An Artificial Life View to the Collatz Problem

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Extended Abstract

The Collatz problem, also known as the $3x + 1$ problem (Lagarias 1985), discusses the behavior of a series that starts with an arbitrary positive integer x_0 and develops according to the following rule:

$$x_{t+1} = \begin{cases} 3x_t + 1 & \text{if } x_t \text{ is odd} \\ x_t/2 & \text{if } x_t \text{ is even} \end{cases}$$

The Collatz conjecture asserts that this series always falls into a $4 \to 2 \to 1$ cycle regardless of x_0, which is believed to be true by many but has defied any formal proof for more than 70 years (Lagarias 2003; Lagarias 2006).

Here I propose a new perspective on the Collatz problem by considering it an ecological process of artificial organisms (1's in bit strings) and studying the spatio-temporal dynamics of their patterns. To make this approach easier, I ignore the second condition of the rule because it only right-shifts bit strings with no influence on their patterns. Ignoring it converts the series into a simpler iterative map with no ifs:

$$x_{t+1} = 3x_t + \mathrm{LSNB}(x_t)$$

Here $\mathrm{LSNB}(x)$ is the Least Significant Nonzero Bit of x (e.g., $\mathrm{LSNB}(172) = \mathrm{LSNB}(\mathit{10101100}) = \mathit{100} = 4$; *italics* are binary representations).

The above formula can be interpreted in ecological terms. A bit string of x_t represents the population distribution at time t, where 1's are living organisms and 0's are empty sites. $3x_t$ represents the replication of those organisms because it literally replicates each single bit (Fig. 1(a)). This causes leftward growth of the bit string as well as overcrowding of bits whose effects propagate leftward, depending on the carry rule. Also, $\mathrm{LSNB}(x_t)$ represents an external perturbation continuously introduced to the population, which causes extinction of the living organisms residing at the rightmost end, making the non-zero region of the bit string shrink from the right (Fig. 1(b)).

These interpretations suggest that the Collatz problem is about a competition between growth and extinction of the non-zero region in their speeds (Fig. 1(c)). The maximal speed of the leftward growth of the non-zero region is 2 bits/step, which can be sustained only if the population consists of a single 1, while its average speed is approximately $\log_2 3 \approx 1.58$ bits/step. In the meantime, there is no maximum regarding the speed of extinction of the non-zero region from the right. Assuming the equal probability of 0's and 1's in bit patterns, the average speed of extinction is analytically calculated to be 2 bits/step, which was confirmed by computer simulations. This indicates that the extinction from the right is "faster" than the population growth to the left, providing an ecological explanation of why the series always fall into a single-bit cycle.

Note that the above argument is still not a rigorous proof because it assumes stochasticity in bit patterns. The Artificial Life community could also contribute to this problem by attempting to design counter-examples to the conjecture. It may be possible to create, or even evolve, specific bit patterns that are able to "slow down" the extinction by continuously producing "barriers", which might be possible with very large initial conditions.

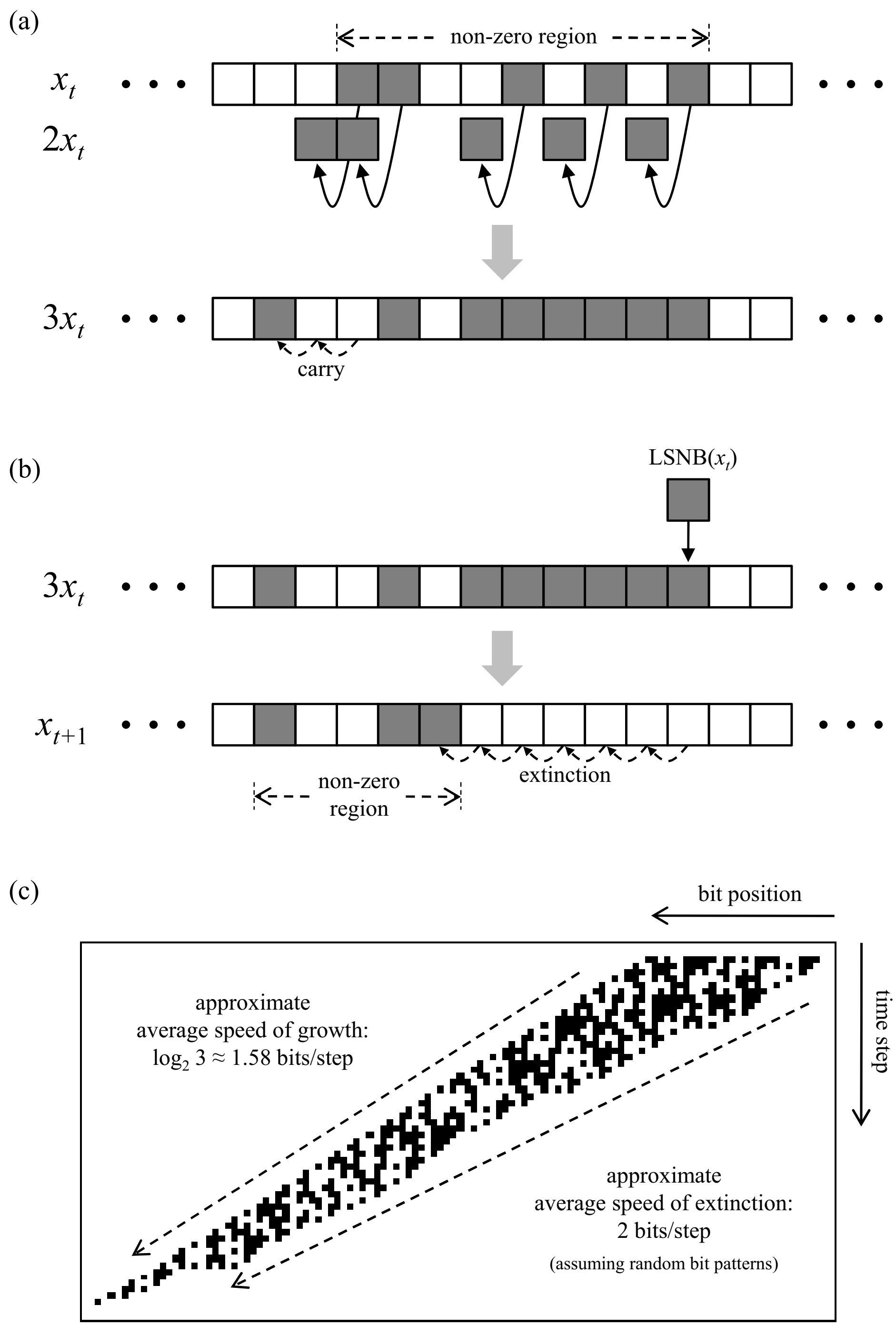

Figure 1: The Collatz problem as an ecological process of artificial organisms represented in bit strings. (a) Replication of 1's (gray cells) and growth of the non-zero region caused by $3x_t$. (b) Extinction from the right caused by LSNB(x_t). (c) Spatio-temporal dynamics of a sample series ($x_0 = 111111111$) visualized as bit patterns.

References

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