\theta_1/2)\cos(\theta_2/2) - \sin(\theta_1/2)\sin(\theta_2/2)\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_1, \tag{114}$$

$$\hat{\mathbf{n}}_3 = \frac{\sin(\theta_1/2)\cos(\theta_2/2)}{\sin(\theta_3/2)}\hat{\mathbf{n}}_1 + \frac{\sin(\theta_2/2)\cos(\theta_1/2)}{\sin(\theta_3/2)}\hat{\mathbf{n}}_2 - \frac{\sin(\theta_1/2)\sin(\theta_2/2)}{\sin(\theta_3/2)}\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1. \tag{115}$$

Equation (115) follows most simply from the quaternion composition rule discussed below. Note again that the ambiguity in the sign of θ_3 is unimportant physically. Likewise, if two successive rotations are represented in terms of the rotation vector,

$$R(\boldsymbol{\theta}_3) = R(\boldsymbol{\theta}_2)R(\boldsymbol{\theta}_1), \tag{116}$$

then the relevant composition rules follow trivially from those of the axis and angle of rotation.

Note that the two rotation vectors which appear in the right member of equation (115) are not referred to the same coordinate system. The first rotation vector $\boldsymbol{\theta}_1$ is referred to the initial coordinate system (as is $\boldsymbol{\theta}_3$), but $\boldsymbol{\theta}_2$ is referred to the intermediate coordinate system reached after the first rotation. If it is desired to combine two rotation vectors which are both referred to the initial coordinate system, then the second rotation vector must be transformed to the intermediate

frame, and the composition rule becomes instead

$$R((\boldsymbol{\theta}_3)_{\mathcal{E}}) = R((\boldsymbol{\theta}_2)_{\mathcal{E}'})R((\boldsymbol{\theta}_1)_{\mathcal{E}}) = R(R((\boldsymbol{\theta}_1)_{\mathcal{E}})(\boldsymbol{\theta}_2)_{\mathcal{E}})R((\boldsymbol{\theta}_1)_{\mathcal{E}})$$
$$= R((\boldsymbol{\theta}_1)_{\mathcal{E}})R((\boldsymbol{\theta}_2)_{\mathcal{E}}), \tag{117}$$

where equations (106) and (107) have been used. Similar arguments apply to the representation in terms of the axis and angle of rotation. Equation (117) is known as *Rodrigues' Transposition Theorem*. The net result when both rotation vectors or both axes of rotations are referred to the initial basis is to change the sign of the term in $\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1$ in equation (115).

The rotation matrices form a group. The group operation is matrix multiplication and the group identity element of the group (the *identity rotation matrix*) is the 3×3 identity matrix, and the inverse of the rotation matrix is given by the transpose, which, since it is proper orthogonal, is necessarily a rotation matrix also. This group is usually denoted by SO(3), the group of special orthogonal matrices in 3 dimensions, and is a subgroup of the group O(3), the group of orthogonal matrices in three dimensions.

The Euler Angles

Consider a rotation matrix defined in terms of three consecutive rotations about body-referenced axes of the form

$$R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2', \hat{\mathbf{n}}_3''; \theta_1, \theta_2, \theta_3) \equiv R(\hat{\mathbf{n}}_3'', \theta_3)R(\hat{\mathbf{n}}_2', \theta_2)R(\hat{\mathbf{n}}_1, \theta_1), \tag{118}$$

where

$$\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2' \text{ and } \hat{\mathbf{n}}_3'' = \hat{\mathbf{1}}, \hat{\mathbf{2}} \text{ or } \hat{\mathbf{3}}. \tag{119}$$

To understand the significance of this parameterization and the primes, consider the result of the three consecutive rotations, which can be written in terms of the bases as

$$\mathcal{E}''' \xleftarrow{R(\hat{\mathbf{n}}_3'', \theta_3)} \mathcal{E}'' \xleftarrow{R(\hat{\mathbf{n}}_2', \theta_2)} \mathcal{E}' \xleftarrow{R(\hat{\mathbf{n}}_1, \theta_1)} \mathcal{E}$$

The representation of $\hat{\mathbf{n}}_1$ must be with respect to the basis $\mathcal{E}$ (or $\mathcal{E}'$, since from equation (91) $\hat{\mathbf{n}}_1$ has the same representation with respect to both anterior and posterior bases), and similarly for $\hat{\mathbf{n}}_2'$ and $\hat{\mathbf{n}}_3''$. Thus,

$$\hat{\mathbf{n}}_1 \equiv (\hat{\mathbf{n}}_1)_{\mathcal{E}}, \qquad \hat{\mathbf{n}}_2' \equiv (\hat{\mathbf{n}}_2')_{\mathcal{E}'}, \qquad \hat{\mathbf{n}}_3'' \equiv (\hat{\mathbf{n}}_3'')_{\mathcal{E}''}. \tag{120}$$

If the successive rotations are considered as physical rotations of the body (read "vehicle"), then at each stage $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{E}''$, and $\mathcal{E}'''$ are the body-fixed axes of the vehicle. Frequently, $\mathcal{E}$ is the basis of an inertial coordinate system but may be any coordinate system not fixed in the body. In general, we shall refer to $\mathcal{E}$ as the primary reference system. Hence, θ_1, θ_2 and θ_3 are the *body-referenced Euler angles* [22], which are more commonly written as φ, ϑ, and ψ.[4]

[4]The aeronautics community and a segment of the space community prefer the reverse order for naming the three Euler angles. General books on Mechanics lean mostly toward the convention of this survey. Many authors prefer to simply number the angles.

(The use of the variant form of the Greek *theta*, ϑ, avoids the possibility of confusion with the (total) angle of rotation θ.) The column vectors $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2'$ and $\hat{\mathbf{n}}_3''$ are called the *body-fixed Euler axes*. Since these are restricted by convention to the values $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$, the body-fixed Euler axes (or simply Euler axes) correspond to the coordinate axes of the body-fixed bases. In order that the representation in terms of the Euler angles have the required three degees of freedom, it is necessary that

$$\hat{\mathbf{n}}_1 \neq \hat{\mathbf{n}}_2' \quad \text{and} \quad \hat{\mathbf{n}}_2' \neq \hat{\mathbf{n}}_3'' . \tag{121}$$

Given this restriction, there are twelve possible sets of Euler angles: six symmetric sets, whose labels are written as

1–2–1	1–3–1
2–3–2	2–1–2
3–1–3	3–2–3

and six asymmetric sets, designated by

1–2–3	1–3–2
2–3–1	2–1–3
3–1–2	3–2–1

In each label the first (leftmost) integer denotes the first rotation axis. For example, the 1–3–2 set of Euler angles correspond to $\hat{\mathbf{n}}_1 = \hat{\mathbf{1}}$, $\hat{\mathbf{n}}_2' = \hat{\mathbf{3}}$, and $\hat{\mathbf{n}}_3'' = \hat{\mathbf{2}}$. The asymmetric sets of Euler elements have been called variously *Cardan angles*, *Tait angles*, or *Bryant angles*.

The 3–1–3 Euler angles have been particularly popular for the description of spinning tops and atomic nuclei, and it is the set of Euler angles most frequently encountered in the quantum theory of angular momentum [23]. The rotation of coordinate axes in terms of the 3–1–3 Euler angles is depicted in Fig. 2. The six members of each of the two types of Euler-angle sets share many common ana-

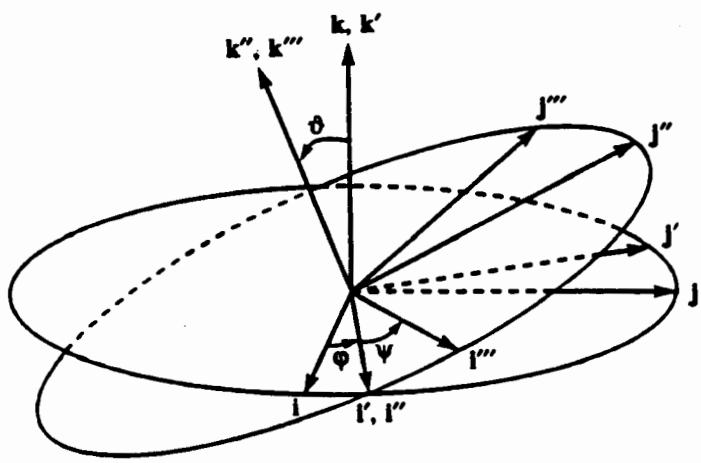

FIG. 2. The 3–1–3 Euler Angles.

lytical properties. Within each type there are minor differences between the members of the two columns, which are grouped according to whether or not the label of the second axis is the cyclic successor of the label of the first axis.

For the 3–1–3 Euler angles the rotation matrix is given by

$$R_{313}(\varphi,\vartheta,\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{bmatrix}\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{122a}$$

$$= \begin{bmatrix} c\psi c\varphi - s\psi c\vartheta s\varphi & c\psi s\varphi + s\psi c\vartheta c\varphi & s\psi s\vartheta \\ -s\psi c\varphi - c\psi c\vartheta s\varphi & -s\psi s\varphi + c\psi c\vartheta c\varphi & c\psi s\vartheta \\ s\vartheta s\varphi & -s\vartheta c\varphi & c\vartheta \end{bmatrix}, \tag{122b}$$

where $s\psi$ has been written in place of $\sin\psi$, et cetera. Every rotation matrix can be written in terms of 3–1–3 Euler angles (or any other set). In contrast to equations (122), the parameterization of the rotation matrix in terms of 3–1–2 Euler angles is given by

$$R_{312}(\varphi,\vartheta,\psi) = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{bmatrix}\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{123a}$$

$$= \begin{bmatrix} c\psi c\varphi - s\psi s\vartheta s\varphi & c\psi s\varphi + s\psi s\vartheta c\varphi & -s\psi c\vartheta \\ -c\vartheta s\varphi & c\vartheta c\varphi & s\vartheta \\ s\psi c\varphi + c\psi s\vartheta s\varphi & s\psi s\varphi - c\psi s\vartheta c\varphi & c\psi c\vartheta \end{bmatrix}. \tag{123b}$$

Note that it is $\sin\vartheta$ which appears in the simplest element. The equation for the rotation matrix as a function of each of the twelve sets of body-referenced Euler angles is presented in several texts [2, 5, 6]. The reader should take note of qualitative differences in the formulas for the symmetric and asymmetirc sets of Euler angles. In actual applications, when computing the rotation matrix from the Euler angles, it is usually better to compute equation (122) or equation (123) directly from "elemental" rotations within the program. Graphical methods also exist [24–26] for computing the rotation matrix and angular velocity as a function of the Euler angles.

For the symmetric sets of Euler angles the axis and angle of rotation are given by

$$\cos\frac{\theta}{2} = \cos\frac{\vartheta}{2}\cos\frac{\varphi+\psi}{2}, \tag{124a}$$

and

$$\hat{\mathbf{n}} = \frac{1}{2\sin\theta}\begin{bmatrix} \sin\vartheta(\cos\varphi + \cos\psi) \\ \sin\vartheta(\sin\varphi - \sin\psi) \\ (1+\cos\vartheta)\sin(\varphi+\psi) \end{bmatrix}. \tag{124b}$$

No similar simple expression exists for the asymmetric sets of Euler angles.

The Euler angles are not unique. In fact,

$$R_{313}(\varphi, \vartheta, \psi) = R_{313}(\varphi + \pi, -\vartheta, \psi - \pi). \tag{125}$$

To maintain a unique set of Euler angles (at least for $\vartheta \neq 0$ or π), one usually demands

$$0 \leq \varphi < 2\pi, \qquad 0 \leq \vartheta \leq \pi, \qquad 0 \leq \psi < 2\pi. \tag{126}$$

The ambiguity stated in equation (125) holds for all six symmetric sets of Euler angles. For the six asymmetric sets the equivalent relation is

$$R_{312}(\varphi, \vartheta, \psi) = R_{312}(\varphi + \pi, \pi - \vartheta, \psi - \pi). \tag{127}$$

In order to maintain a unique set of Euler angles in this case (at least for $\vartheta \neq \pm\pi/2$), one usually demands

$$0 \leq \varphi < 2\pi, \qquad -\pi/2 \leq \vartheta \leq \pi/2, \qquad 0 \leq \psi < 2\pi. \tag{128}$$

The Euler angles from a given rotation matrix are determined from inspection of the elements. Taking as an example the 3–1–3 set of Euler angles we note from equation (122b)

$$R_{33} = \cos\vartheta, \tag{129a}$$

$$R_{31} = \sin\vartheta \sin\varphi, \qquad R_{13} = \sin\vartheta \sin\psi, \tag{129bc}$$

$$R_{32} = -\sin\vartheta \cos\varphi. \qquad R_{23} = \sin\vartheta \cos\psi. \tag{129de}$$

Thus,

$$\vartheta = \arccos(R_{33}), \tag{130}$$

and for $\sin\vartheta > 0$,

$$\varphi = \arctan_2(R_{31}, -R_{32}), \qquad \psi = \arctan_2(R_{13}, R_{23}), \tag{131}$$

where $\arctan_2(y, x)$ is a function which gives the angle whose tangent is y/x and which is in the correct quadrant. In FORTRAN this function has the name "ATAN2."

For $\sin\vartheta = 0$, the rotation matrix for the 3–1–3 set of Euler angles has the form

$$R_{313} = \begin{bmatrix} \cos(\varphi \pm \psi) & \sin(\varphi \pm \psi) & 0 \\ \mp\sin(\varphi \pm \psi) & \pm\cos(\varphi \pm \psi) & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}, \tag{132}$$

so that φ and ψ are not uniquely determined, although depending on whether $\vartheta = 0$ or π, $\varphi + \psi$ or $\varphi - \psi$, respectively, is determined. One usually chooses arbitrarily as a solution

$$\psi = 0, \qquad \varphi = \arctan_2(R_{12}, R_{11}). \tag{133}$$

A slightly different approach to determining the Euler angles from the direction-cosine matrix has been described by Nicholson, Markley, and Seidewitz [27].

Kolve [28] has presented general expressions for the direction-cosine matrix as a function of arbitrary symmetric and asymmetric sequences of Euler angles. If

i, j, i are the indices of the axes of the three successive rotations characterizing a symmetric sequence of Euler angles, and k is the remaining coordinate axis, then the direction-cosine matrix may be written in the form

$$R_{iji}(\varphi,\vartheta,\psi) = \begin{array}{c} \\ i \\ j \\ k \end{array}\!\!\begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{bmatrix} c\vartheta & s\vartheta s\varphi & -\alpha_{ij}s\vartheta c\varphi \\ s\psi s\vartheta & c\psi c\varphi - s\psi c\vartheta s\varphi & \alpha_{ij}(c\psi s\varphi + s\psi c\vartheta c\varphi) \\ \alpha_{ij}c\psi s\vartheta & -\alpha_{ij}(s\psi c\varphi + c\psi c\vartheta s\varphi) & -s\psi s\varphi + c\psi c\vartheta c\varphi \end{bmatrix} \end{array} \tag{134}$$

where

$$\alpha_{ij} \equiv \sum_{l=1}^{3} \epsilon_{ijl} = \begin{cases} +1, & \text{for } j \text{ the cyclic successor of } i, \\ -1, & \text{for } i \text{ the cyclic successor of } j, \\ 0, & \text{otherwise.} \end{cases} \tag{135}$$

Likewise, if i, j, and k are the indices of the three successive axes characterizing an asymmetric sequence of Euler angles, then the direction-cosine matrix may be written

$$R_{ijk}(\varphi,\vartheta,\psi) = \begin{array}{c} \\ i \\ j \\ k \end{array}\!\!\begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{bmatrix} c\psi c\vartheta & \alpha_{ij}s\psi c\varphi + c\psi s\vartheta s\varphi & s\psi s\varphi - \alpha_{ij}c\psi s\vartheta c\varphi \\ -\alpha_{ij}s\psi c\vartheta & c\psi c\varphi - \alpha_{ij}s\psi s\vartheta s\varphi & \alpha_{ij}c\psi s\varphi + s\psi s\vartheta c\varphi \\ \alpha_{ij}s\vartheta & -\alpha_{ij}c\vartheta s\varphi & c\vartheta c\varphi \end{bmatrix} \end{array} \tag{136}$$

Equations (122b) and (123b) are special cases of these equations. Note that for a 3–1–2 sequence of Euler angles the rows and columns of equation (136) are in the order 3–1–2, as indicated by the indices shown in the border of the matrices, and similarly for equation (134).

To extract the Euler angles from the direction-cosine matrix the corresponding general formulas for the symmetric sequence of Euler angles are

$$\vartheta = \arccos(R_{ii}), \tag{137}$$

which for $\sin\vartheta \neq 0$ leads to

$$\varphi = \arctan_2(R_{ij}, -\alpha_{ij}R_{ik}), \qquad \psi = \arctan_2(R_{ji}, \alpha_{ij}R_{ki}), \tag{138}$$

or, for $\sin\vartheta = 0$, we choose arbitrarily

$$\varphi = \arctan_2(\alpha_{ij}R_{jk}, R_{jj}), \qquad \psi = 0. \tag{139}$$

For the asymmetric sequence of Euler angles we have similarly

$$\vartheta = \arcsin(\alpha_{ij}R_{ki}), \tag{140}$$

which for $\cos\vartheta \neq 0$ leads to

$$\varphi = \arctan_2(-\alpha_{ij}R_{kj}, R_{kk}), \qquad \psi = \arctan_2(-\alpha_{ij}R_{ji}, R_{ii}), \tag{141}$$

or, for $\cos \vartheta = 0$, we choose arbitrarily

$$\varphi = \arctan_2(\alpha_{ij} R_{jk}, R_{jj}), \qquad \psi = 0, \tag{142}$$

In equations (137) and (140), as previously, the inverse trigonometric functions assume their principal values to insure uniqueness.

Note that these equations for determining the Euler angles do not necessarily yield values which satisfy equations (126) or (128), nor is the prescription given by equations (126) and (128) necessarily the best, since the imposition of the constraint can lead to discontinuities in the Euler angles even if the rotation matrix varies continuously. We shall see in a later discussion that discontinuities in the Euler angles are unavoidable, even with the relaxation of these prescriptions.

It is possible to combine the symmetric Euler angles for two successive rotations relatively simply. If $(\varphi_1, \vartheta_1, \psi_1)$ and $(\varphi_2, \vartheta_2, \psi_2)$ are two sets of symmetric Euler angles with respect to the same sequence of body axes, say 3–1–3, then the composition of these two symmetric sets of Euler angles is defined by

$$R_{313}(\varphi_3, \vartheta_3, \psi_3) \equiv R_{313}(\varphi_2, \vartheta_2, \psi_2) R_{313}(\varphi_1, \vartheta_1, \psi_1). \tag{143}$$

This is equivalent to [29]

$$R_{313}(\varphi_3 - \varphi_1, \vartheta_2, \psi_3 - \psi_2) = R_{131}(\vartheta_1, \psi_1 + \varphi_2, \vartheta_2), \tag{144}$$

from which we may derive [29, 30]

$$\vartheta_3 = \arccos(\cos \vartheta_1 \cos \vartheta_2 - \sin \vartheta_1 \sin \vartheta_2 \cos(\psi_1 + \varphi_2)), \tag{145a}$$

$$\varphi_3 = \varphi_1 + \arctan_2(\sin \vartheta_1 \sin \vartheta_2 \sin(\psi_1 + \varphi_2), \cos \vartheta_2 - \cos \vartheta_3 \cos \vartheta_1), \tag{145b}$$

$$\psi_3 = \psi_2 + \arctan_2(\sin \vartheta_1 \sin \vartheta_2 \sin(\psi_1 + \varphi_2), \cos \vartheta_1 - \cos \vartheta_3 \cos \vartheta_2). \tag{145c}$$

These equations hold for all six symmetric sets of Euler angles and can be derived also from the spherical triangle of Fig. 3. Apparently, no equally simple expression exists for the asymmetric sets of Euler angles.

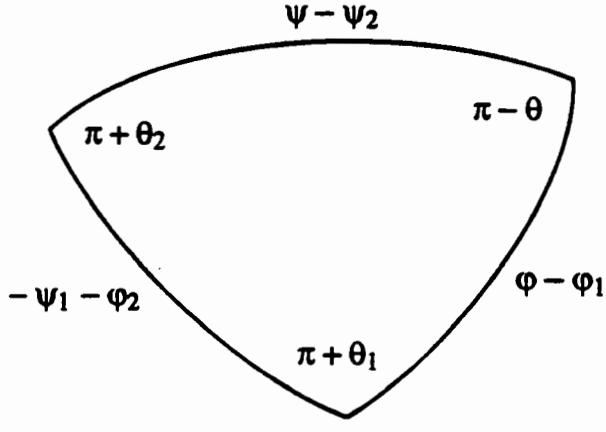

FIG. 3. Geometric Composition of the 3–1–3 Euler Angles.

The Euler angles are singular for certain attitudes. The rotation matrix as a function of the 3–1–3 Euler angles when these are infinitesimally small is given by

$$R_{313}(\Delta\varphi, \Delta\vartheta, \Delta\psi) \approx \begin{bmatrix} 1 & \Delta\varphi + \Delta\psi & 0 \\ -(\Delta\varphi + \Delta\psi) & 1 & \Delta\vartheta \\ 0 & -\Delta\vartheta & 1 \end{bmatrix}. \tag{146}$$

From equation (146) it is obviously impossible to represent an *infinitesimal* rotation about the y-axis in terms of *infinitesimal* 3–1–3 Euler angles. However, for the identity rotation, $R = I$, it is true that

$$R_{313}(0, 0, 0) = I. \tag{147}$$

It follows that if initially the three Euler angles all vanish, and some infinitesimal rotation is performed on the system about the y-axis, the 3–1–3 Euler angles must change instantaneously from $(0, 0, 0)$ to some finite value. A similar phenomenon occurs when $\vartheta = \pi$. (In gyroscopes, this phenomenon is called *gimbal-lock*, and it manifests itself as a seizing of the gimbals, because these, not being massless, cannot move by a finite amount in an infinitesimal time.) Since a very small change in the attitude can lead to a large change in the 3–1–3 Euler angles, these are not accurately determined when ϑ is close to either 0 or 2π. This phenomenon is true for all six symmetric sets of Euler angles.

Note, however, that

$$R_{312}(\Delta\varphi, \Delta\vartheta, \Delta\psi) \approx \begin{bmatrix} 1 & \Delta\varphi & -\Delta\psi \\ -\Delta\varphi & 1 & \Delta\vartheta \\ \Delta\psi & -\Delta\vartheta & 1 \end{bmatrix}, \tag{148}$$

which is well behaved for infinitesimal 3–1–2 Euler angles. However, this representation has similar singularity problems when $\vartheta \approx \pm\pi/2$. Thus, to be able to describe all attitudes accurately in terms of Euler angles, at least two sets must be employed. Note that equation (148) shows that the components of the infinitesimal rotation vector in equation (109) can be interpreted as infinitesimal 3–1–2 (or any other asymmetrical set of) Euler angles.

In addition, the rotation matrix can be expressed also in terms of *space-referenced Euler angles.* Thus, we can write

$$R^{\text{space}}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3; \theta_1, \theta_2, \theta_3) \equiv R((\hat{\mathbf{n}}_3^{\text{space}})_{\mathcal{E}'}, \theta_3) R((\hat{\mathbf{n}}_2^{\text{space}})_{\mathcal{E}'}, \theta_2) R((\hat{\mathbf{n}}_1^{\text{space}})_{\mathcal{E}}, \theta_1), \tag{149}$$

where the axis column vectors in equation (149) are now the representations of inertial or "space"-fixed coordinate axes (i.e., the coordinate axes of $\mathcal{E}$, when these are inertial axes) but still with respect to the current body-axes.[5] These, in general, do not have the values $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$, or $\hat{\mathbf{3}}$ (except for $\hat{\mathbf{n}}_1^{\text{space}}$, since the initial

[5]Here we see illustrated a frequent inconsistency in notation. In equations (106) and (118), the prime distinguishes the abstract vector, while in equation (73) the prime indicates the basis of representation. This ambiguity in notation can be avoided only by indicating both the abstract vector and the basis of representation.

body and inertial reference axes are the same). However, it is possible to show by application of equations (106) and (107) that equation (149) is equivalent to

$$R^{\text{space}}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3; \theta_1, \theta_2, \theta_3) = R((\hat{\mathbf{n}}_1^{\text{space}})_{\mathcal{E}}, \theta_1) R((\hat{\mathbf{n}}_2^{\text{space}})_{\mathcal{E}}, \theta_2) R((\hat{\mathbf{n}}_3^{\text{space}})_{\mathcal{E}}, \theta_3), \tag{150}$$

where the representations of the space axes are now with respect to the space axes and hence have the values $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$, or $\hat{\mathbf{3}}$. This result follows directly from equation (117). (This shows also that from the point of view of parameterization, there is no advantage in using space-referenced Euler angles instead of the body-referenced variety.)

The most obvious advantage of the Euler angles is that they have the minimum dimension. However, the large computational burden imposed by their use and the problem of the singularity make them only infrequently the best choice in real-world applications.

Of particular interest is a new representation offered very recently by Markley [31], which represents the attitude in terms of inertial and body representations of a given abstract vector and an angle parameter. In certain special cases, this representation reduces effectively to the Euler angles. A related seven-dimensional representation has been applied to dynamical studies by Broucke [32].

The Axis-Azimuth Representation

Similar to the Euler angles is the axis-azimuth representation [33, 34], which has been used chiefly in applications dealing with spinning spacecraft. In this representation two of the angles give the *elevation* and *right-ascension* of the spin axis, usually taken to be the z-axis (or $\boldsymbol{k}$-axis) of the spacecraft. The xy-plane of the spacecraft is then the spacecraft equatorial plane. Thus, with respect to inertial axes, $\{\hat{\boldsymbol{I}}, \hat{\boldsymbol{J}}, \hat{\boldsymbol{K}}\}$,

$$\hat{\mathbf{k}}_{\{\hat{I},\hat{J},\hat{K}\}} = \begin{bmatrix} \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{I}} \\ \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{J}} \\ \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{K}} \end{bmatrix} = \begin{bmatrix} \cos\delta \cos\alpha \\ \cos\delta \sin\alpha \\ \sin\delta \end{bmatrix}. \tag{151}$$

The remaining angle, ϕ, the *azmiuthal angle* or simply the *azimuth*, is the angle about the spacecraft z-axis to the spacecraft x-axis (denoted here by $\boldsymbol{i}$, the y-axis is $\boldsymbol{j}$) measured from the projection of the inertial X-axis (typically the vernal equinox, denoted here by $\hat{\boldsymbol{I}}$) on the spacecraft equatorial plane. Thus, if $\boldsymbol{I}_N$ is the projection of the vernal equinox onto the spacecraft equatorial plane, then

$$|\boldsymbol{I}_N| \cos\phi = \boldsymbol{I}_N \cdot \boldsymbol{i} = \hat{\boldsymbol{I}} \cdot \boldsymbol{i}, \tag{152a}$$

$$|\boldsymbol{I}_N| \sin\phi = (\boldsymbol{I}_N \times \boldsymbol{i}) \cdot \hat{\boldsymbol{k}} = -\boldsymbol{I}_N \cdot \boldsymbol{j} = -\hat{\boldsymbol{I}} \cdot \boldsymbol{j}. \tag{152b}$$

In terms of the elements of the rotation matrix,

$$\alpha = \arctan_2(R_{32}, R_{31}), \tag{153a}$$

$$\delta = \arcsin(R_{33}), \tag{153b}$$

$$\phi = \arctan_2(-R_{21}, R_{11}). \tag{153c}$$

The ranges of the axis-azimuth variables are

$$0 \leq \alpha < 2\pi, \qquad -\pi/2 \leq \delta \leq \pi/2, \qquad 0 \leq \phi < 2\pi\,. \tag{154}$$

The choice of different axes for defining the three angles (α, δ, ϕ) leads to different possibilities for the axis-azimuth representation, as it did for the symmetric sets of Euler angles.

The axis-azimuth representation is similar but not identical to the Euler-angle representation. For instance, if $(\varphi, \vartheta, \psi)$ is a 1–2–3 set of Euler angles, then

$$\varphi = \arctan_2(-\cos\delta \sin\alpha, \sin\delta)\,, \tag{155a}$$

$$\vartheta = \arcsin(\cos\delta \cos\alpha)\,, \tag{155b}$$

$$\psi = \phi\,. \tag{155c}$$

Similarly, if $(\varphi, \vartheta, \psi)$ is a 3–1–3 set of Euler angles, then

$$\varphi = \alpha + \pi/2\,, \tag{156a}$$

$$\vartheta = \pi/2 - \delta\,, \tag{156b}$$

$$\psi = \phi - \arctan_2(\sin\delta \cos\alpha, -\sin\alpha)\,. \tag{156c}$$

The Euler-Rodrigues Symmetric Parameters and the Quaternion

Euler's formula, equation (97), can be rewritten as

$$R(\boldsymbol{\eta}, \eta_4) = (\eta_4^2 - |\boldsymbol{\eta}|^2)I + 2\boldsymbol{\eta}\boldsymbol{\eta}^T + 2\eta_4[[\boldsymbol{\eta}]]\,, \tag{157}$$

or,

$$R(\boldsymbol{\eta}, \eta_4) = \begin{bmatrix} \eta_1^2 - \eta_2^2 - \eta_3^2 + \eta_4^2 & 2(\eta_1\eta_2 + \eta_4\eta_3) & 2(\eta_1\eta_3 - \eta_4\eta_2) \\ 2(\eta_2\eta_1 - \eta_4\eta_3) & -\eta_1^2 + \eta_2^2 - \eta_3^2 + \eta_4^2 & 2(\eta_2\eta_3 + \eta_4\eta_1) \\ 2(\eta_3\eta_1 + \eta_4\eta_2) & 2(\eta_3\eta_2 - \eta_4\eta_1) & -\eta_1^2 - \eta_2^2 + \eta_3^2 + \eta_4^2 \end{bmatrix}, \tag{158}$$

where

$$\boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \equiv \sin(\theta/2)\hat{\mathbf{n}}, \qquad \eta_4 \equiv \cos(\theta/2)\,. \tag{159}$$

The quantities η_1, η_2, η_3, and η_4, originally written $\{\xi, \eta, \zeta, \chi\}$, are known as the *Euler-Rodrigues symmetric parameters*, more commonly, *Euler symmetric parameters* [35, 36], or *quaternion of rotation* [37]. These are usually arranged in a four-dimensional column matrix, $\bar{\eta}$, given by

$$\bar{\eta} \equiv \begin{bmatrix} \boldsymbol{\eta} \\ \eta_4 \end{bmatrix} = [\eta_1 \quad \eta_2 \quad \eta_3 \quad \eta_4]^T. \tag{160}$$

The column matrix of Euler-Rodrigues symmetric parameters satisfies a simple normalization constraint

$$\bar{\eta}^T\bar{\eta} = |\eta|^2 + \eta_4^2 = 1. \tag{161}$$

The Euler-Rodrigues symmetric parameters can be extracted from the rotation matrix. Noting

$$\begin{aligned} \text{tr } R &\equiv R_{11} + R_{22} + R_{33}, \\ &= 4\eta_4^2 - 1. \end{aligned} \tag{162}$$

Hence,

$$\eta_4 = \pm\frac{1}{2}\sqrt{1 + \text{tr } R}. \tag{163}$$

If $\eta_4 \neq 0$, then the remaining components can be calculated as

$$\eta_1 = \frac{1}{4\eta_4}(R_{23} - R_{32}), \qquad \eta_2 = \frac{1}{4\eta_4}(R_{31} - R_{13}), \qquad \eta_3 = \frac{1}{4\eta_4}(R_{12} - R_{21}). \tag{164}$$

Note that the sign of $\bar{\eta}$ is not determined by equations (163) and (164). However, since R is a quadratic function of $\bar{\eta}$, the sign of $\bar{\eta}$ is not physically significant. This non-uniqueness of the Euler-Rodrigues symmetric parameters is related to the non-uniqueness of the axis/angle representation.

If η_4 is close to zero, equations (163) and (164) will not be very accurate due to loss of numerical significance in the computation of the square root. To get around this problem, note from the unit normalization of the quaternion, equation (161), that the diagonal elements of equation (158) can be written as

$$R_{11} = 1 - 2\eta_2^2 - 2\eta_3^2, \tag{165a}$$

$$R_{22} = 1 - 2\eta_1^2 - 2\eta_3^2, \tag{165b}$$

$$R_{33} = 1 - 2\eta_1^2 - 2\eta_2^2, \tag{165c}$$

which can be solved for any of the other components and the off-diagonal elements of R used to find the remaining components of $\bar{\eta}$. The results are

$$\eta_1 = \pm\frac{1}{2}\sqrt{1 + R_{11} - R_{22} - R_{33}}, \qquad \eta_2 = \frac{1}{4\eta_1}(R_{12} + R_{21}), \tag{166ab}$$

$$\eta_3 = \frac{1}{4\eta_1}(R_{13} + R_{31}), \qquad \eta_4 = \frac{1}{4\eta_1}(R_{23} - R_{32}), \tag{166cd}$$

$$\eta_2 = \pm\frac{1}{2}\sqrt{1 - R_{11} + R_{22} - R_{33}}, \qquad \eta_1 = \frac{1}{4\eta_2}(R_{21} + R_{12}), \tag{167ab}$$

$$\eta_3 = \frac{1}{4\eta_2}(R_{23} + R_{32}), \qquad \eta_4 = \frac{1}{4\eta_2}(R_{31} - R_{13}), \tag{167cd}$$

$$\eta_3 = \pm\frac{1}{2}\sqrt{1 - R_{11} - R_{22} + R_{33}}, \qquad \eta_1 = \frac{1}{4\eta_3}(R_{31} + R_{13}), \tag{168ab}$$

$$\eta_2 = \frac{1}{4\eta_3}(R_{32} + R_{23}), \qquad \eta_4 = \frac{1}{4\eta_3}(R_{12} - R_{21}). \tag{168cd}$$

Greatest numerical accuracy is obtained for the Euler-Rodrigues symmetric parameters if equations (163) and (164), equations (166), equations (167), or equations (168) are selected for evaluation according to which set has the largest argument in the square root. The four sets of equations above for computing the quaternion from the direction-cosine matrix have a simple geometric interpretation [38].

Unlike the case of the Euler angles, where, depending on the attitude, two different sets of angles were needed in order to achieve an accurate representation of the rotation, the four sets of equations above all yield the same set of Euler-Rodrigues symmetric parameters (within a sign). Since the sum of the squares of the Euler-Rodrigues symmetric parameters is unity, at least one of these parameters must have a value of at least 1/2. Thus, one of the above sets of equations will always yield a suitable solution. Note that this set of equations can be identified before any square roots are calculated.

The best method for extracting the Euler-Rodrigues symmetric parameters from the rotation matrix was the subject of considerable interest in the recent past [39–45]. The above procedure, was revealed at the end of this controversy to be the best.

Unlike the Euler angles, which cannot be combined easily for successive rotations, the Euler-Rodrigues symmetric parameters have a very simple composition rule, which can be obtained by the following steps: the composition of Euler-Rodrigues symmetric parameters, which is denoted by

$$\bar{\eta}'' = \bar{\eta}' \otimes \bar{\eta}, \tag{169}$$

is defined here so that it is also true that

$$R(\bar{\eta}'') = R(\bar{\eta}')R(\bar{\eta}). \tag{170}$$

The steps for extracting the Euler-Rodrigues symmetric parameters are shown diagrammatically in Fig. 4. First the rotation matrices corresponding to the two sets of Euler-Rodrigues symmetric parameters, $\bar{\eta}$ and $\bar{\eta}'$, are computed, then the two rotation matrices are combined by matrix multiplication, and finally the desired set of Euler-Rodrigues symmetric parameters is extracted from the resulting rotation matrix according to equations (163) and (164), or from equations (166), or from equations (167), or from equations (168).

Carrying out analytically the operations indicated in Fig. 4, the resulting expression for the composition of Euler-Rodrigues symmetric parameters is

$$\bar{\eta}' \otimes \bar{\eta} = \pm \begin{bmatrix} \eta_4\boldsymbol{\eta}' + \eta_4'\boldsymbol{\eta} - \boldsymbol{\eta}' \times \boldsymbol{\eta} \\ \eta_4'\eta_4 - \boldsymbol{\eta}' \cdot \boldsymbol{\eta} \end{bmatrix}. \tag{171}$$

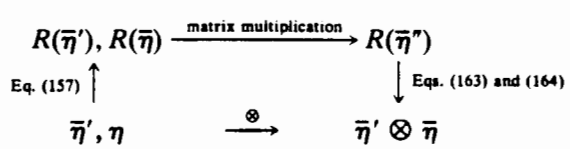

FIG. 4. Composition of the Euler-Rodrigues Symmetric Parameters.

The ambiguity in the sign in equation (171) is the result of the indifference of equation (157) or (158) to the sign of $\bar{\eta}$. Traditionally, one chooses the positive sign in equation (171), so that the multiplication rule for the Euler-Rodrigues symmetric parameters can be written as

$$\bar{\eta}' \otimes \bar{\eta} = \{\bar{\eta}'\}_L \bar{\eta} = \{\bar{\eta}\}_R \bar{\eta}', \tag{172}$$

where

$$\{\bar{\eta}\}_L \equiv \begin{bmatrix} \eta_4 & \eta_3 & -\eta_2 & \eta_1 \\ -\eta_3 & \eta_4 & \eta_1 & \eta_2 \\ \eta_2 & -\eta_1 & \eta_4 & \eta_3 \\ -\eta_1 & -\eta_2 & -\eta_3 & \eta_4 \end{bmatrix}, \qquad \{\bar{\eta}\}_R \equiv \begin{bmatrix} \eta_4 & -\eta_3 & \eta_2 & \eta_1 \\ \eta_3 & \eta_4 & -\eta_1 & \eta_2 \\ -\eta_2 & \eta_1 & \eta_4 & \eta_3 \\ -\eta_1 & -\eta_2 & -\eta_3 & \eta_4 \end{bmatrix}, \tag{173}$$

or, in terms of smaller submatrices,

$$\{\bar{\eta}\}_L = \eta_4 I_{4\times 4} + \begin{bmatrix} [[\boldsymbol{\eta}]] & \boldsymbol{\eta} \\ -\boldsymbol{\eta}^T & 0 \end{bmatrix}, \qquad \{\bar{\eta}\}_R = \eta_4 I_{4\times 4} + \begin{bmatrix} -[[\boldsymbol{\eta}]] & \boldsymbol{\eta} \\ -\boldsymbol{\eta}^T & 0 \end{bmatrix}. \tag{174}$$

Note that the two 4×4 matrices defined in equation (173) are each orthogonal.

The Euler-Rodrigues symmetric parameters have several advantages over the rotation matrix as an attitude representation. First, there are fewer elements (4 instead of 9), so that less storage is required. Secondly, there are fewer constraints (1 instead of 6). Thirdly, the composition rule is simpler (16 multiplications instead of 27). Further reduction in the number of multiplications is possible using the Strasser-Winograd algorithm [46,47]. Lastly, the constraint is very simple to enforce,

$$\bar{\eta} \longrightarrow (\bar{\eta}^T \bar{\eta})^{-1/2} \bar{\eta}, \tag{175}$$

while the "orthogonalization" of a 3×3 matrix is much more difficult [48–55]. It is well known that the orthogonalization of a 3×3 matrix can be related to the problem of optimal attitude estimation [49, 56].

The column vector of Euler-Rodrigues symmetric parameters forms a group under the multiplication rule of equations (172) and (173) with *unique* identity element

$$\bar{1} \equiv [0 \quad 0 \quad 0 \quad 1]^T, \tag{176}$$

and inverse

$$\bar{\eta}^{-1} \equiv \begin{bmatrix} -\boldsymbol{\eta} \\ \eta_4 \end{bmatrix}. \tag{177}$$

Note that although $-\bar{1}$ also corresponds to the identity rotation, it is not the identity element of the group of column vectors of Euler-Rodrigues symmetric parameters. This last statement would not be true, however, had we chosen to define the multiplication rule for Euler-Rodrigues symmetric parameters with the negative sign in equation (171). In that case -1 rather than $+1$ would have appeared in the fourth component of $\bar{1}$.

Note that

$$\bar{\eta} = \{\bar{\eta}\}_L \bar{1} = \{\bar{\eta}\}_R \bar{1}, \tag{178}$$

a fact that has been used by some authors [3] to simplify the evaluation of the Euler-Rodrigues symmetric parameters as a function of the different sets of Euler angles.

Equation (157) can be written equivalently as

$$\{\bar{\eta}\}_L \{\bar{\eta}^{-1}\}_R = \{\bar{\eta}^{-1}\}_R \{\bar{\eta}\}_L = \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}, \tag{179}$$

where R is the upper-left 3×3 partition of the right member of equation (179).

The column vector of Euler-Rodrigues symmetric parameters is a special case of a more general object, the *quaternion*, defined to be any arbitrary four-component vector (i.e., with arbitrary norm) which satisfies the multiplication rule of equations (172) and (173). Thus, every column vector of Euler-Rodrigues symmetric parameters is a quaternion (the *quaternion of rotation*), but the inverse is not true. (The name quaternion, however, is generally used inaccurately to mean the vector of Euler-Rodrigues symmetric parameters.) While the Euler-Rodrigues symmetric parameters form a group, the quaternion $\bar{q}$ forms a skew field (i.e., a field with non-commutative multiplication, also called a division ring [12]) with inverse given by

$$\bar{q}^{-1} \equiv (\bar{q}^T \bar{q})^{-1} \begin{bmatrix} -\mathbf{q} \\ q_4 \end{bmatrix}, \tag{180}$$

for

$$\bar{q} \neq \bar{0} \equiv [0 \quad 0 \quad 0 \quad 0]^T. \tag{181}$$

Given this more general definition, a quaternion corresponding to an arbitrary three-dimensional column vector $\mathbf{v}$ can be defined according to

$$\bar{\mathbf{v}} \equiv \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix}, \tag{182}$$

and equation (179) is then equivalent to

$$\bar{\mathbf{v}}' \equiv \begin{bmatrix} \mathbf{v}' \\ 0 \end{bmatrix} = \begin{bmatrix} R\mathbf{v} \\ 0 \end{bmatrix} = \bar{\eta} \otimes \bar{\mathbf{v}} \otimes \bar{\eta}^{-1}. \tag{183}$$

Thus, the transformation of vectors under rotations becomes an example of quaternion algebra.

The *conjugate quaternion* $\bar{\bar{q}}$ is defined as

$$\bar{\bar{q}} \equiv \begin{bmatrix} -\mathbf{q} \\ q_4 \end{bmatrix}. \tag{184}$$

For the Euler-Rodrigues symmetric parameters the conjugate quaternion is identical to the quaternion inverse. If $\bar{p}$ and $\bar{q}$ are two quaternions, then their conju-

gates satisfy

$$\bar{p}^T\bar{q} = \tilde{\bar{p}}^T\tilde{\bar{q}}, \quad \text{and} \quad \widetilde{\bar{p} \otimes \bar{q}} = \tilde{\bar{q}} \otimes \tilde{\bar{p}}, \tag{185}$$

Note also

$$\{\tilde{\bar{q}}\}_L = \{\bar{q}\}_L^T, \quad \text{and} \quad \{\tilde{\bar{q}}\}_R = \{\bar{q}\}_R^T, \tag{186}$$

and

$$\tilde{\bar{q}}^T\bar{q} = \bar{q}^T\tilde{\bar{q}} = \cos\theta. \tag{187}$$

In the above discussion the Euler-Rodrigues symmetric parameters for successive rotations have been written in the same order as the rotation matrices. This has not always been the convention followed (see, for example, [57] and works listed in the historical section). It was once the convention to write the composition of matrices also in the opposite order to today's usage. The convention changed when interest focused more on the algebra of operators. The quaternion had by this time fallen into disuse and did not succumb to the change in the convention. This historical oddity has persisted in many works up to the present. The need to abandon the older convention becomes apparent when equation (183) is considered for two successive rotations. If $\bar{\eta}$, $\bar{\eta}'$ and $\bar{\eta}''$ satisfy equation (170), then under the old convention the associative law for the transformation of vectors takes the form

$$(\bar{\eta} \odot \bar{\eta}') \odot \bar{\mathbf{v}} \odot (\bar{\eta} \odot \bar{\eta}')^{-1} = \bar{\eta}' \odot (\bar{\eta} \odot \bar{\mathbf{v}} \odot \bar{\eta}^{-1}) \odot (\bar{\eta}')^{-1}! \tag{188}$$

Here "$\odot$" denotes quaternion multiplication with the operands written in the "historical" order, $\bar{\eta} \odot \bar{\eta}' \equiv \bar{\eta}' \otimes \bar{\eta}$. With quaternion multiplication written in the "natural" order, this shifting of position does not take place. Many authors sidestep this confusion by avoiding altogether the writing of an equation like equation (169) (or its equivalent with "$\odot$"). The choice of the "natural" order has other consequences, which will become evident in the section on Cayley-Klein parameters.

There is no universal agreement on the choice of indices for the Euler-Rodrigues symmetric parameters. Most authors prefer the vectorial indices to be 1 through 3 and the scalar index to be 4. Junkins and Turner [3] choose the scalar index to be zero, which causes difficulties for some programming languages. At least one work [58] has made the choice that the scalar index is 1 and the vectorial indices are 2, 3, and 4, which makes the cyclic symmetry of quaternion expressions somewhat difficult to recognize.

The Euler-Rodrigues symmetric parameters can be parameterized also in terms of the Euler angles. From equations (159) and (160)

$$\bar{\eta}(\hat{\mathbf{1}}, \theta) = [\sin(\theta/2) \quad 0 \quad 0 \quad \cos(\theta/2)]^T, \tag{189a}$$

$$\bar{\eta}(\hat{\mathbf{2}}, \theta) = [0 \quad \sin(\theta/2) \quad 0 \quad \cos(\theta/2)]^T, \tag{189b}$$

$$\bar{\eta}(\hat{\mathbf{3}}, \theta) = [0 \quad 0 \quad \sin(\theta/2) \quad \cos(\theta/2)]^T. \tag{189c}$$

In terms of the 3–1–3 set of Euler angles, for example,

$$\bar{\eta}_{313}(\varphi,\vartheta,\psi)=\bar{\eta}(\hat{\mathbf{3}},\psi)\otimes\bar{\eta}(\hat{\mathbf{1}},\vartheta)\otimes\bar{\eta}(\hat{\mathbf{3}},\varphi)=\begin{bmatrix}\mathbf{s}\vartheta\mathbf{c}(\varphi-\psi)\\ \mathbf{s}\vartheta\mathbf{s}(\varphi-\psi)\\ \mathbf{c}\vartheta\mathbf{s}(\varphi+\psi)\\ \mathbf{c}\vartheta\mathbf{c}(\varphi+\psi)\end{bmatrix}, \tag{190a}$$

where $\mathbf{c}\phi=\cos(\phi/2)$, $\mathbf{s}\phi=\sin(\phi/2)$, et cetera. In terms of the 3–1–2 set

$$\bar{\eta}_{312}(\varphi,\vartheta,\psi)=\bar{\eta}(\hat{\mathbf{2}},\psi)\otimes\bar{\eta}(\hat{\mathbf{1}},\vartheta)\otimes\bar{\eta}(\hat{\mathbf{3}},\varphi)=\begin{bmatrix}\mathbf{c}\varphi\mathbf{s}\vartheta\mathbf{c}\psi-\mathbf{s}\varphi\mathbf{c}\vartheta\mathbf{s}\psi\\ \mathbf{c}\varphi\mathbf{c}\vartheta\mathbf{s}\psi+\mathbf{s}\varphi\mathbf{s}\vartheta\mathbf{c}\psi\\ \mathbf{c}\varphi\mathbf{s}\vartheta\mathbf{s}\psi+\mathbf{s}\varphi\mathbf{c}\vartheta\mathbf{c}\psi\\ \mathbf{c}\varphi\mathbf{c}\vartheta\mathbf{c}\psi-\mathbf{s}\varphi\mathbf{s}\vartheta\mathbf{s}\psi\end{bmatrix}, \tag{190b}$$

showing again the qualitatively different character of the symmetric and asymmetric sets of Euler angles. The equations for the Euler-Rodrigues symmetric parameters as functions of the twelve sets of Euler angles is given in [3].

Kolve [28] has presented general expressions also for the Euler-Rodrigues symmetric parameters in terms of the Euler angles. For a symmetric sequence of Euler angles this is

$$\bar{\eta}_{iji}(\varphi,\vartheta,\psi)=\begin{matrix}i\\ j\\ k\\ 4\end{matrix}\begin{bmatrix}\mathbf{c}\vartheta\mathbf{s}(\varphi+\psi)\\ \mathbf{s}\vartheta\mathbf{c}(\varphi-\psi)\\ \alpha_{ij}\mathbf{s}\vartheta\mathbf{s}(\varphi-\psi)\\ \mathbf{c}\vartheta\mathbf{c}(\varphi+\psi)\end{bmatrix}, \tag{191}$$

and the asymmetric sequence of Euler angles

$$\bar{\eta}_{ijk}(\varphi,\vartheta,\psi)=\begin{matrix}i\\ j\\ k\\ 4\end{matrix}\begin{bmatrix}\mathbf{c}\psi\mathbf{c}\vartheta\mathbf{s}\varphi+\alpha_{ij}\mathbf{s}\psi\mathbf{s}\vartheta\mathbf{c}\varphi\\ \mathbf{c}\psi\mathbf{s}\vartheta\mathbf{c}\varphi-\alpha_{ij}\mathbf{s}\psi\mathbf{c}\vartheta\mathbf{s}\varphi\\ \mathbf{s}\psi\mathbf{c}\vartheta\mathbf{c}\varphi+\alpha_{ij}\mathbf{c}\psi\mathbf{s}\vartheta\mathbf{s}\varphi\\ \mathbf{c}\psi\mathbf{c}\vartheta\mathbf{c}\varphi-\alpha_{ij}\mathbf{s}\psi\mathbf{s}\vartheta\mathbf{s}\varphi\end{bmatrix}, \tag{192}$$

Again, the indices in the border label the rows. These lead to simple expressions for the extraction of the Euler angles from the Euler-Rodrigues symmetric parameters. These are for the symmetric set

$$\vartheta=\arccos(\eta_i^2-\eta_j^2-\eta_k^2+\eta_4^2), \tag{193}$$

and for $\sin\vartheta\neq 0$

$$\varphi=\arctan_2(\eta_i,\eta_4)+\arctan_2(\alpha_{ij}\eta_k,\eta_j), \tag{194a}$$

$$\psi=\arctan_2(\eta_i,\eta_4)-\arctan_2(\alpha_{ij}\eta_k,\eta_j). \tag{194b}$$

Otherwise, if $\sin\vartheta=0$,

$$\psi=0,\qquad \varphi=\begin{cases}\arctan_2(\eta_i,\eta_4), & \text{if } \vartheta=0,\\ \arctan_2(\alpha_{ij}\eta_k,\eta_j), & \text{if } \vartheta=\pi.\end{cases} \tag{195}$$

For the asymmetric set of Euler angles these become

$$\vartheta = \arcsin(2(\eta_4 \eta_j + \alpha_{ij} \eta_k \eta_i)), \tag{196}$$

and for $\cos \vartheta \neq 0$

$$\varphi = \arctan_2(\eta_i + \eta_k, \eta_4 + \alpha_{ij} \eta_j) + \arctan_2(\eta_i - \eta_k, \eta_4 - \alpha_{ij} \eta_j), \tag{197a}$$

$$\psi = \arctan_2(\eta_i + \eta_k, \eta_4 + \alpha_{ij} \eta_j) - \arctan_2(\eta_i - \eta_k, \eta_4 - \alpha_{ij} \eta_j). \tag{197b}$$

Otherwise, if $\cos \vartheta = 0$,

$$\psi = 0, \qquad \varphi = \begin{cases} \arctan_2(\eta_i + \alpha_{ij} \eta_k, \eta_4 + \alpha_{ij} \eta_j), & \text{for } \vartheta = +\pi/2, \\ \arctan_2(\eta_i - \alpha_{ij} \eta_k, \eta_4 - \alpha_{ij} \eta_j), & \text{for } \vartheta = -\pi/2. \end{cases} \tag{198}$$

The vectorial components of the Euler-Rodrigues parameters by themselves can be used as a representation with the convention that $\eta_4 \geq 0$ [59]. The composition rule then is,

$$\boldsymbol{\eta}'' = (\sqrt{1 - |\boldsymbol{\eta}|^2}\, \boldsymbol{\eta}' + \sqrt{1 - |\boldsymbol{\eta}'|^2}\, \boldsymbol{\eta} - \boldsymbol{\eta}' \times \boldsymbol{\eta}) \operatorname{sgn}(\boldsymbol{\eta}, \boldsymbol{\eta}'), \tag{199a}$$

where

$$\operatorname{sgn}(\boldsymbol{\eta}, \boldsymbol{\eta}') \equiv \operatorname{sign}(\sqrt{(1 - |\boldsymbol{\eta}|^2)(1 - |\boldsymbol{\eta}'|^2)} - \boldsymbol{\eta} \cdot \boldsymbol{\eta}'). \tag{199b}$$

The sign factor insures that the ignored fourth component is always non-negative. This representation is always finite but discontinuous as a function of the rotation matrix. It becomes inaccurate when the angle of rotation is close to π. Note that as η_4 passes through 0, the sign of $\boldsymbol{\eta}$ may change abruptly. Thus, $\boldsymbol{\eta}$ as a three-vector is not a convenient representation for interpolation or filtering. For continuous attitude motion, the full 4×1 column vector of Euler-Rodrigues parameters can always be made continuous.

Different arrangements of terms than as in equation (157) are also used. A different choice may reduce the number of computations or improve numerical significance [60]. In this respect, note also equations (255a) and (387) below.

The Rodrigues Parameters

Closely related to the Euler-Rodrigues symmetric parameters is another attitude representation, the *Rodrigues vector* [35] or Gibbs vector [61, 62], which is defined as

$$\boldsymbol{\rho} \equiv \boldsymbol{\eta}/\eta_4 = \tan(\theta/2)\hat{\mathbf{n}}. \tag{200}$$

The three components of $\boldsymbol{\rho}$, (ρ_1, ρ_2, ρ_3), are the Rodrigues parameters. Thus,

$$\bar{\eta} = \frac{1}{\sqrt{1 + |\boldsymbol{\rho}|^2}} \begin{bmatrix} \boldsymbol{\rho} \\ 1 \end{bmatrix}, \tag{201}$$

and

$$R(\boldsymbol{\rho}) = \frac{1}{1 + |\boldsymbol{\rho}|^2}((1 - |\boldsymbol{\rho}|^2)I + 2\boldsymbol{\rho}\boldsymbol{\rho}^T + 2[[\boldsymbol{\rho}]]) \tag{202}$$

$$= (I + [[\boldsymbol{\rho}]])(I - [[\boldsymbol{\rho}]])^{-1}, \tag{203}$$

$$\boldsymbol{\rho} = \frac{1}{1 + \text{tr } R}\begin{bmatrix} R_{23} - R_{32} \\ R_{31} - R_{13} \\ R_{12} - R_{21} \end{bmatrix}. \tag{204}$$

Equation (203) is generally known as Cayley's formula [63].

The composition rule for the Rodrigues vector, which follows directly from that for the Euler-Rodrigues symmetric parameters, is simply

$$\boldsymbol{\rho}'' = \frac{\boldsymbol{\rho}' + \boldsymbol{\rho} - \boldsymbol{\rho}' \times \boldsymbol{\rho}}{1 - \boldsymbol{\rho}' \cdot \boldsymbol{\rho}}. \tag{205}$$

The Rodrigues vector has the minimum dimension but the disadvantage that $|\boldsymbol{\rho}| \to \infty$ as $\theta \to \pi$. Thus, rotations through π cannot be represented (except formally as $\infty\hat{\mathbf{n}}$).

The Cayley-Klein Parameters

Also related to the quaternion is the *Cayley-Klein matrix*, Q, a complex 2×2 matrix defined according to [64,65]

$$Q \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} q_4 + \mathsf{i}q_3 & q_2 + \mathsf{i}q_1 \\ -q_2 + \mathsf{i}q_1 & q_4 - \mathsf{i}q_3 \end{bmatrix}, \tag{206a}$$

where $\mathsf{i} \equiv \sqrt{-1}$, and the *sans serif* character has been used to distinguish $\sqrt{-1}$ from the index i. When the quaternion has unit norm, i.e., when it is equivalent to the vector of Euler-Rodrigues symmetric parameters, we write

$$\mathrm{H} \equiv \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \equiv \begin{bmatrix} \eta_4 + \mathsf{i}\eta_3 & \eta_2 + \mathsf{i}\eta_1 \\ -\eta_2 + \mathsf{i}\eta_1 & \eta_4 - \mathsf{i}\eta_3 \end{bmatrix}. \tag{206b}$$

This Cayley-Klein matrix, H (the symbol is upper-case eta, distinguishable in mathematical formulae from Roman H by the lack of italics), is *unitary*. When the Cayley-Klein matrix is unitary, the elements, α, β, γ, δ, are called the *Cayley-Klein parameters*.[6] Thus, if the *Hermitian conjugate* of Q (and, therefore, also of H) is defined as

$$Q^\dagger \equiv Q^{*T}, \tag{207}$$

where the asterisk denotes complex conjugation, then

$$\mathrm{H}^\dagger\mathrm{H} = I_{2\times 2} \quad \text{or} \quad \mathrm{H}^{-1} = \mathrm{H}^\dagger, \tag{208}$$

where the implied operation is matrix multiplication. For the general Cayley-

[6]Our notation for the Cayley-Klein matrix is not standard. Other authors consider only the unitary Cayley-Klein matrix and denote this matrix by Q (and similarly do not distinguish in notation between $\bar{q}$ and $\bar{\eta}$). The reader offended by our dogged consistency can readily restore the more standard though less precise notation.

Klein matrix

$$Q^{-1} = (\det Q)^{-1} Q^{\dagger}. \tag{209}$$

The determinant of the Cayley-Klein matrix is given by

$$\det Q = ad - bc = \bar{q}^T \bar{q}, \tag{210}$$

from which it follows that H is *unimodular*, i.e., det H = 1. Matrices which are both unitary and unimodular are said to be *special unitary*.

The unitary Cayley-Klein matrices are isomorphic to the Euler-Rodrigues symmetric parameters. Thus,

$$\mathrm{H}(\bar{\eta}')\mathrm{H}(\bar{\eta}) = \mathrm{H}(\bar{\eta}' \otimes \bar{\eta}), \tag{211}$$

and similarly for the isomorphism between $\bar{q}$ and Q. Like the Euler-Rodrigues symmetric parameters, the unitary Cayley-Klein matrices form a group, denoted by SU(2), the group of special unitary matrices in 2 dimensions.[7] The group identity element is $I_{2\times 2}$, and the inverse is given by the Hermitian conjugate. Like the Euler-Rodrigues symmetric parameters, there are two sets of the Cayley-Klein parameters corresponding to every rotation, which differ from one another only by an overall sign.

Noting the similarities in the elements of the Cayley-Klein matrix, this has the general form

$$Q = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}, \quad \text{and} \quad \mathrm{H} = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}, \tag{212}$$

with

$$\alpha\alpha^* + \beta\beta^* = 1. \tag{213}$$

In terms of the Pauli matrices [23], given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{214}$$

the Cayley-Klein matrix can be written as

$$Q = \mathrm{i}q_1\sigma_1 + \mathrm{i}q_2\sigma_2 + \mathrm{i}q_3\sigma_3 + q_4 I_{2\times 2} \equiv q_4 I_{2\times 2} + \mathrm{i}\mathbf{q} \cdot \boldsymbol{\sigma}. \tag{215}$$

For the unitary Cayley-Klein matrix, we have in particular

$$\mathrm{H} = \eta_4 I_{2\times 2} + \mathrm{i}\boldsymbol{\eta} \cdot \boldsymbol{\sigma} = \cos(\theta/2) I_{2\times 2} + \mathrm{i} \sin(\theta/2)\,(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) \tag{216}$$

$$= \exp\left\{\mathrm{i}\frac{\theta}{2}(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})\right\} = \exp\left\{\frac{\mathrm{i}}{2}(\boldsymbol{\theta} \cdot \boldsymbol{\sigma})\right\}. \tag{217}$$

Noting equation (209) we can write the relationship between the unitary Cayley-

[7] Note that each of the representations of the rotations forms a group, because the physical rotations form a group. However, the group operation (composition rule) may not be easy to write down, as is the case, for example, for the Euler angles.

Klein matrix and the Rodrigues-Gibbs vector in analogy with equation (203) as

$$\mathrm{H} = (I_{2\times 2} + \mathrm{i}\boldsymbol{\rho}\cdot\boldsymbol{\sigma})(I_{2\times 2} - \mathrm{i}\boldsymbol{\rho}\cdot\boldsymbol{\sigma})^{-1}. \tag{218}$$

For rotations about individual coordinate axes the unitary Cayley-Klein matrices have the form

$$\mathrm{H}(\hat{\mathbf{1}},\theta) = \begin{pmatrix} \cos(\theta/2) & \mathrm{i}\sin(\theta/2) \\ \mathrm{i}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \tag{219a}$$

$$\mathrm{H}(\hat{\mathbf{2}},\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \tag{219b}$$

$$\mathrm{H}(\hat{\mathbf{3}},\theta) = \begin{pmatrix} e^{\mathrm{i}\theta/2} & 0 \\ 0 & e^{-\mathrm{i}\theta/2} \end{pmatrix}. \tag{219c}$$

For the 3–1–3 set of Euler angles the Cayley-Klein parameters are given by

$$\alpha_{313} = e^{\mathrm{i}(\psi+\varphi)/2}\cos(\vartheta/2), \qquad \beta_{313} = \mathrm{i}\, e^{\mathrm{i}(\psi-\varphi)/2}\sin(\vartheta/2), \tag{220}$$

and for the 3–1–2 set of Euler angles,

$$\alpha_{312} = e^{\mathrm{i}\varphi/2}(\cos\psi\cos\vartheta + \mathrm{i}\sin\psi\sin\vartheta), \tag{221a}$$

$$\beta_{312} = e^{-\mathrm{i}\varphi/2}(\sin\psi\cos\vartheta + \mathrm{i}\cos\psi\sin\vartheta). \tag{221b}$$

From the algebra of the Pauli matrices,

$$\sigma_l\sigma_m = \delta_{lm}I_{2\times 2} + \mathrm{i}\sum_{n=1}^{3}\varepsilon_{lmn}\sigma_n, \tag{222}$$

or equivalently,

$$(\mathbf{u}\cdot\boldsymbol{\sigma})(\mathbf{v}\cdot\boldsymbol{\sigma}) = (\mathbf{u}\cdot\mathbf{v})I_{2\times 2} + \mathrm{i}(\mathbf{u}\times\mathbf{v})\cdot\boldsymbol{\sigma}. \tag{223}$$

The rotation matrix is related to the unitary Cayley-Klein matrices according to

$$R_{kl} = \frac{1}{2}\,\mathrm{tr}(\sigma_k\mathrm{H}\sigma_l\mathrm{H}^{\dagger}), \tag{224}$$

which is equivalent to equation (157). Evaluating equation (224) leads to

$$R = \begin{bmatrix} \frac{1}{2}(\alpha^2-\beta^2-\gamma^2+\delta^2) & -\frac{\mathrm{i}}{2}(\alpha^2+\beta^2-\gamma^2-\delta^2) & -(\alpha\beta-\gamma\delta) \\ \frac{\mathrm{i}}{2}(\alpha^2-\beta^2+\gamma^2-\delta^2) & \frac{1}{2}(\alpha^2+\beta^2+\gamma^2+\delta^2) & -\mathrm{i}(\alpha\beta+\gamma\delta) \\ -(\alpha\gamma-\beta\delta) & \mathrm{i}(\alpha\gamma+\beta\delta) & (\alpha\delta+\beta\gamma) \end{bmatrix}, \tag{225}$$

and can be inverted to yield

$$\mathrm{H} = \frac{1}{\sqrt{4(1+\mathrm{tr}\,R)}}\left[I_{2\times 2} + \sum_{l=1}^{3}\sum_{m=1}^{3}R_{lm}\sigma_l\sigma_m\right]. \tag{226}$$

Defining now the 3×3 matrices,

$$\mathbf{i} \equiv \mathrm{i}\,\sigma_1, \quad \mathbf{j} = \mathrm{i}\,\sigma_2, \quad \mathbf{k} = \mathrm{i}\,\sigma_3 \quad \text{and} \quad \mathbf{1} = I_{2\times 2}, \tag{227}$$

the Cayley-Klein matrix can be written more familiarly as

$$Q = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} + q_4\mathbf{1}, \tag{228}$$

and these new basis "vectors" satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \tag{229}$$

$$\mathbf{ji} = -\mathbf{ij} = \mathbf{k}, \qquad \mathbf{kj} = -\mathbf{jk} = \mathbf{i}, \qquad \mathbf{ik} = -\mathbf{ki} = \mathbf{j}. \tag{230}$$

Apart from an overall sign in the rightmost members of equations (230), this is Hamilton's quaternion algebra. The origin of the sign difference is simply the convention of this work that the quaternion multiplication is written in the "natural" order. Equations (229) and (230) hold equally well, obviously, if we identify **i**, **j**, **k**, and **1** with the appropriate quaternions $([1,0,0,0]^T, \ldots, [0,0,0,1]^T)$ and identity multiplication with quaternion multiplication. It should be noted that equations (157), (170) and (228) cannot be made consistent with $\mathbf{ij} = \mathbf{k}$. Thus, for instance, Battin [66], who preserves Hamilton's convention for the value of **ij** but also writes equations (170) and (228) in the same form as this work, must define the rotation matrix as a different function of the Euler-Rodrigues symmetric parameters. Likewise, Junkins and Turner [3], who also prefer the natural order of quaternion multiplication, note that in order to satisfy Hamilton's convention for the vector multiplication they must alter the sign of the quaternion components.

In analogy with the antisymmetric 3×3 matrices $[[\mathbf{u}]]$, the complex 2×2 matrices can be defined as

$$[[\mathbf{u}]]_{2\times 2} \equiv u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3 \equiv \mathbf{u}\cdot\boldsymbol{\sigma}. \tag{231}$$

It follows that

$$\mathrm{H}[[\mathbf{u}]]_{2\times 2}\mathrm{H}^\dagger = [[R\mathbf{u}]]_{2\times 2}, \tag{232}$$

which corresponds in three dimensions to

$$R[[\mathbf{u}]]_{3\times 3}R^T = [[R\mathbf{u}]]_{3\times 3}, \tag{233}$$

a special case of equation (80).

If the representation of a vector $\boldsymbol{v}$ is written in terms of the basis "vectors" of equation (227) as

$$\mathbf{V} \equiv v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \equiv \mathrm{i}\begin{pmatrix} v_3 & v_1 - \mathrm{i}v_2 \\ v_1 + \mathrm{i}v_2 & -v_3 \end{pmatrix}, \tag{234}$$

then equation (232) becomes

$$\mathbf{V}' = \mathbf{HVH}^\dagger = \mathbf{HVH}^{-1}, \tag{235}$$

which is just equation (183) in terms of Cayley-Klein matrices. In this algebra of Cayley-Klein matrices, scalars are represented as quantities of the form

$$S \equiv s\mathbf{1}, \tag{236}$$

where s is a real number if the underlying vector space is real.[8] The vector product in this algebra can now be defined in terms of the multiplication rules of equations (229) and (230) as

$$\mathbf{U} \times \mathbf{V} \equiv -\frac{1}{2}[\mathbf{UV} - \mathbf{VU}]. \tag{237}$$

Likewise, it is evident that the scalar product can be defined as

$$\mathbf{U} \cdot \mathbf{V} \equiv -\frac{1}{2}[\mathbf{UV} + \mathbf{VU}], \tag{238}$$

which has the value $(\mathbf{u} \cdot \mathbf{v})\mathbf{1}$. Apart from the different sign in the quaternion multiplication rule, equations (228) through (230) and equations (234) through (238) constitute the quaternionization of vector spaces by Hamilton and others, although Hamilton never accepted equation (235).[9] In this representation scalars and vectors exist on an almost equal footing and can be added together to form new objects, which are the quaternions.

The algebra of the 3×3 and 2×2 representations of the rotations can be made more similar by defining antisymmetric 3×3 matrices according to

$$[\xi_k]_{ij} = \varepsilon_{ijk}, \qquad i, j, k = 1, 2, 3, \tag{239}$$

or,

$$\xi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = [[\hat{\mathbf{1}}]], \qquad \xi_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = [[\hat{\mathbf{2}}]], \tag{240ab}$$

$$\xi_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [[\hat{\mathbf{3}}]], \tag{240c}$$

so that

$$[[\mathbf{u}]]_{3\times 3} = u_1\xi_1 + u_2\xi_2 + u_3\xi_3 \equiv \mathbf{u} \cdot \boldsymbol{\xi}, \tag{241}$$

which more closely parallels equation (231). If one defines further

$$\mathrm{S}_k^{3\times 3} \equiv \frac{1}{\mathrm{i}}\xi_k, \qquad \mathrm{S}_k^{2\times 2} \equiv \frac{1}{2}\sigma_k, \qquad k = 1, 2, 3, \tag{242}$$

then $\mathrm{S}_k^{3\times 3}$ and $\mathrm{S}_k^{2\times 2}$ both satisfy [23]

[8] Most presentations of quaternion algebra, beginning with Hamilton, never consider the basis "vectors" as 2×2 matrices but simply write $\bar{q} = q_1 i + q_2 j + q_3 k + q_4$ and impose the multiplication rule (generally with Hamilton's sign convention) upon the orthonormal basis.

[9] See the historical note below.

$$\text{tr } S_k = 0, \tag{243}$$

$$S_k^\dagger = S_k, \tag{244}$$

$$S_l S_m - S_m S_l = i \sum_{k=1}^{3} \varepsilon_{lmk} S_k, \tag{245}$$

and finally

$$R(\hat{\mathbf{n}}, \theta) = \exp\{i\,\theta\,\hat{\mathbf{n}} \cdot \mathbf{S}^{3\times 3}\}, \tag{246}$$

$$H(\hat{\mathbf{n}}, \theta) = \exp\{i\,\theta\,\hat{\mathbf{n}} \cdot \mathbf{S}^{2\times 2}\}, \tag{247}$$

where in each case

$$\mathbf{S} \equiv \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}, \tag{248}$$

a "column vector" of which every component is a matrix. The scalar product in these equations should be understood in the same sense as in equation (231) or equation (241). The relationships for the Cayley-Klein parameters can be developed in the opposite order, beginning with the unitary transformations of complex two-dimensional vectors [67].

The Modified Rodrigues Parameters

Related to the Rodrigues vector is the modified Rodrigues vector [68, 69], which exists in two forms, the positive form, defined in terms of the Euler-Rodrigues symmetric parameters as

$$\mathbf{p} \equiv \frac{\boldsymbol{\eta}}{1 + \eta_4}, \tag{249}$$

and the negative form, defined by

$$\mathbf{m} \equiv \frac{\boldsymbol{\eta}}{1 - \eta_4}. \tag{250}$$

These two formulas represent three-dimensional stereographic projections of the sphere of Euler-Rodrigues symmetric parameters, the positive form having the point at infinity correspond to $-\bar{1}$ and the negative form vector to $\bar{1}$. The components of $\mathbf{p}$ or $\mathbf{m}$ we can regard as the modified Rodrigues parameters.

If $\bar{\eta}$ is given by equations (159) and (160) then

$$\mathbf{p}(\hat{\mathbf{n}}, \theta) = \tan(\theta/4)\hat{\mathbf{n}}. \tag{251}$$

For the parameter values $(\hat{\mathbf{n}}, \theta + 2\pi)$, which must represent the same rotation,

$$\mathbf{p}(\hat{\mathbf{n}}, \theta + 2\pi) = -\cot(\theta/4)\hat{\mathbf{n}}. \tag{252}$$

Thus,

$$\mathbf{p} \quad \text{and} \quad -\frac{1}{|\mathbf{p}|^2}\mathbf{p}$$

represent the same rotation and correspond to $\bar{\eta}$ and $-\bar{\eta}$. The Euler-Rodrigues symmetric parameters and the Rodrigues vector can be computed from the modified Rodrigues vector as

$$\bar{\eta} = \pm \frac{1}{1 + |\mathbf{p}|^2} \begin{bmatrix} 2\mathbf{p} \\ 1 - |\mathbf{p}|^2 \end{bmatrix}, \qquad \boldsymbol{\rho} = \frac{2\mathbf{p}}{1 - |\mathbf{p}|^2}, \tag{253}$$

where the positive sign is consistent with both equations (251) and equations (159) and (160).

For the negative form with the same conventions,

$$\mathbf{m}(\hat{\mathbf{n}}, \theta) = \cot(\theta/4)\hat{\mathbf{n}}, \qquad \mathbf{m}(\hat{\mathbf{n}}, \theta + 2\pi) = -\tan(\theta/4)\hat{\mathbf{n}}. \tag{254}$$

Thus, the values of the negative form of the modified Rodrigues vectors are the negatives of the values of the positive form. The remainder of this survey, therefore, will treat only the positive form, since it is the more similar in its conventions to the Rodrigues vector. It is clearly advantageous also for the zero vector to be an acceptable value.

The Rodrigues vector maps rotations bijectively onto all of space with rotations through π corresponding to points at infinity. Thus, to every rotation whose angle of rotation is less than π there corresponds a single Rodrigues vector, and to each finite Rodrigues vectors there corresponds a unique rotation. However, the representation is singular and discontinuous at $\theta = \pi$. The modified Rodrigues vector, on the other hand, maps rotations bijectively onto either the closed unit sphere $|\mathbf{p}| \leq 1$, or the closed exterior of the unit sphere $|\mathbf{p}| \geq 1$. Thus, if the angle of rotation is bounded, i.e., for librational motion, and the maximum angle of rotation is less than 2π, the modified Rodrigues vector provides a continuous single-valued and analytic representation of rotations. (For this to be true for the Rodrigues vector, the maximum angle of rotation must be no greater than π.) If the angle of rotation is not bounded, however, as in the case of a spinning spacecraft or a very agile aircraft, the singularity or the discontinuity of the modified Rodrigues vector is unavoidable.

In terms of the modified Rodrigues vector the rotation matrix can be written as

$$R = I + 2\eta_4[[\boldsymbol{\eta}]] + 2[[\boldsymbol{\eta}]]^2 \tag{255a}$$

$$= I + \frac{4(1 - |\mathbf{p}|^2)}{(1 + |\mathbf{p}|^2)^2}[[\mathbf{p}]] + \frac{8}{(1 + |\mathbf{p}|^2)^2}[[\mathbf{p}]]^2, \tag{255b}$$

and the Rodrigues vector by

$$\boldsymbol{\rho} = \frac{2\mathbf{p}}{1 - |\mathbf{p}|^2}. \tag{256}$$

The composition rule for the modified Rodrigues vector is somewhat complicated and reads

$$\mathbf{p}'' = \frac{(1 - |\mathbf{p}|^2)\mathbf{p}' + (1 - |\mathbf{p}'|^2)\mathbf{p} - 2\mathbf{p}' \times \mathbf{p}}{1 + |\mathbf{p}'|^2|\mathbf{p}|^2 - 2\mathbf{p}' \cdot \mathbf{p}}, \tag{257}$$

which is considerably more burdensome than that for the Rodrigues vector.

Attitude Kinematics

The kinematic relations, i.e., the relations between the temporal derivative of the attitude representation and the angular velocity, are presented here in the approximate order in which the representations were introduced in the preceding text.

The Rotation Matrix

If the attitude is changing with time, then $R(t + \Delta t)$, the rotation matrix representing the attitude at time $t + \Delta t$, will differ from $R(t)$, the rotation matrix at time t. Thus,

$$R(t + \Delta t) = \Phi(t + \Delta t, t)R(t), \tag{258}$$

where $\Phi(t + \Delta t, t)$ must also be a rotation matrix, and for Δt sufficiently small the rotation $\Phi(t)$ must also be small. Hence, from equation (109)

$$\Phi(t + \Delta t, t) = I + [[\Delta\xi(t)]] + O(|\Delta\xi(t)|^2), \tag{259}$$

where $\Delta\xi(t)$ is some small three-vector which tends to zero as Δt tends to zero, and

$$\frac{1}{\Delta t}(R(t + \Delta t) - R(t)) = \frac{1}{\Delta t}[[\Delta\xi(t)]]R(t) + O(|\Delta\xi(t)|). \tag{260}$$

Taking the limit as Δt tends to zero leads to

$$\frac{d}{dt}R(t) = [[\omega(t)]]R(t), \tag{261}$$

where $\omega(t)$, the *body-referenced angular velocity*, or simply *angular velocity*, is defined as

$$\omega(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta\xi(t)}{\Delta t}. \tag{262}$$

The angular velocity appearing in equation (261) is necessarily referred to body axes, because $\Phi(t + \Delta t, t)$ is the small rotation which carries representations of vectors with respect to the body axes at time t into the representations of these same vectors with respect to the body axes at time $t + \Delta t$. (For emphasis, the body-referenced angular velocity will sometimes be denoted in this work by ω_b.) Thus, $\xi(t)$ is the rotation vector of that small rotation, referred to body axes (at either time t or time $t + \Delta t$).

The *space-referenced angular velocity* can also be defined. This is given by

$$\omega^{\text{space}} = R^T\omega, \tag{263}$$

i.e., the representation of the angular velocity with respect to the space (or inertial) axes. In order that equations involving the space-referenced angular velocity not become overly cumbersome, this quantity will be denoted sometimes by ω_i. Then,

$$\frac{d}{dt}R(t) = R(t)[[\omega^{\text{space}}(t)]]. \tag{264}$$

The space-referenced angular velocity receives considerably less use than the body-referenced angular velocity, due largely to the fact that for a rigid body the space-referenced inertia tensor is time-varying, while the body-referenced inertia tensor is constant in time. When no subscript or superscript indicates which angular-velocity is intended, the body-referenced vector will always be meant.

Likewise, the angular velocity can be reconstructed from the rate of change of the rotation matrix. From equations (261) and (264)

$$[[\boldsymbol{\omega}]] = \dot{R}R^T, \qquad [[\boldsymbol{\omega}^{\text{space}}]] = R^T\dot{R}, \tag{265}$$

or in component form

$$\omega_i = \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{ijk}\dot{R}_{jl}R_{kl}, \qquad \omega_i^{\text{space}} = \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{ijk}R_{lj}\dot{R}_{lk}, \tag{266}$$

where the raised dot indicates the temporal derivative. Note also

$$|\boldsymbol{\omega}^{\text{space}}|^2 = |\boldsymbol{\omega}|^2 \equiv \omega^2 = \frac{1}{2}\,\text{tr}(\dot{R}\dot{R}^T). \tag{267}$$

If $R = R''R'$ is a composite rotation and

$$\frac{d}{dt}R' = [[\boldsymbol{\omega}']]R', \quad \text{and} \quad \frac{d}{dt}R'' = [[\boldsymbol{\omega}'']]R'', \tag{268}$$

then the angular velocities combine according to the rule

$$\boldsymbol{\omega} = \boldsymbol{\omega}'' + R''\boldsymbol{\omega}'. \tag{269}$$

The Axis and Angle of Rotation

The kinematic relations for the axis and angle of rotation are [2, 70, 71]

$$\frac{d}{dt}\theta = \hat{\mathbf{n}}\cdot\boldsymbol{\omega}, \tag{270}$$

$$\frac{d}{dt}\hat{\mathbf{n}} = \frac{1}{2}[\hat{\mathbf{n}}\times\boldsymbol{\omega} - \cot(\theta/2)\,\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\boldsymbol{\omega})]. \tag{271}$$

In terms of the rates of change of the angle and the axis of rotation, the angular velocity becomes

$$\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{n}} + \sin\theta\,\dot{\hat{\mathbf{n}}} - (1-\cos\theta)\,\hat{\mathbf{n}}\times\dot{\hat{\mathbf{n}}}. \tag{272}$$

Expressed in terms of the space-referenced angular velocity these become

$$\frac{d}{dt}\theta = \hat{\mathbf{n}}\cdot\boldsymbol{\omega}_I, \tag{273}$$

$$\frac{d}{dt}\hat{\mathbf{n}} = -\frac{1}{2}[\hat{\mathbf{n}}\times\boldsymbol{\omega}_I + \cot(\theta/2)\,\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\boldsymbol{\omega}_I)]. \tag{274}$$

and

$$\boldsymbol{\omega}_I = \dot{\theta}\hat{\mathbf{n}} + \sin\theta\,\dot{\hat{\mathbf{n}}} + (1-\cos\theta)\,\hat{\mathbf{n}}\times\dot{\hat{\mathbf{n}}}. \tag{275}$$

The Rotation Vector

Similarly, the kinematic relations for the rotation vector are [2, 71–77]

$$\frac{d}{dt}\boldsymbol{\theta} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1}{\theta^2}(1 - (\theta/2)\cot(\theta/2))\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega}), \tag{276}$$

$$\boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt} - \left(\frac{1 - \cos\theta}{\theta^2}\right)\boldsymbol{\theta} \times \frac{d\boldsymbol{\theta}}{dt} + \left(\frac{\theta - \sin\theta}{\theta^3}\right)\boldsymbol{\theta} \times \left(\boldsymbol{\theta} \times \frac{d\boldsymbol{\theta}}{dt}\right). \tag{277}$$

Expressed in terms of the space-referenced angular velocity these become

$$\frac{d}{dt}\boldsymbol{\theta} = \boldsymbol{\omega}_I - \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\omega}_I + \frac{1}{\theta^2}(1 - (\theta/2)\cot(\theta/2))\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega}_I), \tag{278}$$

$$\boldsymbol{\omega}_I = \frac{d\boldsymbol{\theta}}{dt} + \left(\frac{1 - \cos\theta}{\theta^2}\right)\boldsymbol{\theta} \times \frac{d\boldsymbol{\theta}}{dt} + \left(\frac{\theta - \sin\theta}{\theta^3}\right)\boldsymbol{\theta} \times \left(\boldsymbol{\theta} \times \frac{d\boldsymbol{\theta}}{dt}\right). \tag{279}$$

The Euler Angles

The kinematic relationships for the Euler angles are more complicated. The angular velocity in terms of the Euler angle rates and the corresponding instantaneous intermediate axes of rotation is given by

$$\boldsymbol{\omega} = \dot{\varphi}(\hat{\mathbf{n}}_1)_{\mathcal{E}''} + \dot{\vartheta}(\hat{\mathbf{n}}_2')_{\mathcal{E}''} + \dot{\psi}(\hat{\mathbf{n}}_3'')_{\mathcal{E}''}. \tag{280}$$

The representation of the Euler axis vectors is now with respect to $\mathcal{E}'''$ for all three vectors, since we desire a representation of $\boldsymbol{\omega}$ with respect to $\mathcal{E}'''$. However, the representation of the Euler axes is simple only with respect to the intermediate bases (except for $(\hat{\mathbf{n}}_3'')_{\mathcal{E}''}$, which is equal to $(\hat{\mathbf{n}}_3'')_{\mathcal{E}'}$). Thus, the simple representations of the Euler axes with respect to the intermediate bases must be transformed to representations with respect to $\mathcal{E}'''$. Carrying out the necessary transformations equation (280) becomes

$$\boldsymbol{\omega} = \dot{\psi}(\hat{\mathbf{n}}_3'')_{\mathcal{E}''} + \dot{\vartheta}R((\hat{\mathbf{n}}_3'')_{\mathcal{E}''}, \psi)(\hat{\mathbf{n}}_2')_{\mathcal{E}'} + \dot{\varphi}R((\hat{\mathbf{n}}_3'')_{\mathcal{E}''}, \psi)R((\hat{\mathbf{n}}_2')_{\mathcal{E}'}, \vartheta)(\hat{\mathbf{n}}_1)_{\mathcal{E}}, \tag{281}$$

or, simply remembering the basis conventions for the body-referenced Euler axes,

$$\boldsymbol{\omega} = \dot{\psi}\hat{\mathbf{n}}_3'' + \dot{\vartheta}R(\hat{\mathbf{n}}_3'', \psi)\hat{\mathbf{n}}_2' + \dot{\varphi}R(\hat{\mathbf{n}}_3'', \psi)R(\hat{\mathbf{n}}_2', \vartheta)\hat{\mathbf{n}}_1. \tag{282}$$

Equation (282) is, in fact, a consequence of equation (269). It follows that

$$\omega^2 = \dot{\varphi}^2 + \dot{\vartheta}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}(\hat{\mathbf{n}}_3''^T R(\hat{\mathbf{n}}_2', \vartheta)\hat{\mathbf{n}}_1), \tag{283}$$

and

$$\boldsymbol{\omega} = R(\hat{\mathbf{n}}_3'', \psi)S(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2', \hat{\mathbf{n}}_3''; \vartheta)\begin{bmatrix}\dot{\varphi}\\ \dot{\vartheta}\\ \dot{\psi}\end{bmatrix} \equiv M(\varphi, \vartheta, \psi)\begin{bmatrix}\dot{\varphi}\\ \dot{\vartheta}\\ \dot{\psi}\end{bmatrix}, \tag{284}$$

with $S(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2', \hat{\mathbf{n}}_3''; \vartheta)$ represented in terms of column vectors as

$$S(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2', \hat{\mathbf{n}}_3''; \vartheta) = [R(\hat{\mathbf{n}}_2', \vartheta)\hat{\mathbf{n}}_1 \mid \hat{\mathbf{n}}_2' \mid \hat{\mathbf{n}}_3'']. \tag{285}$$

For the case of a 3–1–3 set of Euler angles, the matrix, $S(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2', \hat{\mathbf{n}}_3''; \vartheta)$, is given by [78]

$$S_{313}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2', \hat{\mathbf{n}}_3''; \vartheta) = \begin{bmatrix} 0 & 1 & 0 \\ \sin\vartheta & 0 & 0 \\ \cos\vartheta & 0 & 1 \end{bmatrix}, \tag{286}$$

whence,

$$M_{313}(\varphi, \vartheta, \psi) = \begin{bmatrix} \sin\vartheta\sin\psi & \cos\psi & 0 \\ \sin\vartheta\cos\psi & -\sin\psi & 0 \\ \cos\vartheta & 0 & 1 \end{bmatrix}, \tag{287}$$

and

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \dot{\varphi}\sin\vartheta\sin\psi + \dot{\vartheta}\cos\psi \\ \dot{\varphi}\sin\vartheta\cos\psi - \dot{\vartheta}\sin\psi \\ \dot{\varphi}\cos\vartheta + \dot{\psi} \end{bmatrix}, \tag{288}$$

which is the familiar relation. For the case of a 3–1–2 set of Euler angles,

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} -\dot{\varphi}\cos\vartheta\sin\psi + \dot{\vartheta}\cos\psi \\ \dot{\varphi}\sin\vartheta + \dot{\psi} \\ \dot{\varphi}\cos\vartheta\cos\psi + \dot{\vartheta}\sin\psi \end{bmatrix}. \tag{289}$$

The equations for the matrix $M(\varphi, \vartheta, \psi)$ for the twelve sets of (body-referenced) Euler angles is given in [3, 6].

Relative to the space axes the angular velocity vector can be written

$$\boldsymbol{\omega}^{\text{space}} \equiv \boldsymbol{\omega}_s = \dot{\varphi}\hat{\mathbf{n}}_1 + \dot{\vartheta}R^T(\hat{\mathbf{n}}_1, \varphi)\hat{\mathbf{n}}_2 + \dot{\psi}R^T(\hat{\mathbf{n}}_1, \varphi)R^T(\hat{\mathbf{n}}_2', \vartheta)\hat{\mathbf{n}}_3'', \tag{290}$$

where the column vectors representing the rotation axes are also chosen from the set $\{\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}}\}$), but correspond, obviously, to different abstract vectors than the column vectors in equation (282). The superscript *space* does not indicate the space-referenced Euler angles in the present context.[10] Thus,

$$\boldsymbol{\omega}^{\text{space}} = M^{\text{space}}(\varphi, \vartheta, \psi)\begin{bmatrix} \dot{\varphi} \\ \dot{\vartheta} \\ \dot{\psi} \end{bmatrix} = R^T(\hat{\mathbf{n}}_1, \varphi)S^{\text{space}}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2', \hat{\mathbf{n}}_3''; \vartheta)\begin{bmatrix} \dot{\varphi} \\ \dot{\vartheta} \\ \dot{\psi} \end{bmatrix}, \tag{291}$$

with

$$S^{\text{space}}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2', \hat{\mathbf{n}}_3''; \vartheta) = [\hat{\mathbf{n}}_1 \mid \hat{\mathbf{n}}_2' \mid R^T(\hat{\mathbf{n}}_2', \vartheta)\hat{\mathbf{n}}_3'']. \tag{292}$$

For the case of a 3–1–3 set of Euler angles, equation (290) or (291) becomes

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}^{\text{space}} = \begin{bmatrix} \dot{\vartheta}\cos\varphi + \dot{\psi}\sin\vartheta\sin\varphi \\ \dot{\vartheta}\sin\varphi - \dot{\psi}\sin\vartheta\cos\varphi \\ \dot{\psi}\cos\vartheta + \dot{\varphi} \end{bmatrix}. \tag{293}$$

[10] For the 3–1–3 space-referenced Euler angles, the expressions for the body-referenced and space-referenced angular velocities would be identical to equations (288) and (293) but with the interchange of φ and ψ, and $\dot{\varphi}$ and $\dot{\psi}$, respectively, and with the axes of rotation being understood to be represented with respect to space axes.

To obtain the Euler angle rates from the angular velocity, equation (284) can be inverted to give

$$\begin{bmatrix} \dot{\varphi} \\ \dot{\vartheta} \\ \dot{\psi} \end{bmatrix} = M^{-1}(\varphi, \vartheta, \psi)\boldsymbol{\omega}, \tag{294}$$

and from equations (284) and (285)

$$M^{-1}(\varphi, \vartheta, \psi) = S^{-1}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2', \hat{\mathbf{n}}_3''; \vartheta) R^T(\hat{\mathbf{n}}_3'', \psi), \tag{295}$$

with

$$S^{-1}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2', \hat{\mathbf{n}}_3''; \vartheta) = \frac{1}{(R(\hat{\mathbf{n}}_2', \vartheta)\hat{\mathbf{n}}_1) \cdot (\hat{\mathbf{n}}_2' \times \hat{\mathbf{n}}_3'')} \begin{bmatrix} [\hat{\mathbf{n}}_2' \times \hat{\mathbf{n}}_3'']^T \\ [\hat{\mathbf{n}}_3'' \times R(\hat{\mathbf{n}}_2', \vartheta)\hat{\mathbf{n}}_1]^T \\ [(R(\hat{\mathbf{n}}_2', \vartheta)\hat{\mathbf{n}}_1) \times \hat{\mathbf{n}}_2']^T \end{bmatrix}. \tag{296}$$

The right member of equation (296) can be reduced to

$$\frac{1}{\cos\vartheta(\hat{\mathbf{n}}_1 \cdot (\hat{\mathbf{n}}_2' \times \hat{\mathbf{n}}_3'')) - \sin\vartheta(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_3'')} \times \begin{bmatrix} [\hat{\mathbf{n}}_2' \times \hat{\mathbf{n}}_3'']^T \\ [\cos\vartheta(\hat{\mathbf{n}}_3'' + \hat{\mathbf{n}}_1) - \sin\vartheta(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2') \times \hat{\mathbf{n}}_3'']^T \\ [\cos\vartheta\,\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2' - \sin\vartheta\,\hat{\mathbf{n}}_1]^T \end{bmatrix} \tag{297}$$

For the symmetric sequences of Euler angles this formula reduces further to

$$S^{-1}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2', \hat{\mathbf{n}}_3''; \vartheta) = \frac{1}{\sin\vartheta} \begin{bmatrix} (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2')^T \\ \sin\vartheta\,\hat{\mathbf{n}}_2'^T \\ \sin\vartheta\,\hat{\mathbf{n}}_1^T - \cos\vartheta(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2')^T \end{bmatrix}, \tag{298a}$$

and for the asymmetric sequence the formula reduces to

$$S^{-1}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2', \hat{\mathbf{n}}_3''; \vartheta) = \frac{1}{\cos\vartheta} \begin{bmatrix} \hat{\mathbf{n}}_1^T \\ \cos\vartheta\,\hat{\mathbf{n}}_2'^T \\ \cos\vartheta\,\hat{\mathbf{n}}_3''^T - \sin\vartheta(\hat{\mathbf{n}}_2' \times \hat{\mathbf{n}}_3'')^T \end{bmatrix}. \tag{298b}$$

For the case of the 3–1–3 Euler angles

$$S_{313}^{-1}(\vartheta) = \frac{1}{\sin\vartheta} \begin{bmatrix} 0 & 1 & 0 \\ \sin\vartheta & 0 & 0 \\ 0 & -\cos\vartheta & \sin\vartheta \end{bmatrix}, \tag{299}$$

and

$$M_{313}^{-1}(\varphi,\vartheta,\psi)=\frac{1}{\sin\vartheta}\begin{bmatrix}\sin\psi & \cos\psi & 0\\ \sin\vartheta\cos\psi & -\sin\vartheta\sin\psi & 0\\ -\cos\vartheta\sin\psi & -\cos\vartheta\cos\psi & \sin\vartheta\end{bmatrix} \tag{300}$$

Recursive implementations of these equations are also possible [79]. The equation for the matrix $M^{-1}(\varphi,\vartheta,\psi)$ for the twelve sets of (body-referenced) Euler angles is given in [2,3,6].

The Euler-Rodrigues Symmetric Parameters

The relation for the Euler-Rodrigues symmetric parameters is very similar to that for the rotation matrix. Note that

$$\begin{aligned}\bar{\eta}(t+\Delta t)&=\delta\bar{\eta}(t)\otimes\bar{\eta}(t)\\ &=\{\delta\bar{\eta}(t)\}_L\bar{\eta}(t)\,,\end{aligned} \tag{301}$$

where, since $\delta\bar{\eta}(t)$ describes a small rotation,

$$\delta\bar{\eta}(t)=\begin{bmatrix}\Delta\xi(t)/2\\ 1\end{bmatrix}+O(|\Delta\xi(t)|^2)\,. \tag{302}$$

It follows that

$$\{\delta\bar{\eta}(t)\}_L=I_{4\times4}+\frac{1}{2}\Omega_L(\Delta\xi(t))+O(|\Delta\xi(t)|^2)\,, \tag{303}$$

where

$$\Omega_L(\mathbf{v})\equiv\begin{bmatrix}0 & v_3 & -v_2 & v_1\\ -v_3 & 0 & v_1 & v_2\\ v_2 & -v_1 & 0 & v_3\\ -v_1 & -v_2 & -v_3 & 0\end{bmatrix}=\{\bar{\mathbf{v}}\}_L\,. \tag{304}$$

Carrying out the same steps to calculate the time derivative of the Euler-Rodrigues symmetric parameters as for the rotation matrix leads directly to

$$\frac{d}{dt}\bar{\eta}=\frac{1}{2}\Omega_L(\boldsymbol{\omega})\bar{\eta}=\frac{1}{2}\bar{\boldsymbol{\omega}}\otimes\bar{\eta}\,, \tag{305}$$

where $\bar{\boldsymbol{\omega}}$ is to be understood in the sense of equation (182). Equation (305) can also be written as [63]

$$\frac{d}{dt}\bar{\eta}=\frac{1}{2}\Xi(\bar{\eta})\boldsymbol{\omega}\,, \tag{306}$$

where

$$\Xi(\bar{\eta})=\begin{bmatrix}\eta_4 & -\eta_3 & \eta_2\\ \eta_3 & \eta_4 & -\eta_1\\ -\eta_2 & \eta_1 & \eta_4\\ -\eta_1 & -\eta_2 & -\eta_3\end{bmatrix}=\begin{bmatrix}\eta_4 I_{3\times3}-[[\boldsymbol{\eta}]]\\ -\boldsymbol{\eta}^T\end{bmatrix}, \tag{307}$$

is a submatrix of $\{\bar{\eta}\}_R$,

$$\{\bar{\eta}\}_R \equiv [\Xi(\bar{\eta}) \,|\, \bar{\eta}]\,. \tag{308}$$

Mayo [80] has developed an alternate form for equations (304) and (305) which are better suited to cases when the Euler-Rodrigues symmetric parameters represent a rotation relative to a non-inertial frame. A similar form must be used also in the propagation of the attitude error quaternion in the Kalman filter [81].

For the Euler-Rodrigues symmetric parameters, equation (305) can be inverted readily to yield

$$\bar{\omega} = \begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} = 2\frac{d\bar{\eta}}{dt} \otimes \bar{\eta}^{-1}. \tag{309}$$

Likewise, equation (306) leads readily to [82, 83]

$$\boldsymbol{\omega} = 2\Xi^T(\bar{\eta})\dot{\bar{\eta}} \tag{310}$$

$$= 2\begin{bmatrix} \eta_4\dot{\eta}_1 + \eta_3\dot{\eta}_2 - \eta_2\dot{\eta}_3 - \eta_1\dot{\eta}_4 \\ -\eta_3\dot{\eta}_1 + \eta_4\dot{\eta}_2 + \eta_1\dot{\eta}_3 - \eta_2\dot{\eta}_4 \\ \eta_2\dot{\eta}_1 - \eta_1\dot{\eta}_2 + \eta_4\dot{\eta}_3 - \eta_3\dot{\eta}_4 \end{bmatrix} \tag{311}$$

$$= 2(\eta_4\dot{\boldsymbol{\eta}} - \dot{\eta}_4\boldsymbol{\eta} - \boldsymbol{\eta} \times \dot{\boldsymbol{\eta}})\,, \tag{312}$$

$$\omega^2 = 4\dot{\bar{\eta}}^T\dot{\bar{\eta}}\,. \tag{313}$$

In similar fashion, the kinematic equation for the Euler-Rodrigues symmetric parameters may be expresessed in terms of the space-referenced angular velocity. Thus,

$$\frac{d}{dt}\bar{\eta} = \frac{1}{2}\bar{\eta} \otimes \bar{\omega}_I = \frac{1}{2}\Omega_R(\boldsymbol{\omega}_I)\bar{\eta}\,, \tag{314}$$

where now

$$\Omega_R(\mathbf{v}) \equiv \begin{bmatrix} 0 & -\nu_3 & \nu_2 & \nu_1 \\ \nu_3 & 0 & -\nu_1 & \nu_2 \\ -\nu_2 & \nu_1 & 0 & \nu_3 \\ -\nu_1 & -\nu_2 & -\nu_3 & 0 \end{bmatrix} = \{\bar{\mathbf{v}}\}_R\,. \tag{315}$$

Writing

$$\{\bar{\eta}\}_L \equiv [\Psi(\bar{\eta}) \,|\, \bar{\eta}]\,, \tag{316}$$

or

$$\Psi(\bar{\eta}) = \begin{bmatrix} \eta_4 & \eta_3 & -\eta_2 \\ -\eta_3 & \eta_4 & \eta_1 \\ \eta_2 & -\eta_1 & \eta_4 \\ -\eta_1 & -\eta_2 & -\eta_3 \end{bmatrix} = \begin{bmatrix} \eta_4 I_{3\times 3} + [[\boldsymbol{\eta}]] \\ -\boldsymbol{\eta}^T \end{bmatrix}, \tag{317}$$

it follows that

$$\frac{d}{dt}\bar{\eta} = \frac{1}{2}\Psi(\bar{\eta})\boldsymbol{\omega}_I\,. \tag{318}$$

Inverting equation (314) leads directly to

$$\bar{\omega}_I = \begin{bmatrix} \omega_I \\ 0 \end{bmatrix} = 2\bar{\eta}^{-1} \otimes \frac{d\bar{\eta}}{dt}, \tag{319}$$

or

$$\omega_I = 2\Psi^T(\bar{\eta})\dot{\bar{\eta}} \tag{320}$$

$$= 2\begin{bmatrix} \eta_4\dot{\eta}_1 - \eta_3\dot{\eta}_2 + \eta_2\dot{\eta}_3 - \eta_1\dot{\eta}_4 \\ \eta_3\dot{\eta}_1 + \eta_4\dot{\eta}_2 - \eta_1\dot{\eta}_3 - \eta_2\dot{\eta}_4 \\ -\eta_2\dot{\eta}_1 + \eta_1\dot{\eta}_2 + \eta_4\dot{\eta}_3 - \eta_3\dot{\eta}_4 \end{bmatrix} \tag{321}$$

$$= 2(\eta_4\dot{\boldsymbol{\eta}} - \dot{\eta}_4\boldsymbol{\eta} + \boldsymbol{\eta} \times \dot{\boldsymbol{\eta}}), \tag{322}$$

$$\omega_I^2 = 4\dot{\bar{\eta}}^T\dot{\bar{\eta}}. \tag{323}$$

Note also

$$\frac{d}{dt}\bar{\eta}^{-1} = -\frac{1}{2}\bar{\eta}^{-1} \otimes \bar{\omega} = -\frac{1}{2}\Omega_R(\omega)\bar{\eta}^{-1} = -\frac{1}{2}\Psi(\bar{\eta}^{-1})\omega, \tag{324}$$

$$\frac{d}{dt}\bar{\eta}^{-1} = \frac{1}{2}\bar{\omega}_I \otimes \bar{\eta}^{-1} = -\frac{1}{2}\Omega_L(\omega_I)\bar{\eta}^{-1} = -\frac{1}{2}\Xi(\bar{\eta}^{-1})\omega_I. \tag{325}$$

Three-Vector of Euler-Rodrigues Symmetric Parameters

For the vector components of the column matrix of the Euler Rodrigues symmetric parameters, the kinematic relations when $|\boldsymbol{\eta}| < 1$ (if $|\boldsymbol{\eta}| = 1$, the expressions may not be defined) are:

$$\frac{d}{dt}\boldsymbol{\eta} = \frac{1}{2}(\sqrt{1 - |\boldsymbol{\eta}|^2}\, I_{3\times 3} - [[\boldsymbol{\eta}]])\boldsymbol{\omega}, \tag{326}$$

$$\boldsymbol{\omega} = 2(\sqrt{1 - |\boldsymbol{\eta}|^2}\,\dot{\boldsymbol{\eta}} + \frac{\boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}}}{\sqrt{1 - |\boldsymbol{\eta}|^2}}\boldsymbol{\eta} - \boldsymbol{\eta} \times \dot{\boldsymbol{\eta}}). \tag{327}$$

Likewise, in terms of the space-referenced angular velocity (provided $|\boldsymbol{\eta}| < 1$)

$$\frac{d}{dt}\boldsymbol{\eta} = \frac{1}{2}(\sqrt{1 - |\boldsymbol{\eta}|^2}\, I_{3\times 3} + [[\boldsymbol{\eta}]])\boldsymbol{\omega}_I, \tag{328}$$

$$\boldsymbol{\omega}_I = 2(\sqrt{1 - |\boldsymbol{\eta}|^2}\,\dot{\boldsymbol{\eta}} + \frac{\boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}}}{\sqrt{1 - |\boldsymbol{\eta}|^2}}\boldsymbol{\eta} + \boldsymbol{\eta} \times \dot{\boldsymbol{\eta}}). \tag{329}$$

The Rodrigues Parameters

The kinematic relation for the Rodrigues vector follows from that for the Euler-Rodrigues symmetric parameters and is [63]

$$\frac{d}{dt}\boldsymbol{\rho} = \frac{1}{2}[\boldsymbol{\omega} - \boldsymbol{\omega} \times \boldsymbol{\rho} + (\boldsymbol{\omega} \cdot \boldsymbol{\rho})\boldsymbol{\rho}], \tag{330}$$

which may be written equivalently as

$$\frac{d}{dt}[[\boldsymbol{\rho}]] = \frac{1}{2}(I - [[\boldsymbol{\rho}]])\,[[\boldsymbol{\omega}]]\,(I - [[\boldsymbol{\rho}]])^T. \tag{331}$$

Inverting this equation leads to [83]

$$\boldsymbol{\omega} = 2(I - [[\boldsymbol{\rho}]] + \boldsymbol{\rho}\boldsymbol{\rho}^T)^{-1}\dot{\boldsymbol{\rho}} = \frac{2}{1 + |\boldsymbol{\rho}|^2}(\dot{\boldsymbol{\rho}} - \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}). \tag{332}$$

In terms of the space-referenced angular velocity these become

$$\frac{d}{dt}\boldsymbol{\rho} = \frac{1}{2}[\boldsymbol{\omega}_s + \boldsymbol{\omega}_s \times \boldsymbol{\rho} + (\boldsymbol{\omega}_s \cdot \boldsymbol{\rho})\boldsymbol{\rho}], \tag{333a}$$

and

$$\boldsymbol{\omega}_s = 2(I + [[\boldsymbol{\rho}]] + \boldsymbol{\rho}\boldsymbol{\rho}^T)^{-1}\dot{\boldsymbol{\rho}} = \frac{2}{1 + |\boldsymbol{\rho}|^2}(\dot{\boldsymbol{\rho}} + \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}). \tag{333b}$$

The Cayley-Klein Parameters

In analogy with the Euler-Rodrigues symmetric parameters, the kinematic equation for the unitary Cayley-Klein matrix is given by

$$\frac{d}{dt}\mathrm{H} = \frac{\mathrm{i}}{2}\Gamma(\boldsymbol{\omega})\mathrm{H} = \frac{\mathrm{i}}{2}\mathrm{H}\Gamma(\boldsymbol{\omega}_s), \tag{334}$$

where

$$\Gamma(\boldsymbol{\omega}) \equiv \begin{bmatrix} \omega_3 & \omega_1 - \mathrm{i}\omega_2 \\ \omega_1 + \mathrm{i}\omega_2 & -\omega_3 \end{bmatrix} = \boldsymbol{\omega} \cdot \boldsymbol{\sigma}. \tag{335}$$

The matrix Γ is Hermitian, i.e.,

$$\Gamma^\dagger = \Gamma. \tag{336}$$

Using equation (222)

$$\boldsymbol{\omega} = -\mathrm{i}\,\mathrm{tr}\left(\mathrm{H}^\dagger \boldsymbol{\sigma} \frac{d\mathrm{H}}{dt}\right), \qquad \boldsymbol{\omega}_s = -\mathrm{i}\,\mathrm{tr}\left(\frac{d\mathrm{H}}{dt}\boldsymbol{\sigma}\mathrm{H}^\dagger\right). \tag{337}$$

The Modified Rodrigues Parameters

For the modified Rodrigues vector

$$\frac{d\mathbf{p}}{dt} = \frac{1}{4}[(1 - |\mathbf{p}|^2)\boldsymbol{\omega} - 2\boldsymbol{\omega} \times \mathbf{p} + 2(\boldsymbol{\omega} \cdot \mathbf{p})\mathbf{p}], \tag{338}$$

from which

$$\boldsymbol{\omega} = \frac{4}{(1 + |\mathbf{p}|^2)^2}\left[(1 - |\mathbf{p}|^2)\frac{d\mathbf{p}}{dt} - 2\mathbf{p} \times \frac{d\mathbf{p}}{dt} + 2\left(\mathbf{p} \cdot \frac{d\mathbf{p}}{dt}\right)\mathbf{p}\right]. \tag{339}$$

These may be recast as

$$\frac{d\mathbf{p}}{dt} = \frac{1 + |\mathbf{p}|^2}{4}\left(\frac{I - [[\mathbf{p}]]}{I + [[\mathbf{p}]]}\right)\boldsymbol{\omega}, \tag{340}$$

and

$$\boldsymbol{\omega} = \frac{4}{1 + |\mathbf{p}|^2}\left(\frac{I + [[\mathbf{p}]]}{I - [[\mathbf{p}]]}\right)\frac{d\mathbf{p}}{dt}. \tag{341}$$

In terms of the space-referenced angular velocity these become

$$\frac{d\mathbf{p}}{dt} = \frac{1}{4}[(1 - |\mathbf{p}|^2)\boldsymbol{\omega}_I + 2\boldsymbol{\omega}_I \times \mathbf{p} + 2(\boldsymbol{\omega}_I \cdot \mathbf{p})\mathbf{p}] \tag{342a}$$

$$= \frac{1 + |\mathbf{p}|^2}{4}\left(\frac{I + [[\mathbf{p}]]}{I - [[\mathbf{p}]]}\right)\boldsymbol{\omega}_I, \tag{342b}$$

$$\boldsymbol{\omega}_I = \frac{4}{(1 + |\mathbf{p}|^2)^2}\left[(1 - |\mathbf{p}|^2)\frac{d\mathbf{p}}{dt} + 2\mathbf{p} \times \frac{d\mathbf{p}}{dt} + 2\left(\mathbf{p} \cdot \frac{d\mathbf{p}}{dt}\right)\mathbf{p}\right] \tag{343a}$$

$$= \frac{4}{1 + |\mathbf{p}|^2}\left(\frac{I - [[\mathbf{p}]]}{I + [[\mathbf{p}]]}\right)\frac{d\mathbf{p}}{dt}. \tag{343b}$$

Attitude Dynamics

Related to these results is the rate of change of matrix representations in rotating coordinate systems. Let $\mathbf{v}_I(t)$ denote the representation of a vector with respect to the space reference system, which is assumed to be inertial, and let $\mathbf{v}_B(t)$ denote the representation of this same vector with respect to the reference system fixed in the vehicle body, which is assumed to be rotating. Thus,

$$\mathbf{v}_B(t) = A(t)\mathbf{v}_I(t), \tag{344}$$

where the direction-cosine matrix is denoted by A. The possibility is allowed that $\mathbf{v}_I$ is not a constant column vector. Differentiating with respect to the time yields

$$\frac{d}{dt}\mathbf{v}_B(t) = [[\boldsymbol{\omega}(t)]]\mathbf{v}_B(t) + A(t)\frac{d}{dt}\mathbf{v}_I(t), \tag{345}$$

the first term representing the rate of change due to the rotation of the coordinate system, the second due to intrinsic changes in $\mathbf{v}_I(t)$.

Similar to the definition of the abstract axis of rotation of equation (100) we can define an abstract angular velocity vector according to

$$\boldsymbol{\varpi} = \sum_{i=1}^{3} \omega_i \hat{\boldsymbol{b}}_i, \tag{346}$$

where $\boldsymbol{\omega}$ is the body-referenced angular velocity and the body-fixed basis has been written as $\mathcal{B} = \{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$. Then

$$\frac{d}{dt}\hat{\boldsymbol{b}}_i = \boldsymbol{\varpi} \times \hat{\boldsymbol{b}}_i, \qquad i = 1, 2, 3. \tag{347}$$

Since the representation of the body coordinate axes do not change in the body

coordinate system,

$$\frac{d}{dt}(\hat{\mathbf{b}}_i(t))_s = \mathbf{0}, \qquad i = 1,2,3. \tag{348}$$

It follows as a consequence of equation (345) that the inertial representations of the body axes satisfy

$$\frac{d}{dt}(\hat{\mathbf{b}}_i(t))_I = \boldsymbol{\omega}_I \times (\hat{\mathbf{b}}_i(t))_I, \qquad i = 1,2,3. \tag{349}$$

Writing the inertial basis as $I = \{\hat{\imath}_1, \hat{\imath}_2, \hat{\imath}_3\}$ we have correspondingly

$$\boldsymbol{\varpi} = \sum_{i=1}^{3} (\omega_I)_i \hat{\imath}_i, \tag{350a}$$

$$\frac{d}{dt}\hat{\imath}_i = 0, \qquad i = 1,2,3, \tag{350b}$$

$$\frac{d}{dt}(\hat{\mathbf{i}}_i)_I = \mathbf{0}, \qquad i = 1,2,3, \tag{350c}$$

$$\frac{d}{dt}(\hat{\mathbf{i}}_i)_s = -\boldsymbol{\omega}_s \times (\hat{\mathbf{i}}_i)_s, \qquad i = 1,2,3. \tag{350d}$$

Note the different signs in the right members of equations (349) and (350d). Note also that the direction of $\boldsymbol{\varpi}$ is the instantaneous axis of rotation, which is not the same as the axis of rotation characterizing a finite rotation.

Likewise, the angular momentum of a rigid body, in the inertial frame is

$$\mathbf{L}_I = \int \mathbf{r}_I \times \mathbf{v}_I \, dm = \mathbf{I}_I \boldsymbol{\omega}_I, \tag{351}$$

where $\mathbf{I}_I$ is the *inertia tensor* (in inertial coordinates) defined as

$$\mathbf{I}_I \equiv \int [(\mathbf{r}_I \cdot \mathbf{r}_I) I - \mathbf{r}_I \mathbf{r}_I^T] \, dm. \tag{352}$$

If in the inertial reference frame

$$\frac{d}{dt}\mathbf{L}_I = \mathbf{N}_I, \tag{353}$$

where $\mathbf{N}_I$ is the torque in inertial coordinates, in body coordinates this becomes

$$\frac{d}{dt}\mathbf{L}_s = [[\boldsymbol{\omega}_s]]\mathbf{L}_s + \mathbf{N}_s, \quad \text{or} \quad \frac{d}{dt}\mathbf{L}_s + \boldsymbol{\omega}_s \times \mathbf{L}_s = \mathbf{N}_s, \tag{354}$$

which is *Euler's Equation*, and

$$\mathbf{N}_s = A\mathbf{N}_I. \tag{355}$$

For a rigid body

$$\mathbf{L}_s = \mathbf{I}_s \boldsymbol{\omega}_s, \tag{356}$$

where $\mathbf{I}_s$, the inertia tensor in the body-fixed representation, is given by

$$\mathbf{I}_s = A\mathbf{I}_i A^T = \int [(\mathbf{r}_s \cdot \mathbf{r}_s)I - \mathbf{r}_s\mathbf{r}_s^T]\, dm\,, \tag{357}$$

and (for a rigid body) is constant in time. Thus, for a rigid body equation (354) becomes

$$\mathbf{I}_s \frac{d}{dt}\boldsymbol{\omega}_s + \boldsymbol{\omega}_s \times (\mathbf{I}_s\boldsymbol{\omega}_s) = \mathbf{N}_s\,. \tag{358}$$

Properties of Quaternion Transformations

The quaternion composition operators $\{\bar{q}\}_L$ and $\{\bar{q}\}_R$ also have interesting algebraic and kinematic properties [81, 84]. If $\bar{p}$ is an arbitrary quaternion, and

$$\bar{q}'' = \bar{q}' \otimes \bar{q} \otimes \bar{p}\,, \tag{359}$$

it follows that

$$\bar{q}'' = \{\bar{q}' \otimes \bar{q}\}_L \bar{p} = \{\bar{q}'\}_L \{\bar{q}\}_L \bar{p}\,. \tag{360}$$

Hence,

$$\{\bar{q}' \otimes \bar{q}\}_L = \{\bar{q}'\}_L \{\bar{q}\}_L\,. \tag{361}$$

and $\{\bar{q}\}_L$ has the same group properties as $\bar{q}$, with the operation being matrix multiplication and the identity element and inverses being given by

$$\{\bar{1}\}_L = I_{4\times 4}, \quad \text{and} \quad \{\bar{q}\}_L^{-1} = \{\bar{q}^{-1}\}_L = (\bar{q}^T\bar{q})^{-1}\{\bar{q}\}_L^T\,. \tag{362}$$

Similarly,

$$\{\bar{q}' \otimes \bar{q}\}_R = \{\bar{q}\}_R \{\bar{q}'\}_R\,, \tag{363}$$

with identity element and inverses given by

$$\{\bar{1}\}_R = I_{4\times 4}, \quad \text{and} \quad \{\bar{q}\}_R^{-1} = \{\bar{q}^{-1}\}_R = (\bar{q}^T\bar{q})^{-1}\{\bar{q}\}_R^T\,. \tag{364}$$

Note that

$$\bar{q}'' = \{\bar{q}'\}_L \bar{q} \quad \text{implies} \quad \{\bar{q}''\}_L = \{\bar{q}'\}_L \{\bar{q}\}_L\,, \tag{365a}$$

$$\bar{q}'' = \{\bar{q}\}_R \bar{q}' \quad \text{implies} \quad \{\bar{q}''\}_R = \{\bar{q}\}_R \{\bar{q}'\}_R\,, \tag{365b}$$

and $\bar{q}^T\bar{q}' = 0$ implies

$$\{\bar{q}'\}_L^T\bar{q} = -\{\bar{q}\}_L^T\bar{q}', \quad \text{and} \quad \{\bar{q}'\}_R^T\bar{q} = -\{\bar{q}\}_R^T\bar{q}'. \tag{365c}$$

From the associativity of quaternion multiplication, it follows that

$$\{\bar{q}\}_L \{\bar{p}\}_R = \{\bar{p}\}_R \{\bar{q}\}_L \tag{366}$$

for arbitrary quaternions $\bar{q}$ and $\bar{p}$.

For the special case of the Euler-Rodrigues symmetric parameters, $\{\bar{\eta}\}_L$ and $\{\bar{\eta}\}_R$ are orthogonal as well. It is interesting to note that the set of all $\{\bar{\eta}\}_L$ and the set of all $\{\bar{\eta}\}_R$ are also representations of the rotation group. In fact, they are subgroups of a larger group of 4×4 matrices which contains the Lorentz transfor-

mations (see, for example, [1]). The group of Lorentz transformations can be described by Cayley-Klein matrices [1, 17, 85] or quaternions provided these are allowed to be complex [86–88]. Ebert [89] has shown that any 4×4 proper orthogonal matrix O can be written in the form

$$O = \{\bar{\alpha}\}_L \{\bar{\beta}\}_R , \tag{367}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are unit 4-vectors, which are unique within a common sign. A special case of this result is equation (179). This is a reflection of the fact (well-known to theoretical physicists [90]) that the group SO(4) is homomorphic to the group SO(3) $\otimes$ SO(3).

For the Euler-Rodrigues symmetric parameters, it follows immediately from equations (361) and (362) that these satisfy the kinematic relations

$$\frac{d}{dt}\{\bar{\eta}(t)\}_L = \frac{1}{2}\Omega_L(\boldsymbol{\omega}(t))\,\{\bar{\eta}(t)\}_L , \tag{368}$$

and, recalling equation (314),

$$\frac{d}{dt}\{\bar{\eta}(t)\}_L = \frac{1}{2}\{\eta(t)\}_L \Omega_L(\boldsymbol{\omega}_l(t)) . \tag{369}$$

Likewise, from equations (363) and (364)

$$\frac{d}{dt}\{\bar{\eta}(t)\}_R = \frac{1}{2}\{\bar{\eta}(t)\}_R \Omega_R(\boldsymbol{\omega}(t)) \tag{370}$$

$$= \frac{1}{2}\Omega_R(\boldsymbol{\omega}_l(t))\,\{\bar{\eta}(t)\}_R . \tag{371}$$

Note also the relations

$$\{\bar{\eta}\}_L \Omega_L(\mathbf{v})\,\{\bar{\eta}^{-1}\}_L = \Omega_L(R(\bar{\eta})\mathbf{v}) , \tag{372a}$$

$$\{\bar{\eta}^{-1}\}_R \Omega_R(\mathbf{v})\,\{\bar{\eta}\}_R = \Omega_R(R(\bar{\eta})\mathbf{v}) , \tag{372b}$$

which are the four-dimensional counterparts of equations (232) and (233).

The matrix $\Xi(\bar{\eta})$ has interesting representational and kinematic properties. From the definitions, equation (307) or equation (308), it follows that

$$\Xi^T(\bar{\eta})\Xi(\bar{\eta}) = I_{3\times 3}, \quad \text{and} \quad \Xi(\bar{\eta})\Xi^T(\bar{\eta}) = I_{4\times 4} - \bar{\eta}\bar{\eta}^T, \tag{373}$$

and

$$\Xi^T(\bar{\eta})\bar{\eta} = \mathbf{0} , \tag{374}$$

$$\Xi(\bar{\eta})\mathbf{v} = \Omega_L(\mathbf{v})\bar{\eta} . \tag{375}$$

From equations (304), (308), and (373) it follows that

$$\Xi(\bar{\eta})\,[[\mathbf{v}]] - \Omega_L(\mathbf{v})\Xi(\bar{\eta}) = \bar{\eta}\,\mathbf{v}^T. \tag{376}$$

The matrix $\Xi(\bar{\eta})$ satisfies several kinematic relations. From the definition of $\Xi(\bar{\eta})$ it follows trivially that

$$\frac{d}{dt}\Xi(\bar{\eta}) = \frac{1}{2}\Omega_R(\boldsymbol{\omega}_I)\Xi(\bar{\eta})\,. \tag{377}$$

On the other hand, it follows from equation (179) that

$$\{\bar{\eta}^{-1}\}_L\Xi(\bar{\eta})R(\bar{\eta}) = \begin{bmatrix} I_{3\times3} \\ \mathbf{0}^T \end{bmatrix}, \tag{378}$$

whence

$$\frac{d}{dt}\Xi(\bar{\eta}) = \frac{1}{2}\Omega_L(\boldsymbol{\omega})\Xi(\bar{\eta}) - \Xi(\bar{\eta})[[\boldsymbol{\omega}]] \tag{379a}$$

$$= -\frac{1}{2}\bar{\eta}\boldsymbol{\omega}^T - \frac{1}{2}\Xi(\bar{\eta})[[\boldsymbol{\omega}]]\,. \tag{379b}$$

The three matrices, $\{\bar{\eta}\}_L$, $\Xi(\bar{\eta})$ and $R(\bar{\eta})$, satisfy other relationships, which may be derived from equation (378). If $\bar{\eta}$ and $\bar{\varepsilon}$ are any two column vectors of Euler-Rodrigues symmetric parameters, then [81]

$$\{\bar{\eta}\otimes\bar{\varepsilon}^{-1}\}_L = \Xi(\bar{\eta})R(\bar{\eta}\otimes\bar{\varepsilon}^{-1})\Xi^T(\bar{\varepsilon}) + \bar{\eta}\bar{\varepsilon}^T, \tag{380}$$

$$\{\bar{\eta}\otimes\bar{\varepsilon}^{-1}\}_L\Xi(\bar{\varepsilon}) = \Xi(\bar{\eta})R(\bar{\eta}\otimes\bar{\varepsilon}^{-1}), \tag{381}$$

$$\Xi^T(\bar{\eta})\{\bar{\eta}\otimes\bar{\varepsilon}^{-1}\}_L = R(\bar{\eta}\otimes\bar{\varepsilon}^{-1})\Xi^T(\bar{\varepsilon})\,, \tag{382}$$

$$\Xi^T(\bar{\eta})\{\bar{\eta}\otimes\bar{\varepsilon}^{-1}\}_L\Xi(\bar{\varepsilon}) = R(\bar{\eta}\otimes\bar{\varepsilon}^{-1}). \tag{383}$$

Similarly to equation (373), the matrix $\Psi(\bar{\eta})$, defined by equation (316) or equation (317), satisfies

$$\Psi^T(\bar{\eta})\Psi(\bar{\eta}) = I_{3\times3}, \quad \text{and} \quad \Psi(\bar{\eta})\Psi^T(\bar{\eta}) = I_{4\times4} - \bar{\eta}\bar{\eta}^T. \tag{384}$$

Note also

$$\Psi^T(\bar{\eta})\bar{\eta} = \mathbf{0}\,, \tag{385}$$

$$\psi(\bar{\eta})\mathbf{v} = \Omega_R(\mathbf{v})\bar{\eta}\,, \tag{386}$$

$$\Xi^T(\bar{\eta})\Psi(\bar{\eta}) = R(\bar{\eta})\,. \tag{387}$$

In analogy with equation (376), we have

$$\Omega_R(\mathbf{v})\Psi(\bar{\eta}) + \Psi(\bar{\eta})\,[[\mathbf{v}]] = -\bar{\eta}\mathbf{v}^T. \tag{388}$$

The matrix $\Psi(\bar{\eta})$ also satisfies several kinematic relations. From the definition of $\Psi(\bar{\eta})$ it follows trivially that

$$\frac{d}{dt}\Psi(\bar{\eta}) = \frac{1}{2}\Omega_L(\boldsymbol{\omega})\Psi(\bar{\eta})\,. \tag{389}$$

Likewise, from equation (179) it follows that

$$\{\bar{\eta}^{-1}\}_R\Psi(\bar{\eta})R^{-1}(\bar{\eta}) = \begin{bmatrix} I_{3\times3} \\ \mathbf{0}^T \end{bmatrix}, \tag{390}$$

whence

$$\frac{d}{dt}\Psi(\bar{\eta}) = -\frac{1}{2}\Omega_R(\boldsymbol{\omega}_I)\Psi(\bar{\eta}) - \Psi(\bar{\eta})\,[[\boldsymbol{\omega}_I]] \tag{391a}$$

$$= \frac{1}{2}\bar{\eta}\boldsymbol{\omega}_I^T - \frac{1}{2}\Psi(\bar{\eta})\,[[\boldsymbol{\omega}_I]]\,. \tag{392b}$$

In analogy to equations (380) through (383)

$$\{\bar{\eta} \otimes \bar{\varepsilon}^{-1}\}_R = \Psi(\bar{\eta})R(\bar{\eta}^{-1} \otimes \bar{\varepsilon})\Psi^T(\bar{\varepsilon}) + \bar{\eta}\bar{\varepsilon}^T, \tag{392}$$

$$\{\bar{\eta} \otimes \bar{\varepsilon}^{-1}\}_R\Psi(\bar{\varepsilon}) = \Psi(\bar{\eta})R(\bar{\eta}^{-1} \otimes \bar{\varepsilon}), \tag{393}$$

$$\Psi^T(\bar{\eta})\{\bar{\eta} \otimes \bar{\varepsilon}^{-1}\}_R = R(\bar{\eta}^{-1} \otimes \bar{\varepsilon})\Psi^T(\bar{\varepsilon})\,, \tag{394}$$

$$\Psi^T(\bar{\eta})\,\{\bar{\eta} \otimes \bar{\varepsilon}^{-1}\}_R\Psi(\bar{\varepsilon}) = R(\bar{\eta}^{-1} \otimes \bar{\varepsilon}). \tag{395}$$

Attitude Errors

Since there are many different parameterizations of the attitude, there are equally many different parameterizations of attitude errors. However, in the same way that there is one particularly natural way to express the time derivative of the attitude, namely, the body-referenced angular velocity, there is a corresponding natural representation for attitude probabilities, the (body-referenced) attitude error vector.

Suppose that A^* is some measurement or estimate of the attitude matrix, i.e., a random attitude matrix, and A is its true value. It is assumed that with very large probability A^* is close to A. Thus, in general, A^*A^T is expected to be a small rotation.

In analogy to equations (258) and (259)

$$A^* = (\delta A)A\,, \tag{396}$$

with

$$\delta A = I + [[\Delta\boldsymbol{\xi}]] + O(|\Delta\boldsymbol{\xi}|^2)\,, \tag{397}$$

where $\Delta\boldsymbol{\xi}$ is some small three-vector, which is a measure of the attitude error. We call $\Delta\boldsymbol{\xi}$ the *attitude error vector.* The random attitude matrix can be written formally as

$$A^* = e^{[[\Delta\boldsymbol{\xi}]]}A\,, \tag{398}$$

and $\Delta\boldsymbol{\xi}$ is, therefore, the rotation vector characterizing the random infinitesimal rotation. Like the angular velocity in equation (261), the attitude error vector is referred to the body axes.

In most practical situations one expects that

$$E\{\Delta\boldsymbol{\xi}\} = \mathbf{0}\,, \tag{399}$$

though this need not be the case if the attitude measurements are subject to systematic errors (as from uncompensated sensor biases or misalignments). The *atti-*

tude covariance matrix is defined in terms of the attitude error vector as

$$P_{\xi\xi} \equiv \text{Cov}\{\Delta\boldsymbol{\xi}\} = E\{\Delta\boldsymbol{\xi}\Delta\boldsymbol{\xi}^T\} - E\{\Delta\boldsymbol{\xi}\}E\{\Delta\boldsymbol{\xi}^T\}. \tag{400}$$

The error in the elements of the attitude matrix is given, therefore, by

$$\Delta A \equiv A^* - A = [[\Delta\boldsymbol{\xi}]]A, \tag{401}$$

which has an analytical form similar to the corresponding kinematic relation, equation (261), the only difference being whether $\Delta\boldsymbol{\xi}$ expresses random error or actual motion. This permits the error relations for the other representations to be written by inspection from the corresponding kinematic relation.

It is sometimes useful to work in terms of

$$P_{AA} \equiv E\{\Delta A\Delta A^T\} - E\{\Delta A\}E\{\Delta A^T\}. \tag{402}$$

Then

$$P_{AA} = (\text{tr } P_{\xi\xi})I - P_{\xi\xi}, \quad \text{and} \quad P_{\xi\xi} = \frac{1}{2}(\text{tr } P_{AA})I - P_{AA}. \tag{403}$$

The errors in the Euler-Rodrigues symmetric parameters can be written in terms of the attitude error vector in the same way that the rate of change of the quaternion was written in terms of the angular velocity. Thus,

$$\bar{\eta}^* = (\delta\bar{\eta}) \otimes \bar{\eta} = \{\bar{\eta}\}_R\delta\bar{\eta}, \tag{404}$$

and

$$\delta\bar{\eta} = \begin{bmatrix} \Delta\boldsymbol{\xi}/2 \\ 1 \end{bmatrix} + O(|\Delta\boldsymbol{\xi}|^2). \tag{405}$$

It follows that

$$\Delta\bar{\eta} \equiv \bar{\eta}^* - \bar{\eta} = \{\bar{\eta}\}_R\begin{bmatrix} \Delta\boldsymbol{\xi}/2 \\ 0 \end{bmatrix} + O(|\Delta\boldsymbol{\xi}|^2), \tag{406}$$

whence, in analogy with equation (306)

$$\Delta\bar{\eta} = \frac{1}{2}\Xi(\bar{\eta})\Delta\boldsymbol{\xi}, \quad \text{and} \quad \Delta\boldsymbol{\xi} = 2\Xi^T(\bar{\eta})\Delta\bar{\eta}. \tag{407}$$

Defining

$$P_{\eta\eta} \equiv \text{Cov}\{\Delta\bar{\eta}\}, \tag{408}$$

it follows that to lowest order

$$P_{\eta\eta} = \frac{1}{4}\Xi(\bar{\eta})P_{\xi\xi}\Xi^T(\bar{\eta}), \tag{409}$$

and, because of the constraint on the Euler-Rodrigues symmetric parameters,

$$P_{\eta\eta}\bar{\eta} = \bar{0}. \tag{410}$$

In the same way, if the relation between the angular velocity and the Euler angle rates is written as in equation (284), then the corresponding relationship for the attitude error is

$$\Delta\boldsymbol{\xi} = M(\varphi, \vartheta, \psi)\begin{bmatrix}\delta\varphi\\ \delta\vartheta\\ \delta\psi\end{bmatrix}, \tag{411}$$

whence,

$$\begin{bmatrix}\delta\varphi\\ \delta\vartheta\\ \delta\psi\end{bmatrix} = M^{-1}(\varphi, \vartheta, \psi)\Delta\boldsymbol{\xi}, \tag{412}$$

Denoting the column vector of Euler angles by $\boldsymbol{\phi}$, the covariance matrix of the errors in the Euler angles is then given by

$$P_{\phi\phi} = M^{-1}P_{\xi\xi}(M^{-1})^T. \tag{413}$$

Since M is a singular matrix for certain values of the Euler angles, the covariance matrix of the Euler angle errors may be infinite, even though the attitude errors are small.

The attitude error vector has an interpretation in terms of Euler angles. If we parameterize the infinitesimal rotation matrix $\exp\{[[\Delta\boldsymbol{\xi}]]\}$ in terms of an asymmetric sequence of Euler angles, then the Euler angle about the i-axis is just $\Delta\xi_i$. Hence, these attitude error parameters are also called *attitude error angles* or *Cartesian error angles*, because they are referred to the three Cartesian coordinate axes of the body frame, rather than being differences of Euler angles.

Likewise, the errors in the remaining representations are obtained straightforwardly. These are for the Rodrigues vector:

$$\Delta\boldsymbol{\rho} = \frac{1}{2}[I_{3\times3} - [[\boldsymbol{\rho}]] + \boldsymbol{\rho}\boldsymbol{\rho}^T]\Delta\boldsymbol{\xi}, \tag{414}$$

the rotation angle:

$$\Delta\theta = \hat{\mathbf{n}}^T\Delta\boldsymbol{\xi}, \tag{415}$$

the rotation axis:

$$\Delta\hat{\mathbf{n}} = -\frac{1}{2}[[[\hat{\mathbf{n}}]] + \cot(\theta/2)\,[[\hat{\mathbf{n}}]]^2]\Delta\boldsymbol{\xi}, \tag{416}$$

and the rotation vector:

$$\Delta\boldsymbol{\theta} = \left[I_{3\times3} - \frac{1}{2}[[\boldsymbol{\theta}]] + \frac{1}{\theta^2}(1 - (\theta/2)\cot(\theta/2))\,[[\boldsymbol{\theta}]]^2\right]\Delta\boldsymbol{\xi}. \tag{417}$$

From these the related covariance matrices are readily constructed. Note that $\Delta\hat{\mathbf{n}}$ and $\Delta\theta$ are correlated.

The attitude and the attitude covariance matrix can be combined into a single representation, a 3×3 matrix given by

$$B = f(P_{\mathfrak{qq}})A\,, \tag{418}$$

where $f(P)$ is any invertible real-symmetric 3×3 matrix function of the covariance matrix. The attitude matrix can then be recovered by any of a number of methods which factor an arbitrary real matrix into a real-symmetric matrix and a matrix which is orthogonal (the *polar decomposition* [91]), and the covariance matrix extracted by inverting f. This property has been exploited in attitude estimation [92]. Ebert [89] has developed a careful treatment of probabilities defined on the Euler-Rodrigues symmetric parameters which take rigorously correct account of the group properties of the representation. These become important when the standard deviations of the attitude become large.

Alias and Alibi

Two descriptions are in use for the transformation of coordinates: the *passive* (known also as the *alias* description, from the Latin word for "otherwise" but in the common sense of "otherwise known as") and the *active* (or *alibi* from the Latin word for "elsewhere"). In spacecraft attitude work the passive description has been more common. The works of Goldstein [1], Hughes [2], Junkins and Turner [3], Wertz [5], this author [7], Thomson [93], Kaplan [94], Rimrott [95], and this survey have chosen the passive description. Kane, Likins and Levinson [6], and Wiesel [96], to cite only the most recent examples, have chosen the active description. The two descriptions are equivalent, although they express a somewhat different point of view and lead to formulas which are different in detail.

Recall that the definition of the rotation matrix in the passive description led to the result

$$(\hat{\mathbf{e}}_i)_{\mathcal{E}'} = R^{\text{passive}}(\hat{\mathbf{3}}, \theta)\,(\hat{\mathbf{e}}_i)_{\mathcal{E}}\,, \qquad i = 1,2,3\,, \tag{419}$$

as the equation for the transformation for the representations (of basis vectors or otherwise) from one frame to another. Thus, in the passive description one generally represents the same vector with respect to more than one frame of reference (the observer *passively* adjusts his frame while the vector remains immutable). For emphasis, we have written *passive* explicitly in equation (419). This survey up to now has treated rotations only in the passsive description.

In the active description, on the other hand, one takes the point of view that the observer is fixed (usually inertial as well) and examines vectors whose direction changes in the observer's frame of reference. If, for example, the changing vectors are the coordinate axes of a rotating vehicular frame $\mathcal{E}'$ then the rotation matrix in the active description is defined so that

$$(\hat{\mathbf{e}}_i')_{\mathcal{E}} = R^{\text{active}}(\hat{\mathbf{n}}, \theta)\,(\hat{\mathbf{e}}_i)_{\mathcal{E}}, \qquad i = 1,2,3\,. \tag{420}$$

Thus, in the active description, the rotation matrix *actively* changes the vector. It follows that

$$R^{\text{active}} = (R^{\text{passive}})^T, \tag{421}$$

or

$$R_{ij}^{\text{active}} \equiv \hat{\boldsymbol{e}}_j' \cdot \hat{\boldsymbol{e}}_i, \tag{422}$$

the transpose of equation (67). This seemingly minor difference in the definition of the rotation matrix leads to numerous subtle differences in sign between equations expressed in the passive and the active descriptions.

The choice of whether to use the active or the passive description of rotations is largely one of taste. There are obvious advantages to using the passive description for remote-sensing applications such as attitude determination, where the rotation matrix representing spacecraft attitude must be reconstructed from observations of the same vectors made from the ground and from the spacecraft. On the other hand, for purely dynamical studies it is often more natural to present dynamics and control studies from the point of view of the active description. For example, in the active description the abstract linear operator which transforms inertial basis vectors into body basis vectors has the rotation matrix (rather than the transpose) as its matrix representation. Likewise, in Quantum Mechanics, where for obvious reasons, the universe is never described from the point of view of an observer seated on an electron or on a photon, the active representation has always been used. For this reason, physicists engaged in spacecraft attitude determination (such as the author of this survey) often have needed some retooling in this area.

The author with some amusement hastens to point out that this survey and the other works which prefer the passive description have not escaped totally a deeply rooted inclination toward the point of view of an external observer. This is seen most clearly in the definition of the body-referenced Euler angles. Here, the body reference frames were distinguished at different states ($\mathcal{E}$, $\mathcal{E}'$, and $\mathcal{E}''$), while the space axes were always regarded as remaining the same. A truly passive treatment might have followed a very different path and have considered the space axes as changing at each stage of the Euler rotations. Such a choice, however, would have entailed other inconveniences, since there are obvious advantages to regarding the inertial axes as fixed.

The most recognizable change in the formulas using the active and the passive rotations is that the angle of rotation and the Euler angles, which generally have the same physical definitions in both descriptions, enter the rotation matrix with different signs because of equation (421). The Euler-Rodrigues symmetric parameters and the rotation vector, however, generally have the same definition in terms of the axis and angle of rotation in both descriptions. Therefore, the expression for the rotation matrix in terms of the Euler-Rodrigues symmetric parameters usually differs by a matrix transpose in the two descriptions. There is some danger, clearly, in transposing relations from this work into the active description without great care.

Historical Note

A spate of recent publications [97–99] on the origin of the various attitude representations and their composition rules has shown that many of the commonly held beliefs about the origins of the attitude representations are not totally cor-

rect. Of these publications Altmann [97, 98], whose interest centered on quaternions, focused his attention on the division of priorities between Sir William Rowan Hamilton and Olinde Rodrigues. Cheng and Gupta [99] investigated in addition the correct attribution of results associated with the name of Leonard Euler. Additional information can be found in the recent article of Bar-Itzhack and Markley [100] and in the books of Boyer [101], Van der Waerden [102], and Crowe [103]. We report here with extreme brevity the principal historical facts as they are currently known.

The first notions of representing rotations by complex numbers begins with the identification of complex numbers with the plane (the Argand diagram, which was not known by that name until 1806) and Euler's famous formula [104]

$$(\cos x + \mathrm{i} \sin x)^n = \cos nx + \mathrm{i} \sin nx . \tag{423}$$

It requires little imagination to see in this formula a precursor to Euler's formula for rotations. According to Van der Waerden [102], however, Euler does not seem ever to have regarded complex numbers as vectors but only as points in a plane. The later connection with two-dimensional vector spaces Van der Waerden credits to Wessel (1797), Argand (1806), Warren (1828), and Gauss (1831). The expression "complex number" is apparently also due to Gauss. It would seem that Euler's theorem on the motion of a rigid body is indeed due first to Euler [20] in 1775, as was Euler's formula [18]. Euler's theorem had, in fact been preceded by a much earlier work [105], which showed that any differential movement of a rigid body could be expressed as the sum of a translation and a rotation about an axis. The supposed first publication [20] of the Euler angles in 1775 (as cited in Whittaker [21] and which citation has been repeated by numerous later authors) would seem to be falsely attributed as shown by Cheng and Gupta [99], who point out that the first publication of the derivation of the Euler angles was posthumous [22], though the actual work cannot date later than 1783, the year in which Euler died. Euler's formula for the rotation, given in terms of the angles characterizing the spin axis and the angle of rotation does, however, first appear in [18].

The earliest vector notation for this formula, of the type

$$\boldsymbol{v}' = \cos\theta\, \boldsymbol{v} + (1 - \cos\theta)(\hat{\boldsymbol{n}} \cdot \boldsymbol{v})\hat{\boldsymbol{n}} + \sin\theta\, \hat{\boldsymbol{n}} \times \boldsymbol{v} , \tag{424}$$

seems to originate with Gibbs [61] in or before 1901 (who, however, expressed this relation in terms of dyadics). Euler, although he showed that the combination of two successive rotations must be another rotation, never proposed an algorithm for determining this combined rotation. Thus, he did not appreciate the matrix as any more than as a table. The matrix as a mathematical object came much later as a result of the improved understanding of vectors from the work of Grassman [106, 107], Gibbs [62], and Heaviside [108]. (For the history of matrices and matrix algebra see McDuffee [109].)

The first publication on the Euler-Rodrigues symmetric parameters is now generally credited to Rodrigues in 1840 [35]. Rodrigues, a successful banker and celebrated social reformer, in almost his only mathematical publication (he is known also for a recursion relation for the Legendre polynomials) invented the

Euler-Rodrigues symmetric parameters, the three Rodrigues parameters (more commonly called the Gibbs vector after their popularizer—Gibbs preferred to call them the semi-tangent of version [62]), the rules for their composition, and a geometrical construction for combining two rotations. Rodrigues' paper was seldom cited in later years (it was known to Cayley and Klein though Hamilton never acknowledged it). Not long ago, it was the subject of a detailed appraisal by Gray [110], who gives the impression of its having fallen into deep obscurity and only recently been rediscovered (despite its being cited in Whittaker [21]). With the 1840 publication Rodrigues is, in fact, the first person to show how to combine the representations of two rotations in any form in order to obtain the representation of the combined rotation.

The discovery of the quaternion by Hamilton came only a few years later [111]. Hamilton's original interest was directed not towards rotations but to generalizing the complex numbers, and he hit upon the quaternions only after unsuccessful attempts to construct a three-dimensional system with two complex numbers [112]. Hamilton used i, j, and k as his three "imaginaries," a generalization of the use of i for the imaginary in two dimensions. Their usage as designators for unit basis vectors originates with Hamilton in the context of the quaternion. Van der Waerden [102] gives a succinct account of the discovery as does Altmann [97, 98]. The connection with rotations came through the analogy with the geometry of the complex plane, where the multiplication of a number by i results in its rotation by $\pi/2$. We know now, however, that the quaternion $\bar{\mathbf{i}} \equiv [1 \quad 0 \quad 0 \quad 0]^T$ is associated not with a rotation through $\pi/2$ but through π, which forces upon us a somewhat different understanding. It would appear, however, that Hamilton never understood (or never wanted to understand) the true nature of the quaternion as it is related to rotations, and, although he himself had shown that the transformation of a vector by a quaternion was in general a bilinear operation [111], throughout his career he insisted that the rotation of a vector is accomplished by the multiplication of the vector by a single quaternion, and this credo appears also in Hamilton's magnum opus on the subject [113] published shortly after his death.

The first publication in the English language to correctly present the rotation of vectors by means of quaternions is due to Cayley in 1845 [37], who acknowledged the priority of Rodrigues and that Hamilton had also known this result. It would appear that equations (203), (306), and (330) (or (332)) were first introduced by Cayley [63] during this same period. The first publication in English on the combination of two rotations is by Silvester [114], who duplicated Rodrigues' geometric construction with no mention of Rodrigues. Gibb's work [61] on the Rodrigues parameters, which led to the association with his name, dates from 1884. The development of the Cayley-Klein parameters [64, 65] was published individually by those authors in 1874 and 1879, respectively. It seems also that the quaternion as an algebraic object and as a representation of rotations had been discovered in 1819 by Gauss [115], who never thought to publish his discovery, and that Grassman also knew of them [116].

Why, then have these parameters been known for so long as the Euler symmetric parameters? According to Altmann [97, 98], who cites many authors in his support, Euler never knew of the Euler-Rodrigues symmetric parameters nor ever

treated rotations in terms of half-angles at any time in his career. Roberson [82], however, citing a different work of Euler [36] has claimed otherwise, and the same work of Euler was cited in a posthumous paper of Jacobi [117] in which the priority for the Rodrigues' expression for the rotation matrix in terms of the Euler-Rodrigues symmetric parameters is explicitly and even adamantly given to Euler. Examination of reference [36] shows that Euler had indeed developed the four symmetric parameters for an orthogonal transformation (although without the use of half angles). However, closer inspection of Euler's equivalent of equation (158) discloses some unexpected aberrations in the signs—Euler, apparently, had developed the symmetric parameters for an improper orthogonal matrix! (In later works (e.g., [22]), when Euler had become interested in rotations, he, of course, pays greater attention to proper orthogonal matrices, and we may presume that he had the "Euler symmetric parameters" within his grasp for this case also.) In [36] Euler presented also the equivalent of equation (367), and, thus, he appears to have nearly stumbled upon quaternion multiplication. He had, in fact, had an earlier brush with quaternions in a study of the "four-squares theorem" of Fermat. Here, he showed [118] that for all α, β, γ, δ, α', β', γ', and δ',

$$\begin{aligned}(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)(\alpha'^2 + \beta'^2 + \gamma'^2 + \delta'^2) \\ = (\alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta')^2 + (\alpha\beta' + \beta\alpha' + \gamma\delta' - \delta\gamma')^2 \\ + (\alpha\gamma' + \gamma\alpha' + \delta\beta' - \beta\delta')^2 + (\alpha\delta' + \delta\alpha' + \beta\gamma' - \gamma\beta')^2, \qquad (425)\end{aligned}$$

which we recognize as being essentially

$$(\bar{q}^T\bar{q})(\bar{q}'^T\bar{q}') = (\bar{q} \otimes \bar{q}')^T(\bar{q} \otimes \bar{q}'). \qquad (426)$$

This same relationship is recapitulated in [36]. Apparently, Euler's researches reported in [36] were without any physical motivation whatsoever but only to develop a diophantine[11] algorithm for orthogonal matrices of dimension 2, 3, 4, and 5, that is, to generate these matrices with rational elements!

Applications

Having now presented twelve[12] different representations of the attitude, a word should be said about their domains of application. The literature in attitude dynamics, control, and estimation is vast, and, therefore, this survey can only present the applications very superficially. Many more examples can be found among the references, especially [2, 3, 5, 6].

Of all the representations of rotations, the most "natural" is the direction-cosine matrix. However, apart from analytical studies and for transforming vectors, this representation sees little use owing largely to its high dimension. For this reason, early dynamical studies of rotations in the eighteenth and nineteenth centuries had generally used only the first two columns or rows as the attitude

[11]Diophantus of Alexandria (third century C. E.) developed an algorithm for generating all the integral Pythagorean triads (e.g., $3^2 + 4^2 = 5^2$).

[12]or thirty-six, depending on how the Euler angles and modified Rodrigues parameters are counted, or even forty-one if we consider that there are really six possible ways to define the axis-azimuth representation.

representation, as pointed out by Roberson [82]. This still leads to a representation of dimension six. Thus, the direction-cosine matrix is expensive to store and in computer simulations difficult to maintain orthogonal. Nonetheless, for expressing attitude relationships, say, in defining transformations between coordinate systems as alignment matrices or attitude matrices, and for deriving mathematical identities it has obvious advantages.

The axis and angle of rotation, despite their obvious geometric appeal, do not seem to have received much use except as an intermediate quantity for defining or calculating other representations.

The vector of Euler-Rodrigues symmetric parameters or the quaternion of rotation, on the other hand, is the ideal representation for simulation, since its kinematic equation, equation (305) or (306) or related equations are linear. Thus, either of these equations coupled with an equation like equation (358) for the dynamics finds frequent use, especially as the preferred representation for the prediction step of the Kalman filter [81]. The Euler-Rodrigues symmetric parameters have four components instead of the minimal three, and thus impose a slight extra cost in terms of storage requirements. At the same time they possess none of the analytical headaches of representations of smaller dimension which lead to inaccuracies in certain geometries. They are, therefore, the preferred parameterization for spacecraft attitude control systems and have received considerable use in this area [119–122]. Modern attitude dynamics and control studies become quite elegant when expressed in terms of the Euler-Rodrigues symmetric parameters [123–127]. Deprit and Elipe [128] have used the Euler-Rodrigues symmetric parameters to regularize Androyer variables in a very elegant study of the Poinsot problem.

The Euler-Rodrigues symmetric parameters provide the easiest path for restoring the orthogonality of a rotation matrix, which has been lost, say, due to accumulated numerical round-off error. In this case one computes the Euler-Rodrigues symmetric parameters from the rotation matrix in the standard way, ignoring the lack of orthogonality. If the rotation matrix has deviated slightly from orthogonality, the computed Euler-Rodrigues symmetric parameters will deviate slightly from a column vector with unit norm. In this event, one simply normalizes the column vector and recomputes the rotation matrix. The result is necessarily orthogonal. In some extreme cases, the nonorthogonality of the approximate rotation matrix may yield a complex quaternion. A more general means of orthogonalizing a matrix is to calculate the orthogonal matrix A which maximizes $\mathrm{tr}[B^T A]$, *where* B is the nearly orthogonal matrix. This is equivalent to finding the proper orthogonal matrix which is closest to B in the sense of minimizing the Schurr norm,

$$\|B - A\| \equiv \mathrm{tr}((B - A)^T(B - A)) = \sum_{i=1}^{3}\sum_{j=1}^{3}(B - A)_{ij}^2 . \tag{427}$$

Reference [129] presents an efficient algorithm for accomplishing this which exploits the quaternion and the Rodrigues vector.

The Euler-Rodrigues symmetric parameters, because they are defined over the entire unit sphere in four dimensions, can always be made continuous. Thus, if

one is given two sets of parameters, $\bar{\eta}(t_k)$ and $\bar{\eta}(t_{k+1})$, at two instants adjacent in time, the two quaternion differences,

$$\bar{\eta}(t_k) - \bar{\eta}(t_{k+1}) \quad \text{and} \quad \bar{\eta}(t_k) + \bar{\eta}(t_{k+1}),$$

can be computed and their magnitudes tested. If the two instants are very close in time, generally one value will be very small (on the order of $|\omega\Delta t|$) while the other will have a magnitude close to 2. This makes choosing the sign which makes the quaternion continuous (say for interpolation or smoothing) a simple matter. The situation for the three-dimensional representations is much more complicated or even impossible.

The Euler angles, because of their minimum dimension and long history of application in theoretical physics have enjoyed continued popularity. In formal dynamical studies they are still important. However, their popularity in formal studies has often led to their overuse in data processing applications, when better behaved representations, particularly the Euler-Rodrigues symmetric parameters, are a more logical choice. For certain applications, such as gimballed gyros where each Euler angle corresponds to a physical gimbal angle, or for rapidly spinning spacecraft where two of the angles, which are chosen so as to represent the spin axis, remain nearly constant, while the third undergoes nearly uniform motion, the Euler angles have obvious advantages. Gimballed gyros still have important applications in seacraft but increasingly less use in spacecraft and aircraft. Euler angles are useful also in formulating spacecraft attitude maneuvers as a sequence of single-axis slews. In this application, after determining the rotation matrix characterizing the slew, one examines the values of the Euler angles characterizing this rotation matrix for each of the twelve sets. The set in which the absolute sum of the Euler angles is smallest often gives the preferred sequence of single-axis slews.

Until the end of the 1970s, however, Euler angles saw a great deal of use, both for the archival recording of attitude and for solving least-square problems, where unobservable variables and constraints provided disincentives to using the quaternion or rotation matrix. At the same time, because of the large number of trigonometric computations, these approaches generally lacked speed. Increasingly nowadays, the Euler angles find less use.

The axis-azimuth representation has been used primarily for the description of spinning spacecraft [130, 131]. It has the advantage that it can often be computed directly from the measurements without the need for computationally expensive parameter searches but the disadvantage that the computation of the rotation matrix from this representation is very cumbersome. It sees little use now.

The Rodrigues or Gibbs vector also finds little use except as an intermediate step in dealing with the Euler-Rodrigues symmetric parameters, when the extra element becomes a nuisance. For an example of this see [129]. The Rodrigues vector is also useful when an approximation has yielded the infinitesimal rotation vector and it is desired to increment the attitude using this. The easiest method is to interpret this quantity as twice the Rodrigues vector and then compute the incremental quaternion according to equation (201) or the incremental attitude

matrix according to equation (202). This provides a means of maintaining the norm of the quaternion or the orthogonality of the attitude matrix easily.

The rotation vector receives considerable use when it is known to be small. The kinematic equation (equation (276)) can be expanded to yield

$$\frac{d}{dt}\boldsymbol{\theta} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1}{12}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega}) + \frac{\theta^2}{720}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega}) + \ldots . \tag{428}$$

Thus, the terms of higher than linear in $\boldsymbol{\theta}$ can be neglected when $|\boldsymbol{\theta}|$ is very small. (Generally, the term in $\boldsymbol{\theta} \times \boldsymbol{\omega}$ is known as the "coning" correction, the terms proportional to $\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega})$ have been called the "sculling terms" [75].) Terms above quadratic are surely negligible for small angles of rotation. This property is exploited in strap-down gyro systems. Generally, the gyro data is accumulated at a very high rate with exceedingly small intervals, on the order of a millisecond. Thus, if the rotation vector is initially zero, the linear approximation (or, better, quadratic approximation) of equation (428) can be used. At larger intervals, say 1 second, the rotation angle may have grown to a degree in magnitude. The higher order terms might amount to a few arc seconds. At this point the incremental rotation vector is converted to an incremental quaternion of rotation and combined with the previously accumulated quaternion of rotation. The incremental rotation vector is reset to zero and the fast loop is executed again [132, 133].

Likewise, the infinitesimal rotation vector in the guise of the attitude error vector is the ideal quantity to use as a differential corrector. Attitude measurements generally have the form

$$\mathbf{z} = \mathbf{f}(\mathbf{W}) + \mathbf{v}, \tag{429}$$

where $\mathbf{W}$ is the representation of a vector in the vehicular frame, and $\mathbf{v}$ is measurement noise. Without loss of generality for the discussion which follows, $\mathbf{z}$ may be taken to be a scalar. We write

$$\mathbf{W} = A\mathbf{V} = (\delta A)A_o\mathbf{V} \equiv (\delta A)\mathbf{W}_o, \tag{430}$$

where $\mathbf{V}$ is the corresponding column-vector representation in the inertial or other primary reference system, A_o is an approximate (usually, *a priori*) value of the attitude, and δA is the correction to this approximate attitude, which can be written to first order as

$$\delta A \approx I_{3\times 3} + [[\Delta\boldsymbol{\xi}]], \tag{431}$$

and we have written $\Delta\boldsymbol{\xi}$ for the attitude error vector rather than $\Delta\boldsymbol{\theta}$ to be consistent with equation (397) and to avoid confusion with the difference of two arbitrary rotation vectors. Substituting equations (430) and (431) into equation (429) leads to [134]

$$z \approx f(\mathbf{W}_o) - [\mathbf{W}_o \times \nabla f(\mathbf{W}_o)]^T\Delta\boldsymbol{\xi} + v, \tag{432}$$

where ∇f is the gradient of f. Since the infinitesimal rotation vector is the natural representation for describing a small attitude change, this is also the preferred representation for the update step of a Kalman filter [81]. The use of the quaternion difference, $\Delta\bar{\eta}$ (rather than the quotient $\delta\bar{\eta}$) as a differential corrector has

also been explored [59, 81]. It should be noted that for infinitesimal rotations the rotation vector is indistinguishable (apart for a factor of 2) from the Gibbs vector or the vector portion of the Euler-Rodrigues symmetric parameters.

The great disadvantage of three-dimensional representations, be they the Euler angles, Rodrigues parameters, or rotation vector, is the nonlinear nature of their composition rule, which leads also to a nonlinear kinematic equation. This is a property of all three-dimensional representations as has been demonstrated by Stuelpnagel [73]. Thus, the Euler-Rodrigues symmetric parameters provide the representation of smallest dimension which has a bilinear composition rule and, consequently, a linear kinematic equation.

The Cayley-Klein parameters are valuable for elucidating the properties of the Euler-Rodrigues symmetric parameters. However, they provide no computational advantage on modern computers over the Euler-Rodrigues symmetric parameters, and, because of the required complex arithmetic, they have not received frequent use in aerospace work. They do provide a significant simplification in symbolic operations involving the Euler angles. For that reason, in the mammoth work by Klein and Sommerfeld on the theory of tops [135], written in the era before automatic computers, the Cayley-Klein parameters received frequent use.

The matrix-valued vectors, **S**, of equation (242) have their most important applications in the quantum theory of angular momentum, where they are associated with the intrinsic "spin" of elementary particles. The Quantum Theory of Angular Momentum is the subject of a rich literature [136–141], but of little practical value for aerospace engineering.

The modified Rodrigues vector has been applied to the case of a librating spacecraft [69].

The Euler-angles, the axis-azimuth representation, the Rodrigues/Gibbs vector, the rotation vector, the vectorial components of the quaternion, or the modified Rodrigues vector, while of minimum dimension are each beset with problems of singularity or discontinuity. The Euler-Rodrigues symmetric parameters (or, equivalently, the Cayley-Klein parameters) are both continuous and nonsingular representations of the rotations. However, the dimension of the Euler-Rodrigues symmetric parameters is four (hence not minimal), and there correspond two sets of Euler-Rodrigues symmetric parameters for every rotation. It has been shown by Hopf [142], as cited in Stuelpnagel [73], that no representation of dimension three or four can be topologically equivalent to the group of rotation matrices, and that the dimension of the smallest representation of the rotation group which can be mapped bijectively and continuously onto the group of rotation matrices is five. Stuelpnagel [73] has shown how to construct such a representation of dimension five from two columns of the rotation matrix by stereographic projection. The composition rule of this representation is nonlinear. Hence, it is not very useful for practical calculations.

Interestingly enough, the attitude representations can also be applied to the description of spacecraft orbits. Since the unit vector in the direction of the spacecraft position (the zenith vector), the orbit normal, and their vector product form a right-handed orthonormal basis, their orientation (read attitude) together with the parameters of the elliptical orbit specifies the particle position [143–146].

The resulting "gyroscopic" equations which describe the orbit are very similar to Euler's equations for a rigid body.

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Appendix A: Dyadics

Just as 3×1 column vectors are the representation of abstract vectors, 3×3 matrices are the representations of abstract linear operators. An abstract *linear operator* $\mathcal{L}$ is an operator mapping $\mathcal{V}$ into $\mathcal{V}$ which when acting on an abstract vector space satisfies

$$\mathcal{L} \cdot (\boldsymbol{v}_1 + a\boldsymbol{v}_2) = \mathcal{L} \cdot \boldsymbol{v}_1 + a(\mathcal{L} \cdot \boldsymbol{v}_2)\,. \tag{A1}$$

Linear operators may be written in terms of dyadics. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two abstract vectors, then the *dyadic*, defined in terms of the ordered pair of vectors and written as $\boldsymbol{uv}$, satisfies

$$(\boldsymbol{uv}) \cdot \boldsymbol{w} \equiv (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u} \quad \text{and} \quad \boldsymbol{w} \cdot (\boldsymbol{uv}) \equiv (\boldsymbol{w} \cdot \boldsymbol{u})\boldsymbol{v}\,, \tag{A2}$$

where $\boldsymbol{w}$ is any other abstract vector. Thus, the scalar product of an abstract linear operator and an abstract vector is another abstract vector. (Likewise, the scalar product of two abstract linear operators is another linear operator.)

Because space is three-dimensional, any linear operator $\mathcal{L}$ may be written as

$$\mathcal{L} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} \boldsymbol{v}_i \boldsymbol{v}_j\,, \tag{A3}$$

where $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is any basis for the vector space $\mathcal{V}$, and the $a_{ij}, i, j = 1, 2, 3$, are nine uniquely determined scalars. In particular, if the basis is an orthonormal basis, $\mathcal{E} = \{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$, then from the abstract relation

$$\boldsymbol{u} = \mathcal{M}\boldsymbol{v}\,, \tag{A4}$$

it follows in terms of matrices that

$$\mathbf{u} = M\mathbf{v}\,, \tag{A5}$$

where

$$M_{ij} = \hat{\boldsymbol{e}}_i \cdot \mathcal{M} \cdot \hat{\boldsymbol{e}}_j\,, \tag{A6}$$

or equivalently,

$$\mathcal{M} = \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j\,. \tag{A7a}$$

If a second abstract linear operator is written similarly in terms of the orthonormal basis as

$$\mathcal{N} = \sum_{i=1}^{3}\sum_{j=1}^{3} N_{ij}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j\,, \tag{A7b}$$

it follows that the product of the two linear operators is given by

$$\mathcal{M}\cdot\mathcal{N} = \sum_{i=1}^{3}\sum_{j=1}^{3} (MN)_{ij}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j\,, \tag{A8}$$

where the implied operation of M and N in the right member is matrix multiplication. Note in particular that the dyadic $\boldsymbol{uv}$ has the matrix representation $\mathbf{u}\mathbf{v}^T$. Thus, the matrix representation of equation (A3) is

$$L_\tau = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}(\mathbf{v}_i)_\tau(\mathbf{v}_j)_\tau^T\,, \tag{A9}$$

and we have indicated the basis explicitly as a subscript on the matrix in the same way as the inertia tensor represented in inertial and body coordinates was indicated as $\boldsymbol{I}_i$ and $\boldsymbol{I}_b$, respectively.

If $\mathcal{E}$ is a right-handed orthonormal basis, then, for example, the antisymmetric matrix $[[\mathbf{u}]]$ corresponds to the linear operator,

$$[[\boldsymbol{u}]] \equiv \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}u_k\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j\,, \tag{A10}$$

where u_k, $k = 1,2,3$, are the components of $\boldsymbol{u}$ with respect to the orthonormal basis $\mathcal{E}$. The operator $[[\boldsymbol{u}]]$ generates abstract vector products in the same way as $[[\mathbf{u}]]$ generates the vector products of column vectors. This may be written as

$$[[\boldsymbol{u}]] = \boldsymbol{u}\cdot\mathcal{A} = \mathcal{A}\cdot\boldsymbol{u}\,, \tag{A11}$$

where $\mathcal{A}$ is the totally antisymmetric "triadic,"

$$\mathcal{A} \equiv \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j\hat{\boldsymbol{e}}_k\,. \tag{A12}$$

Equations (A11) and (A12) have the advantage of being coordinate independent.

We can define the vector products of dyadics and vectors according to

$$(\boldsymbol{ab})\times\boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b}\times\boldsymbol{c}) \quad\text{and}\quad \boldsymbol{c}\times(\boldsymbol{ab}) = (\boldsymbol{c}\times\boldsymbol{a})\boldsymbol{b}\,. \tag{A13}$$

(Note, however, $(\boldsymbol{ab})\times(\boldsymbol{cd})$ does not produce either a dyadic or a vector.)

By definition, the identity operator satisfies

$$\boldsymbol{1}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{1} \equiv \boldsymbol{v} \quad\text{and}\quad \boldsymbol{1}\cdot(\boldsymbol{uv}) = (\boldsymbol{uv})\cdot\boldsymbol{1} = (\boldsymbol{uv})\,. \tag{A14}$$

In terms of an orthonormal basis $\mathcal{E}$ we may write the identity dyadic $\boldsymbol{1}$ as

$$\mathbf{1} = \sum_{i=1}^{3} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_i , \tag{A15}$$

from which equation (A10) or (A11) may be rewritten as

$$[[\boldsymbol{u}]] = \boldsymbol{u} \times \mathbf{1} . \tag{A16}$$

The attitude matrix also has a corresponding abstract operator. From equation (420)

$$C^T = \sum_{i=1}^{3} (\hat{\mathbf{e}}_i')_{\mathcal{I}} (\hat{\mathbf{e}}_i)_{\mathcal{I}}^T , \tag{A17a}$$

or

$$C = \sum_{i=1}^{3} (\hat{\mathbf{e}}_i)_{\mathcal{I}} (\hat{\mathbf{e}}_i')_{\mathcal{I}}^T , \tag{A17b}$$

Hence, noting equation (A7a)

$$\mathcal{C} = \sum_{i=1}^{3} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_i' = \sum_{i=1}^{3} \sum_{j=1}^{3} C_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j = \sum_{i=1}^{3} \sum_{j=1}^{3} C_{ij} \hat{\mathbf{e}}_i' \hat{\mathbf{e}}_j' . \tag{A18}$$

It follows that

$$\mathcal{C} \cdot \hat{\mathbf{e}}_i' = \hat{\mathbf{e}}_i , \tag{A19}$$

so that the abstract attitude operator transforms the orthonormal basis of the final system into the orthonormal basis of the initial system. The fact that the transformation in the dyadic description occurs in the opposite direction to what one might expect naively is due to the fact that dyadics are inherently "active" objects.

Following Gibbs [62] we may write the dyadic description of Euler's formula as

$$\mathcal{C} = \cos\theta \mathbf{1} + (1 - \cos\theta)\hat{\boldsymbol{n}}\hat{\boldsymbol{n}} + \sin\theta \hat{\boldsymbol{n}} \times \mathbf{1} \tag{A20a}$$

$$= \mathbf{1} + \sin\theta \hat{\boldsymbol{n}} \times \mathbf{1} + (1 - \cos\theta)(\hat{\boldsymbol{n}} \times \mathbf{1})^2 \tag{A20b}$$

Dyadic descriptions of abstract linear operators,[13] although cumbersome, have the advantage that they are independent of the choice of basis. Data, of course, must always be represented as matrices.

Appendix B: Vectrices

Many of the results connecting abstract vectors with column vectors, abstract linear operators with matrices, and the attitude itself can be neatly summarized in terms of vectrices [2, 147, 148]. If we consider the basis, $\mathcal{I} = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, then the *vectrix* of the basis $\mathcal{F}_{\mathcal{I}}$ is defined by

$$\mathcal{F}_{\mathcal{I}} \equiv \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{bmatrix} , \tag{B1}$$

[13]We write "dyadic description" rather than "dyadic representation," because in the conventions of this survey dyadics are also abstract objects.

that is, by a column vector whose elements are abstract vectors. The transpose of a vectrix, $\mathcal{F}_\mathcal{I}^T$, is defined as

$$\mathcal{F}_\mathcal{I}^T \equiv [\hat{e}_1 \quad \hat{e}_2 \quad \hat{e}_3], \tag{B2}$$

so that the matrix is transposed but no action is taken on the abstract vectors which fill the matrix. Clearly,

$$\mathcal{F}_\mathcal{I} \cdot \mathcal{F}_\mathcal{I}^T = I, \quad \text{and} \quad \mathcal{F}_\mathcal{I}^T \mathcal{F}_\mathcal{I} = I, \tag{B3}$$

$$\mathcal{F}_\mathcal{I} \times \mathcal{F}_\mathcal{I}^T = \begin{bmatrix} 0 & \hat{e}_3 & -\hat{e}_2 \\ -\hat{e}_3 & 0 & \hat{e}_1 \\ \hat{e}_2 & -\hat{e}_1 & 0 \end{bmatrix} \equiv [[\mathcal{F}_\mathcal{I}]], \tag{B4}$$

and

$$\mathcal{F}_\mathcal{I} \times \mathcal{F}_\mathcal{I}^T \cdot \mathcal{F}_\mathcal{I} = \mathbf{0}. \tag{B5}$$

where, like the dyadics, the scalar and vector product operators act between adjacent abstract vectors.

Abstract vectors may be related to vector representations by means of vectrices according to

$$v = \mathcal{F}_\mathcal{I}^T \mathbf{v} = \mathbf{v}^T \mathcal{F}_\mathcal{I}, \tag{B6}$$

$$\mathbf{v} = \mathcal{F}_\mathcal{I} \cdot v = v \cdot \mathcal{F}_\mathcal{I}. \tag{B7}$$

$$\mathbf{v}^T = \mathcal{F}_\mathcal{I}^T \cdot v = v \cdot \mathcal{F}_\mathcal{I}^T. \tag{B8}$$

Equations (B6–B8) together with equations (B3) and (B4) lead to the usual relations between the scalar and vector products of abstract vectors and column vectors.

Equations (A6) and (A7) relating abstract linear operators and 3×3 matrices take the following form in terms of vectrices:

$$\mathcal{M} = \mathcal{F}_\mathcal{I}^T M \mathcal{F}_\mathcal{I}, \tag{B9}$$

and

$$M = \mathcal{F}_\mathcal{I} \cdot \mathcal{M} \cdot \mathcal{F}_\mathcal{I}^T. \tag{B10}$$

The antisymmetric matrix $[[\mathbf{u}]]$ has the vectrix description

$$[[\mathbf{u}]] = u \cdot \mathcal{F}_\mathcal{I} \times \mathcal{F}_\mathcal{I}^T = \mathcal{F}_\mathcal{I} \times \mathcal{F}_\mathcal{I}^T \cdot u. \tag{B11}$$

The vectrix

$$\mathcal{G} \equiv (\mathcal{F}_\mathcal{I} \times \mathcal{F}_\mathcal{I}^T)\mathbf{u}, \tag{B12}$$

on the other hand, is a somewhat more curious beast which satisfies

$$\mathcal{G} \cdot v = \mathbf{u} \times \mathbf{v}, \tag{B13}$$

but

$$\mathcal{G}^T \mathbf{v} = -u \times v. \tag{B14}$$

In analogy with equation (A18) the attitude can be described by means of vectrices belonging to different bases. If two bases, $\mathcal{E}$ and $\mathcal{E}'$, are related as in equations (66), then their respective vectrices, $\mathcal{F}_{\mathcal{E}}$ and $\mathcal{F}_{\mathcal{E}'}$, are related by

$$\mathcal{F}_{\mathcal{E}'} = C\mathcal{F}_{\mathcal{E}} \quad \text{and} \quad \mathcal{F}_{\mathcal{E}} = \mathcal{C} \cdot \mathcal{F}_{\mathcal{E}'}, \tag{B15}$$

whence

$$C = \mathcal{F}_{\mathcal{E}'} \cdot \mathcal{F}_{\mathcal{E}}^{T}, \quad \text{and} \quad \mathcal{C} = \mathcal{F}_{\mathcal{E}}^{T}\mathcal{F}_{\mathcal{E}'}. \tag{B16}$$

Appendix C: Attitude Representations in Higher Dimensions

Many of the results presented here are applicable to the representation of proper orthogonal transformations in higher dimensions. In n dimensions the rotation vector is replaced by an $n \times n$ antisymmetric matrix Θ. In terms of this quantity the $n \times n$ rotation matrix, R, can be written as

$$\mathsf{R} = \exp\{\Theta\}. \tag{C1}$$

Likewise, we define an $n \times n$ generalization of the Rodrigues-Gibbs vector G, [100, 149–151], so that

$$\mathsf{R} = (I + \mathsf{G})(I - \mathsf{G})^{-1}. \tag{C2}$$

The $n \times n$ Rodrigues-Gibbs matrix G is related to the $n \times n$ rotation-angle matrix Θ by

$$\mathsf{G} = \tanh(\Theta/2), \tag{C3}$$

and to the $n \times n$ rotation matrix R by

$$\mathsf{G} = (\mathsf{R} - I)(\mathsf{R} + I)^{-1}. \tag{C4}$$

The kinematic equations have straightforward generalizations. Equation (233) generalizes to

$$\frac{d}{dt}\mathsf{R}(t) = \Omega(t)\mathsf{R}(t), \tag{C5}$$

where $\Omega(t)$ is an $n \times n$ antisymmetric matrix (which no longer has a corresponding n-dimensional vector associated with it).

The kinematic equation for the Rodrigues-Gibbs matrix in n dimensions, which follows directly from equations (C4) and (C5), is

$$\frac{d}{dt}\mathsf{G}(t) = \frac{1}{2}(I - \mathsf{G}(t))\Omega(t)(I - \mathsf{G}(t))^{T}, \tag{C6}$$

which is the higher-dimensional analogue of equation (331).

The $n \times n$ attitude matrix possesses n^2 elements which are subject to $n(n+1)/2$ constraints. The $n(n-1)/2$ upper-triangular elements of the Rodrigues-Gibbs matrix, on the other hand, provide an unconstrained representation for the description of the attitude. For $n = 3$, these reduce (within a sign in the second-component) to the usual Rodrigues-Gibbs vector.

The quaternion, apparently, does not have a generalization in higher dimensions. In fact, it was shown by Hurwitz [152] that hypercomplex numbers (including the complex numbers) can only have dimension 2 (the complex numbers), dimension 4 (the quaternion), or dimension 8 (the octonion or Cayley numbers, apparently discovered independently by Graves [153] and Cayley [154]). It is well known that quaternion multiplication is not commutative. Octonion multiplication is not even associative. Hexadeconians and beyond do not even preserve the norm under multiplication. Reference [155] contains a complete discussion of this subject. A modern treatment of octonions can be found in Chevalley [156]. The use of complex quaternions to model rotations in four dimensions, which has found applications in special relatively [86–88], also has precursors in the work of Cayley [157].

Euler angles can be generalized to higher dimensions, and an example was provided by Euler [36, 117], but the number of different possible sets quickly becomes astronomical as the dimension increases. In four dimensions, for example, if it is noted that the group SO(4) is isomorphic to the group SO(3) $\otimes$ SO(3), it can be shown from very simple arguments that there must be at least 2880 different possible sets. In five dimensions there must be many times more than 30 million!

The problem of determining the attitude in higher dimensions from the measurement of unit vectors offers interesting insights into an algorithm used frequently in three dimensions [158]. The attitude representations in spaces of dimension less than three have attracted much less attention for obvious reasons. Nonetheless, instructive single-axis results can be obtained [159, 160].

Appendix D: The Lorentz Transformation

Of particular interest in higher dimensions is the application of Cayley-Klein parameters and quaternions to the Special Theory of Relativity [1, 17, 85–88]. Although quanternions cannot be generalized to have more than four components, with only four components they are able to parameterize a class of rotations in four dimensions, namely, the Lorentz transformations.

In the relativistic description of the universe the position column vector $\mathbf{x}$ is generalized to x, a *Minkowski four-vector* [17],

$$\mathsf{x} \equiv \begin{bmatrix} \mathbf{x} \\ ict \end{bmatrix} \equiv \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \tag{D1}$$

where c is the velocity of light, and $\mathrm{i} \equiv \sqrt{-1}$. The *Lorentz transformation*,

$$\mathsf{x}' = \Lambda \mathsf{x}, \tag{D2}$$

leaves the quadratic form

$$\mathsf{x}^T \mathsf{x} = \mathbf{x} \cdot \mathbf{x} - c^2 t^2, \tag{D3}$$

invariant. Thus, Λ satisfies

$$\Lambda^T \Lambda = I_{4\times 4}. \tag{D4}$$

The *Lorentz quaternion* $\bar{\lambda}$ is a complex 4×1 matrix, $\bar{\lambda} = [\lambda_1, \lambda_2, \lambda_3, \lambda_4]^T$, satisfying

$$\bar{\lambda}^T \bar{\lambda} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1, \tag{D5}$$

but, since the $\lambda_i, i = 1,2,3,4$, are complex, the individual terms in the sum above are neither positive nor necessarily even real, nor are the $|\lambda_i|, i = 1,2,3,4$, necessarily bounded above by unity.

Equation (179) generalizes to

$$\Lambda(\bar{\lambda}) = \{\bar{\lambda}\}_L \{\bar{\lambda}^\ddagger\}_R, \tag{D6}$$

where $\bar{\lambda}^\ddagger$, the *conjugate Lorentz quaternion*, is defined as

$$\bar{\lambda}^\ddagger \equiv (\bar{\lambda}^{-1})^* = \begin{bmatrix} -\boldsymbol{\lambda}^* \\ \lambda_4^* \end{bmatrix}, \tag{D7}$$

where again the asterisk denotes complex conjugation. Since $\Lambda(\bar{\lambda})$ is independent of the phase of $\bar{\lambda}$, λ_4 can always be chosen to be real (or positive).

As an example, consider a *pure Lorentz transformation*, or *boost*, that is, a Lorentz transformation in which the two frames are moving with respect to one another but the coordinate axes are not rotated. The Lorentz quaternion is

$$\bar{\lambda} = \left[\mathrm{i} \sqrt{\frac{\gamma - 1}{2}} \hat{\boldsymbol{\beta}}^T \quad \sqrt{\frac{\gamma + 1}{2}} \right]^T, \tag{D8}$$

where

$$\boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c}, \quad \text{and} \quad \gamma \equiv \frac{1}{\sqrt{1 - |\boldsymbol{\beta}^2|}}, \tag{D9}$$

and $\mathbf{v}$ is the relative velocity between the two frames. The corresponding Lorentz transformation is

$$\Lambda = \begin{bmatrix} I_{3\times 3} + (\gamma - 1)\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}^T & \mathrm{i}\gamma\boldsymbol{\beta} \\ -\mathrm{i}\gamma\boldsymbol{\beta}^T & \gamma \end{bmatrix}. \tag{D10}$$

It is generally true that any Lorentz transformation may be decomposed as

$$\Lambda = \Lambda_R \Lambda_B, \quad \text{or} \quad \Lambda = \Lambda_B' \Lambda_R', \tag{D11}$$

where Λ_R and Λ_R' are pure rotations, and Λ_B and Λ_B' are pure boosts. Van Wijk [161] has shown how to compute this decomposition for Λ when it arises from the composition of two pure boosts.

In analogy to the angle of rotation in three dimensions one can define an equivalent "angle" for the Lorentz formation which is generally complex (and for a boost is pure imaginary),

$$\cos\theta = \gamma, \qquad \sin\theta = \mathrm{i}\gamma|\boldsymbol{\beta}|. \tag{D12}$$

It is often more convenient to work in terms of a real parameter ψ (not to be confused with the Euler angle) for a pure boost, such as

$$\cosh\psi = \gamma, \qquad \sinh\psi = \gamma|\boldsymbol{\beta}|, \tag{D13}$$

so that the Lorentz transformation for a rotation about the z-axis takes the form

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\psi & \mathrm{i}\sinh\psi \\ 0 & 0 & -\mathrm{i}\sinh\psi & \cosh\psi \end{bmatrix}, \tag{D14}$$

This formalism generalizes straightforwardly to the Cayley-Klein parameters. One now defines instead of $\mathbf{V}$ the "vector"

$$\mathbf{V} \equiv \nu_1\mathbf{i} + \nu_2\mathbf{j} + \nu_3\mathbf{k} + \nu_4\mathbf{1} = \mathrm{i}\begin{pmatrix} \nu_4 + \nu_3 & \nu_1 - \mathrm{i}\nu_2 \\ \nu_1 + \mathrm{i}\nu_2 & \nu_4 - \nu_3 \end{pmatrix}. \tag{D15}$$

The Cayley-Klein matrix for a pure boost is simply

$$L = \sqrt{\frac{\gamma+1}{2}}\,\mathbf{1} - \sqrt{\frac{\gamma-1}{2}}\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\beta}}), \tag{D16a}$$

$$= \cosh(\psi/2)\mathbf{1} - \sinh(\psi/2)\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\beta}}), \tag{D16b}$$

in complete analogy with the similar expressions for pure rotations. One must insist, however, that λ_4 be real. Otherwise the Cayley-Klein matrix for a general Lorentz transformation will not be unimodular. Note that L in this case, while unimodular, is no longer unitary. The larger group of restricted orthochronous Lorentz transformations[14] and spatial rotations in the guise of unimodular Cayley-Klein matrix is generally denoted by SL(2, C), the group of special linear transformations in two dimensions over the complex numbers [162, 163]. The group SU(2) is thus a subgroup of SL(2, C). Note also that just as the real quaternions or unimodular Cayley-Klein matrices generate only the proper orthogonal matrices and not the improper orthogonal matrices also, so also the complex quaternions or unimodular Cayley-Klein matrices generate only the restricted Lorentz transformations.

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