Lecture 5: continuity and discontinuities

Calculus I, section 10
September 19, 2023

For the past two weeks, we've talked about functions and then about limits. Now we're ready to combine the two and talk about continuity and the various ways it can fail.

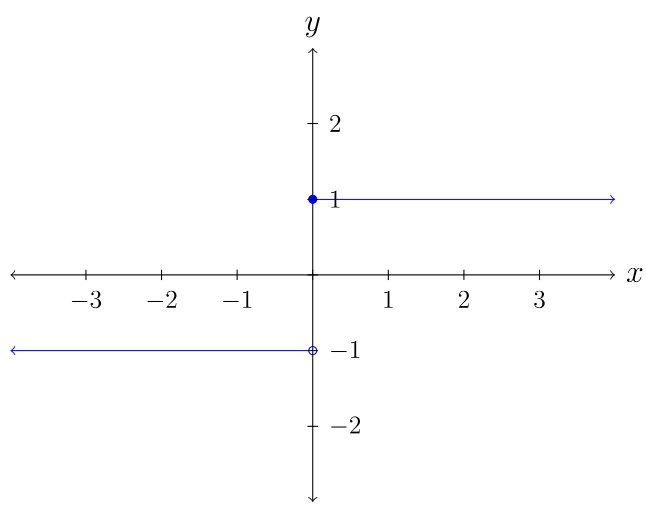
Given a "nice" function $f(x)$, such as $f(x) = x^3+2$, it's fairly straightforward to evaluate limits:

$$\lim_{x\to a} f(x) = \lim_{x\to a}(x^3+2) = a^3+2 = f(a).$$

To know that this is true, we're using some of the good properties of this particular function $f(x)$: for example, it's important that $f(x)$ be defined at a, which in this case is true for every a.

But this isn't the only property of $f(x)$ that we're using. For example, consider the function from last time:

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}.$$



This function exists at $x = 0$, where it has the value 1 since $0 \geq 0$. However, $\lim_{x\to 0} f(x)$ does not exist, since the limits from above and below are different.

So there's some additional property we need in order to make sure we can evaluate limits in this way. This property is called *continuity*.

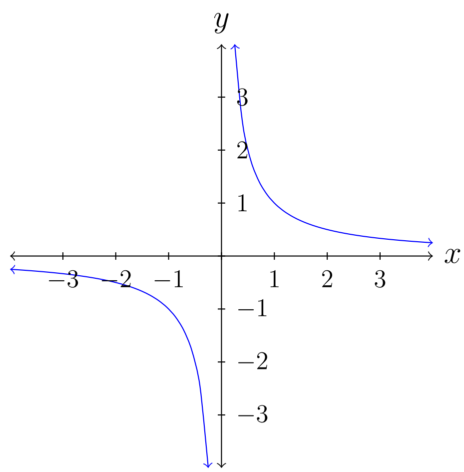
If you've seen continuity before, it might be clear to you why our first function is continuous and our second one isn't. However, the definition we'll use might actually make this less clear: since what we want is to be able to identify the limit of a function with one of its values, we simply make that our definition. A function $f(x)$ is *continuous* at some point a in its domain if

$$\lim_{x\to a} f(x) = f(a).$$

Thus our first example is continuous everywhere, since this is true for any a, and our second example is discontinuous at 0 (but continuous everywhere else). We say that a function is continuous if it is continuous everywhere in its domain.

Another common definition of continuity is something like "the graph can be drawn as a single line, without picking up the pen." We'll see that these two definitions don't precisely agree (the second one isn't completely well-defined, for one thing), but to get some intuition as to why these are roughly the same thing, think about what the limit condition means: as x gets closer and closer to a, $f(x)$ gets closer and closer to $f(a)$. This is what we'd expect to happen if the graph near a is "continuous" in the graphic sense, but not if, for example, there is a break or jump.

However, this new definition doesn't necessarily correlate with our intuition of which functions are continuous. For example, think about the function $f(x) = \frac{1}{x}$.



This clearly has a discontinuity at 0: the limit $\lim_{x\to 0} \frac{1}{x}$ does not exist, and neither does $f(0)$, so we certainly can't say that they're equal if they're not even defined. However, at every other point $f(x)$ is continuous, and since 0 is not in its domain this means $f(x)$ is continuous everywhere in its domain and therefore continuous![1]

In other words, a function can have a discontinuity and still be continuous. Counterintuitive as the terminology is, it's essentially our only choice, and it does address the problem we started with: if $f(a)$ doesn't exist, then we know for sure we need to put in more work to evaluate $\lim_{x\to a} f(x)$ than just plugging in a to f, so the question of continuity doesn't arise.

Let's talk about some kinds of functions, as we did in the first class, and see where some issues might arise.

First up: polynomials, i.e. anything which we can make by adding, subtracting, or multiplying constants together with x, such as $f(x) = x^3 - 2x + 3$. These will always be

[1] You might be tempted to say "Why not drop the 'in its domain' restriction then, and require a continuous function to be continuous everywhere?" The problem is that "everywhere" isn't well-defined, if we don't just mean its domain: we'd need to require that $f(x)$ be continuous at $x =$ an elephant, $x =$ Jupiter, etc., which doesn't make a lot of sense.

continuous, as you can check using the more formal definition of limits, or by staring at that statement for a while: if x changes by a small amount, x^3 is also going to change by a small amount, and similarly for other powers; and then we can combine these together.

This is basically using a version of the limit laws, this time for continuity: if $f(x)$ and $g(x)$ are continuous at a, then $f(x)+g(x)$, $f(x)-g(x)$, and $f(x)\cdot g(x)$ are all continuous at a. If in addition $g(a)\neq 0$, then $\frac{f(x)}{g(x)}$ is also continuous at a.

The division rule suggests adding division to polynomials, which gives rational functions. By the rules above, we already know that any rational function $\frac{f(x)}{g(x)}$ is continuous at a so long as $g(a)\neq 0$, since $f(x)$ and $g(x)$ are polynomials. The question is what happens when $g(a)=0$.

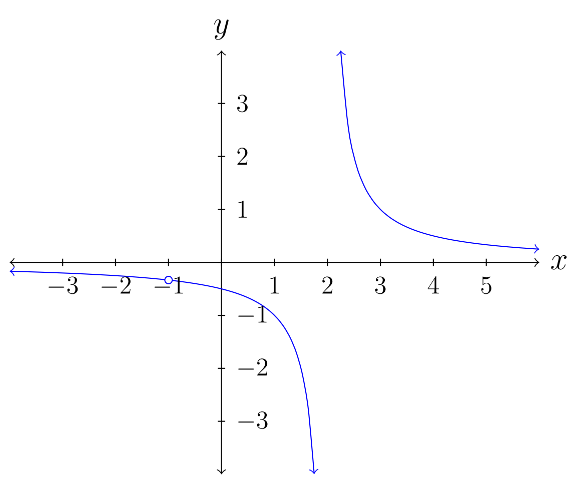
Well, on the one hand, in that case a is not in the domain, and so just as with $\frac{1}{x}$ our function can't be continuous there, but it doesn't matter for the purposes of being called continuous. Nevertheless, we'd still like to be able to say what the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ is. This is actually a kind of limit we're very practiced at: if $f(a)$ is also equal to 0, then we can try to do some cancellation and see what the simplified expression looks like, and if not then we get something of the form $\frac{c}{0}$ for some nonzero number c, which necessarily blows up.

For example, consider the function $f(x)=\frac{x+1}{x^2-x-2}$. The denominator factors as $(x+1)(x-2)$, so for $a\neq -1,2$ the function exists and is continuous at a. At $x=-1$, the numerator and the denominator are both zero, suggesting that we can do some cancellation; indeed, both have a factor of $x+1$, and so away from $x=-1$ our function is the same thing as $\frac{1}{x-2}$, which is continuous at $x=-1$. Therefore

$$\lim_{x\to -1} f(x) = \lim_{x\to -1}\frac{1}{x-2} = -\frac{1}{3}.$$

Thus although $f(x)$ does not exist at $x=-1$, we can reasonably assign it a value which would make it continuous at -1, namely $-\frac{1}{3}$.

At $x=2$ the story is different. Here, the denominator is zero, but the numerator is $2+1=3$, and so the limit approaches $\frac{3}{0}$ and therefore blows up, so there is no value we could give $f(2)$ that would make $f(x)$ continuous at $x=2$. We can confirm this by looking at the graph:



Thus, although $f(x)$ is discontinuous at both $x = -1$ and $x = 2$, the discontinuities are of different natures. The discontinuity at $x = -1$ is called *removable*, or sometimes a "hole discontinuity": there is a hole in the graph at $x = -1$, but we can reasonably fill it in to make the function continuous there (and thus remove the discontinuity). For this kind of discontinuity, the limit $\lim_{x\to a} f(x)$ exists, but either $f(a)$ is different from it or is not defined at all (as in this example).

At $x = 2$, we have an essential discontinuity: unlike the removable discontinuity, there is no way of "fixing" the function to be continuous at that point. Instead, the discontinuity shows that the limit $\lim_{x\to a} f(x)$ does not exist at that point, and indeed neither of the one-sided limits from above or below do either. (We could also have the case where one or the other, but not both, of the one-sided limits existed; this is still an essential discontinuity.)

(This specific kind of discontinuity, where the function blows up to infinity in both directions (though possibly one positive and one negative) is often called a pole, or a vertical asymptote. Essential discontinuities of all types are often referred to as "singularities": they're points at which the function is genuinely very badly-behaved. This is related to the terminology used in physics for black holes or the Big Bang: each one has a "singularity," in the center or at the beginning of time respectively, which makes all our mathematical descriptions break down. Concretely, these are just where the functions describing the behavior of the systems have singularities in this sense.)

The last type of discontinuity is that of our function

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

from above. Here, both the one-sided limits exist, but they are different, resulting in an apparent "jump" in the graph of the function: this is called a jump discontinuity. These will never occur in rational functions, but can occur in general, and are fairly common in piecewise functions like this.

Other kinds of functions we've discussed include exponential and trigonometric functions, as well as their inverses. Trigonometric functions are always continuous; proving this rigorously is more difficult than one might think, but you can convince yourself of it by looking at a triangle and seeing that if you change one angle by a small amount, the ratios of the side lengths should also only change by a small amount. Exponential functions are also continuous, essentially by definition: remember that we only defined exponential functions explicitly for rational numbers. For any real number, we can choose rational numbers getting closer and closer to it (for example by adding more and more digits to the decimal expansion), and then *define* the exponential to be the limit $b^a = \lim_{x\to a} b^x$ for x ranging over these rational numbers.[2]

The inverse functions are more complicated. Like rational functions, they'll be continuous whenever they're defined, essentially because trigonometric and exponential functions are, but we have to be careful since they are not always defined, and logarithms have a pole at $x = 0$.

[2] Some real analysis is necessary to see that these limits exist, which we won't get into.

Finally, we can make more complicated continuous functions from old ones: if $f(x)$ is continuous at a and $g(x)$ is continuous at $f(a)$, then $g(f(x))$ is continuous at a.

Let's think about some examples. Is

$$f(x) = 2^{\frac{\sin(x)}{x}}$$

continuous at $x = 1$? What about at $x = 0$?

Well, let's first look at the fraction inside: $\frac{\sin(x)}{x}$. Since $\sin(x)$ and x are both continuous everywhere, we just need to check that x is nonzero at 1, which it is. So $\frac{\sin(x)}{x}$ is continuous at 1, and since 2^x is continuous everywhere it follows that $2^{\frac{\sin(x)}{x}}$ is continuous at $x = 1$.

At $x = 0$ it's a different story. Here, the fraction $\frac{\sin(x)}{x}$ is not continuous because it doesn't exist, so $2^{\frac{\sin(x)}{x}}$ doesn't exist either. The question is then what kind of discontinuity this function has at $x = 0$. If $\lim_{x\to 0} 2^{\frac{\sin(x)}{x}}$ exists, then this will be a removable discontinuity; if the two one-sided limits both exist but are different, it's a jump discontinuity; and if at least one of them fails to exist, it's an essential discontinuity.

By an amazing coincidence, this is a kind of limit we know how to solve: we use limit laws! Specifically, we can use the function composition limit law: we're composing $\frac{\sin(x)}{x}$ and 2^x. First, we take the inner limit, which we computed last class:

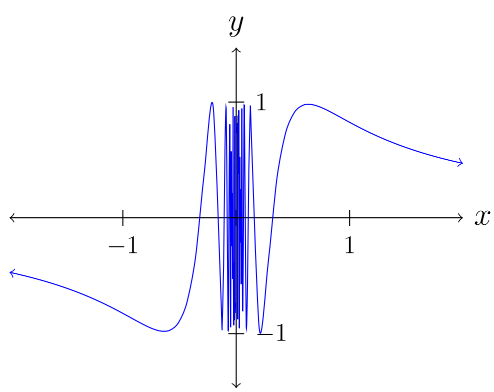
$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

Then by the limit law—and, implicitly, the fact that 2^x is continuous—we have

$$\lim_{x\to 0} 2^{\frac{\sin(x)}{x}} = \lim_{y\to 1} 2^y = 2.$$

Therefore the limit exists, and so this is a removable singularity: we can make $f(x) = 2^{\frac{\sin(x)}{x}}$ continuous at 0 by setting $f(0) = 2$.

Another example was one of our motivating problems for ways in which limits can fail to converge: $f(x) = \sin(\frac{1}{x})$.



For $x \neq 0$, this is a reasonably well-behaved function. At $x = 0$, we know it has a discontinuity of some kind: $f(0) = \sin(\frac{1}{0})$ doesn't exist. From the graph, we might guess that this is an essential discontinuity, which is true: the limit $\lim_{x\to 0} \sin(\frac{1}{x})$ doesn't exist, as we discussed. This is an example of a singularity which is not a pole, and doesn't even go to infinity.