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Geometric properties of generalized Struve functions

Halit Orhan · Nihat Yagmur

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Abstract In the present work our object is to establish some geometric properties (like univalence, starlikeness, convexity and close-to-convexity) for the generalized Struve functions. In order to prove our main results, we use the technique of differential subordinations developed by Miller and Mocanu, some inequalities, and some classical results of Ozaki and Fejer.

Keywords analytic · univalent · starlike · convex · close-to-convex functions · Struve and modified Struve functions

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1 Introduction and preliminary results

It is well known that the special functions (series) play an important role in geometric function theory, especially in the solution by de Branges of the famous Bieberbach conjecture. The surprising use of special functions (hypergeometric functions) has prompted renewed interest in function theory in the last few decades. There is an extensive literature dealing with geometric properties of different types of special functions, especially for generalized, Gaussian, Kummer hypergeometric functions and Bessel functions. Many authors have determined sufficient conditions on the parameters of these functions for belonging to a certain class of univalent functions, such as convex, starlike, close-to-convex, etc. Someone can find more information about

Halit Orhan
Department of Mathematics,
Faculty of Science,
Atatürk University,
Erzurum, 25240, Turkey
E-mail: horhan@atauni.edu.tr

Nihat Yagmur
Department of Mathematics,
Faculty of Science and Art,
Erzincan University,
Erzincan, 24100, Turkey
E-mail: nhtyagmur@gmail.com

geometric properties of special functions in [2–4, 11, 15–18, 14]. In this present investigation our goal is to determine conditions of univalence, starlikeness, convexity and close-to-convexity of generalized Struve functions. In order to achieve our goal in this section, we recall some basic facts and preliminary results.

Let $\mathcal{A}$ denote the class of functions f normalized by

$$f(z) = z + \sum_{n\geq 2} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ which are univalent in $\mathcal{U}$. Also let $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$ and $\mathcal{K}(\alpha)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike, convex and close-to-convex of order α in $\mathcal{U}$ $(0 \leq \alpha < 1)$. Thus we have (see, for details, [5]),

$$\mathcal{S}^*(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, (z \in \mathcal{U};\ 0 \leq \alpha < 1) \right\}, \tag{1.2}$$

$$\mathcal{C}(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, (z \in \mathcal{U};\ 0 \leq \alpha < 1) \right\}, \tag{1.3}$$

$$\mathcal{K}(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{f'(z)}{g'(z)}\right) > \alpha, (z \in \mathcal{U}; 0 \leq \alpha < 1; g \in \mathcal{C}) \right\}, \tag{1.4}$$

where, for convenience,

$$\mathcal{S}^*(0) = \mathcal{S}^*, \quad \mathcal{C}(0) = \mathcal{C}, \quad \text{and} \quad \mathcal{K}(0) = \mathcal{K}. \tag{1.5}$$

We remark that, according to the Alexander duality theorem (see [1]) the function $f : \mathcal{U} \longrightarrow \mathbb{C}$ is convex of order α, where $0 \leq \alpha < 1$ if and only if $z \longrightarrow zf'(z)$ is starlike of order α. We note that every starlike (and hence convex) function of the form (1.1) is in fact close-to-convex, and every close-to-convex function is univalent. However, if a function is starlike then it is not necessary that it will be close-to-convex with respect to a particular convex function. For more details we refer the interested in the papers [5], [7], [13] and the references therein.

Lemma 1.1 ([10]) *Let $\mathbb{E}$ be a set in the complex plane $\mathbb{C}$ and $\psi : \mathbb{C}^3 \times \mathcal{U} \longrightarrow \mathbb{C}$ a function, that satisfies the admissibility condition $\psi(\rho i, \sigma, \mu + vi; z) \notin \mathbb{E}$, where $z \in \mathcal{U}$, $\rho, \sigma, \mu, v \in \mathbb{R}$ with $\mu + \sigma \leq 0$ and $\sigma \leq -(1+\rho^2)/2$. If $h : \mathcal{U} \longrightarrow \mathbb{C}$, which satisfies $h(0) = 1$, is analytic and for all $z \in \mathcal{U}$ we have $\psi(h(z), zh'(z), z^2h''(z); z) \in \mathbb{E}$, then $\Re\{h(z)\} > 0$ for all $z \in \mathcal{U}$. In particular, if we only have $\psi : \mathbb{C}^2 \times \mathcal{U} \longrightarrow \mathbb{C}$ the admissibility condition reduces to $\psi(\rho i, \sigma; z) \notin \mathbb{E}$, for all $z \in \mathcal{U}$ and $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1+\rho^2)/2$.*

Lemma 1.2 ([13]) *If the function $f(z) = z + a_2z^2 + ... + a_nz^n + ...$ is analytic in $\mathcal{U}$ and in addition $1 \geq 2a_2 \geq ... \geq na_n \geq ... \geq 0$ or $1 \leq 2a_2 \leq ... \leq na_n \leq ... \leq 2$, then f is close-to-convex with respect to the convex function $z \longrightarrow -\log(1-z)$. Moreover, if the odd function $g(z) = z + b_3z^3 + ... + b_{2n-1}z^{2n-1} + ...$ is analytic in $\mathcal{U}$ and if $1 \geq 3b_3 \geq ... \geq (2n+1)b_{2n+1} \geq ... \geq 0$ or $1 \leq 3b_3 \leq ... \leq (2n+1)b_{2n+1} \leq ... \leq 2$, then g is univalent in $\mathcal{U}$.*

Lemma 1.3 **([6])** *If the function* $f(z) = a_1 z + a_2 z^2 + ... + a_n z^n + ...,$ *where* $a_1 = 1$ *and* $a_n \geq 0$ *for all* $n \geq 2$, *is analytic in* $\mathcal{U}$, *and if the sequences* $\{na_n\}_{n\geq 1}$, $\{na_n - (n+1)a_{n+1}\}_{n\geq 1}$ *both are decreasing, then* f *is starlike in* $\mathcal{U}$. *Moreover, if for the analytic function* $g(z) = b_1 + b_2 z + ... + b_{n+1} z^n + ...,$ *where* $b_1 = 1$ *and* $b_n \geq 0$ *for all* $n \geq 2$, *we have that* $\{b_n\}_{n\geq 1}$, *is a convex decreasing sequence, i.e.,* $b_n - 2b_{n+1} + b_{n+2} \geq 0$ *and* $b_n - b_{n+1} \geq 0$ *for all* $n \geq 1$, *then* $\Re\{g(z)\} > 1/2$ *for all* $z \in \mathcal{U}$.

Theorem 1.4 **([19])** *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M, \tag{1.6}$$

where M *is the solution of the equation* $\cos M = M$, *then* $\Re\{f'(z)\} > 0$.

Theorem 1.5 **([8])** *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\frac{f(z)}{z} - 1\right| < 1, \qquad (z \in \mathcal{U}), \tag{1.7}$$

then $f(z)$ *is univalent and starlike for* $|z| < \frac{1}{2}$.

Theorem 1.6 **([9])** *If* $f \in \mathcal{A}$ *satisfies*

$$\left|f'(z) - 1\right| < 1, \qquad (z \in \mathcal{U}), \tag{1.8}$$

then $f(z)$ *is convex for* $|z| < \frac{1}{2}$.

Theorem 1.7 **([12])** *If* $f \in \mathcal{A}$ *satisfies the inequality*

$$\left|zf''(z)\right| < \frac{1-\alpha}{4}, \qquad (z \in \mathcal{U};\ 0 \leq \alpha < 1), \tag{1.9}$$

then

$$\Re\left\{f'(z)\right\} > \frac{1+\alpha}{4}, \qquad (z \in \mathcal{U};\ 0 \leq \alpha < 1).$$

2 Univalence, convexity and starlikeness of generalized Struve functions

Let us consider the second-order inhomogeneous differential equation ([20], p.341)

$$z^2 w''(z) + zw'(z) + (z^2 - p^2)w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\,\Gamma(p+\frac{1}{2})} \tag{2.1}$$

whose homogeneous part is Bessel's equation, where p is an unrestricted real (or complex) number. The function H_p, which is called the Struve function of order p, is defined as a particular solution of (2.1). This function has the form

$$H_p(z) = \sum_{n\geq 0} \frac{(-1)^n}{\Gamma(n+3/2)\,\Gamma(p+n+3/2)} \left(\frac{z}{2}\right)^{2n+p+1}, \text{ for all } z \in \mathbb{C}. \tag{2.2}$$

The differential equation

$$z^2w''(z) + zw'(z) - (z^2 + p^2)w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma\left(p + 1/2\right)}, \tag{2.3}$$

which differs from (2.1) only in the coefficient of w. The particular solution of (2.3) is called the modified Struve function of order p, and is defined by the formula ([20], p.353)

$$\begin{aligned} L_p(z) &= -ie^{-ip\pi/2}H_p(iz) \\ &= \sum_{n\geq 0} \frac{1}{\Gamma\left(n + 3/2\right)\Gamma\left(p + n + 3/2\right)} \left(\frac{z}{2}\right)^{2n+p+1}, \text{ for all } z \in \mathbb{C}. \end{aligned} \tag{2.4}$$

Now, let us consider the second-order inhomogeneous linear differential equation

$$z^2w''(z) + bzw'(z) + \left[cz^2 - p^2 + (1-b)p\right]w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma\left(p + b/2\right)}, \tag{2.5}$$

where $b, c, p \in \mathbb{C}$. If we choose $b = 1$, $c = 1$ then we get the equation (2.1) and if we choose $b = 1$, $c = -1$ then we get the equation (2.3). So this generalizes the equations (2.1) and (2.3). Moreover, this permits to study the Struve and modified Struve functions together. A particular solution of the differential equation (2.5), which is denoted by $w_p(z)$, is called the generalized Struve function of order p. In fact we have the following series representation for the function $w_p(z)$:

$$w_p(z) = \sum_{n\geq 0} \frac{(-1)^n c^n}{\Gamma\left(n + 3/2\right)\Gamma(p + n + \frac{b+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \text{ for all } z \in \mathbb{C}. \tag{2.6}$$

In the study of geometric properties of these generalized Struve functions an interesting method is the technique of differential subordinations, i.e. the application of Lemma 1.1. Thus, we would like to apply Lemma 1.1 for the analytic funtion $h : \mathcal{U} \longrightarrow \mathbb{C}$, defined by $h(z) = w_p(z)$ and for the function $\psi : \mathbb{C}^3 \times \mathcal{U} \longrightarrow \mathbb{C}$, defined by

$$\begin{aligned} \psi(h(z), zh'(z), z^2h''(z); z) &= z^2h''(z) + bzh'(z) \\ &\quad + \left[cz^2 - p^2 + (1-b)p\right]h(z) - \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p + \frac{b}{2})}, \end{aligned} \tag{2.7}$$

with $\mathbb{E} = \{0\}$. But we have that $w_p(0) = 0$, and therefore, we consider the transformation

$$u_p(z) = 2^p\sqrt{\pi}\Gamma(p + \frac{b+2}{2})z^{\frac{-p-1}{2}}w_p(\sqrt{z}) \tag{2.8}$$

to obtain $u_p(z) = b_0 + b_1z + b_2z^2 + ... + b_nz^n + ...$, where for all $n \geq 0$

$$b_n = \frac{(-1)^n c^n \Gamma\left(3/2\right)\Gamma(p + \frac{b+2}{2})}{4^n\Gamma\left(n + 3/2\right)\Gamma(p + n + \frac{b+2}{2})}. \tag{2.9}$$

Using the Pochhammer (or Appell) symbol, defined in terms of Euler's gamma functions, by $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda) = \lambda(\lambda+1)...(\lambda+n-1)$, we obtain for the function u_p the following form

$$u_p(z) = \sum_{n\geq 0} \frac{(-c/4)^n}{(3/2)_n\,(\kappa)_n} z^n, \tag{2.10}$$

where $\kappa = p + (b+2)/2 \neq 0, -1, -2, ...$ This function is analytic in $\mathbb{C}$, satisfies the condition $u_p(0) = 1$ and satisfies also the differential equation

$$4z^2 u''(z) + 2(2p+b+3)zu'(z) + (cz+2p+b)u(z) = 2p+b. \tag{2.11}$$

The next proposition will be applied for the study of the univalence of the function including u_p.

Proposition 2.1 *If $b, c, p \in \mathbb{C}$, $\kappa = p + (b+2)/2 \neq 1, 0, -1, -2, ...,$ and $z \in \mathbb{C}$, then for the generalized Struve function of order p the following recursive relations hold:*

(i) $zw_{p-1}(z) + czw_{p+1}(z) = (2\kappa - 3)w_p(z) + \frac{2(z/2)^{p+1}}{\sqrt{\pi}\Gamma(\kappa)}$;
(ii) $zw_p'(z) + (p+b-1)w_p(z) = zw_{p-1}(z)$;
(iii) $zw_p'(z) + czw_{p+1}(z) = pw_p(z) + \frac{2(z/2)^{p+1}}{\sqrt{\pi}\Gamma(\kappa)}$;
(iv) $[z^{-p}w_p(z)]' = -cz^{-p}w_{p+1}(z) + \frac{1}{2^p\sqrt{\pi}\Gamma(\kappa)}$;
(v) $u_p(z) + 2zu_p'(z) + \frac{cz}{2\kappa}u_{p+1}(z) = 1.$

Proof. (i) If we compute the expression $w_{p-1}(z) + w_{p+1}(z)$, then we have that

$$\begin{aligned}
w_{p-1}(z) + w_{p+1}(z) &= \sum_{n\geq 0} \frac{(-1)^n c^n}{\Gamma(n+3/2)\,\Gamma(p+n+\frac{b}{2})}\left(\frac{z}{2}\right)^{2n+p} \\
&+ \sum_{n\geq 0} \frac{(-1)^n c^n}{\Gamma(n+3/2)\,\Gamma(p+n+\frac{b+2}{2}+1)}\left(\frac{z}{2}\right)^{2n+p+2} \\
&= \frac{1}{\Gamma(3/2)\,\Gamma(\kappa-1)}\left(\frac{z}{2}\right)^{p} \\
&+ \sum_{n\geq 1}\left[\frac{(-1)^n c^n}{\Gamma(n+3/2)\,\Gamma(\kappa+n-1)} + \frac{(-1)^{n-1}c^{n-1}}{\Gamma(n+1/2)\,\Gamma(\kappa+n)}\right]\left(\frac{z}{2}\right)^{2n+p} \\
&= \frac{2\kappa-3}{z}\left[\frac{2}{\Gamma(3/2)\,\Gamma(\kappa-1)\,(2\kappa-3)}\left(\frac{z}{2}\right)^{p+1}\right. \\
&\left. + \sum_{n\geq 0} \frac{(-1)^n c^n}{\Gamma(n+3/2)\,\Gamma(\kappa+n)}\left(\frac{z}{2}\right)^{2n+p}\right] \\
&+ \sum_{n\geq 0} \frac{(-1)^{n+1} c^n (c-1)}{\Gamma(n+3/2)\,\Gamma(\kappa+n+1)}\left(\frac{z}{2}\right)^{2n+p+2},
\end{aligned}$$

where $\kappa = p + (b+2)/2$.

Consequently, we obtain that

$$w_{p-1}(z)+w_{p+1}(z)=\frac{2\kappa-3}{z}\left[\frac{1}{(2\kappa-3)\Gamma\left(3/2\right)\Gamma\left(\kappa\right)}\left(\frac{z}{2}\right)^{p+1}+w_p(z)\right]$$
$$+(1-c)w_{p+1}(z)$$

which implies that $zw_{p-1}(z)+czw_{p+1}(z)=(2\kappa-3)w_p(z)+\frac{2(z/2)^{p+1}}{\sqrt{\pi}\Gamma(\kappa)}$ holds, as we required.

(ii) Analogously, if we compute the expression $w_{p-1}(z)-w_{p+1}(z)$, then we have

$$\begin{aligned} w_{p-1}(z)-w_{p+1}(z) &= \sum_{n\geq 0}\frac{(-1)^n c^n}{\Gamma\left(n+3/2\right)\Gamma(p+n+\frac{b}{2})}\left(\frac{z}{2}\right)^{2n+p} \\ &-\sum_{n\geq 0}\frac{(-1)^n c^n}{\Gamma\left(n+3/2\right)\Gamma(p+n+\frac{b+2}{2}+1)}\left(\frac{z}{2}\right)^{2n+p+2} \\ &=\frac{1}{\Gamma(\frac{3}{2})\Gamma\left(\kappa-1\right)}\left(\frac{z}{2}\right)^{p} \\ &+\sum_{n\geq 1}\left[\frac{(-1)^n c^n\left(\kappa+n-1\right)+(-1)^{n-1}c^{n-1}(n+1/2)}{\Gamma\left(n+3/2\right)\Gamma\left(\kappa+n\right)}\right]\left(\frac{z}{2}\right)^{2n+p} \\ &=w_p'(z)+\frac{(p+b-1)}{z}w_p(z)-w_{p+1}(z) \end{aligned}$$

and thus we obtain the second recursive relation.

(iii) Combining the recursive relations (i) and (ii), we get that

$$zw_p'(z)+(p+b-1)w_p(z)+czw_{p+1}(z)=(2p+b-1)w_p(z)+\frac{2(z/2)^{p+1}}{\sqrt{\pi}\Gamma\left(\kappa\right)}$$

which implies that $zw_p'(z)+czw_{p+1}(z)=pw_p(z)+\frac{2(z/2)^{p+1}}{\sqrt{\pi}\Gamma(\kappa)}$.

(iv) Using the third recursive relation we obtain

$$\begin{aligned} \left[z^{-p}w_p(z)\right]' &= z^{-2p}\left[w_p'(z)z^p-pz^{p-1}w_p(z)\right] \\ &= z^{-p-1}\left[zw_p'(z)-pw_p(z)\right] \\ &= z^{-p-1}\left[-czw_{p+1}(z)+\frac{2(z/2)^{p+1}}{\sqrt{\pi}\Gamma\left(\kappa\right)}\right] \\ &= -cz^{-p}w_{p+1}(z)+\frac{1}{2^p\sqrt{\pi}\Gamma\left(\kappa\right)}. \end{aligned}$$

(v) For convenience, we use part (iv). Since from definition and from part (iv) we have $w_p(z)=\frac{z^{p+1}}{2^p\sqrt{\pi}\Gamma(\kappa)}u_p(z^2)$ and $[z^{-p}w_p(z)]'=-cz^{-p}w_{p+1}(z)+\frac{1}{2^p\sqrt{\pi}\Gamma(\kappa)}$, we get $[z^{-p}w_p(z)]'=\frac{1}{2^p\sqrt{\pi}\Gamma(\kappa)}[zu_p(z^2)]'$. We have $w_{p+1}(z)=\frac{z^{p+2}}{2^{p+1}\sqrt{\pi}\Gamma(\kappa+1)}u_{p+1}(z^2)$, so $\left[zu_p(z^2)\right]'=\frac{-cz^2}{2\kappa}u_{p+1}(z^2)+1$. Consequently, we get $u_p(z)+2zu_p'(z)=\frac{-cz}{2\kappa}u_{p+1}(z)+1$, thus the proof is complete. $\square$

The next result contains conditions for the function $g_p(z) = zu_p(z)$ to be univalent, convex and starlike in the unit disk.

Theorem 2.2 *If $b, c, p \in \mathbb{R}$, $\kappa = p + (b+2)/2$ then the functions u_p and g_p satisfy the following properties:*

(i) If $\kappa \geq \frac{5+\sqrt{1+2c^2}}{8}$, then $\Re\{u_p(z)\} > 0$ for all $z \in \mathcal{U}$;

(ii) If $\kappa > \frac{7M+2+\sqrt{M^2+12M+4}}{24M}|c|$, where M is the solution of the equation $\cos M = M$, then $\Re\{g_p'(z)\} > 0$ for all $z \in \mathcal{U}$ and hence $g_p(z)$ is univalent in $\mathcal{U}$;

(iii) If $\kappa > \frac{9+\sqrt{17}}{24}|c|$, then $g_p(z)$ is starlike in $\mathcal{U}$;

(iv) If $\kappa > \frac{13}{12}|c|$, then $g_p(z)$ is convex in $\mathcal{U}$;

(v) If $\kappa > \frac{1}{3}|c|$, then $g_p(z)$ is starlike for $|z| < \frac{1}{2}$;

(vi) If $\kappa > \frac{7}{12}|c|$, then $g_p(z)$ is convex for $|z| < \frac{1}{2}$.

Proof. (i) Clearly when $c = 0$ we have $u_p(z) \equiv 1$, thus $\Re\{u_p(z)\} > 0$ for all $z \in \mathcal{U}$. Now suppose that $\kappa \geq \frac{5+\sqrt{1+2c^2}}{8}$ and $c \neq 0$. Put $h = u_p$. Since h satisfies (2.11), we have

$$4z^2h''(z) + 2(2p+b+3)zh'(z) + (cz+2p+b)h(z) - 2p - b = 0. \tag{2.12}$$

If we let $\psi(r,s,t;z) = 4t + 2(2p+b+3)s + (cz+2p+b)r - 2p - b$ and $\mathbb{E} = \{0\}$, then (2.12) can be written as $\psi(h(z), zh'(z), z^2h''(z); z) \in \mathbb{E}$ for all $z \in \mathcal{U}$. Now we use Lemma 1.1 to prove that $\Re\{u_p(z)\} > 0$, for all $z \in \mathcal{U}$. If we put $z = x + iy$, where $x, y \in \mathbb{R}$, then $\Re\{\psi(\rho i, \sigma, \mu + vi; x+iy)\} = 4(\mu+\sigma) + 2(2\kappa-1)\sigma - c\rho y - 2(\kappa-1)$ for all $\rho, \sigma, \mu, v \in \mathbb{R}$. Let $\rho, \sigma, \mu, v \in \mathbb{R}$ satisfy $\mu + \sigma \leq 0$ and $\sigma \leq -(1+\rho^2)/2$. Since $\kappa > 1/2$, we have $\Re\{\psi(\rho i, \sigma, \mu+vi; x+iy)\} \leq -(2\kappa-1)\rho^2 - c\rho y - (4\kappa-3)$. Set $Q(p) = -(2\kappa-1)\rho^2 - c\rho y - (4\kappa-3)$. This value will be strictly negative for all real ρ, because the discriminant Δ of $Q(p)$ satisfies $\Delta = c^2y^2 - 4(2\kappa-1)(4\kappa-3) \leq 0$, whenever $y \in (-1,1)$.Consequently ψ satisfies the admissibility condition of Lemma 1.1. Hence by Lemma 1.1 we conclude that $\Re\{h(z)\} = \Re\{u_p(z)\} > 0$, for all $z \in \mathcal{U}$.

(ii) By using the well-known triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$ and the inequalities $\left(\frac{3}{2}\right)_n \geq \frac{3}{2}n$, $(\kappa)_n \geq \kappa^n$ $(n \in \mathbb{N})$, we have

$$\begin{aligned}\left|g_p'(z) - \frac{g_p(z)}{z}\right| &= \left|\sum_{n\geq 1} \frac{n(-c/4)^n}{(3/2)_n(\kappa)_n} z^n\right| \leq \sum_{n\geq 1} \frac{|-c/4|^n}{\frac{3}{2}\kappa^n} \\ &= \frac{2}{3}\frac{|c|}{4\kappa}\sum_{n\geq 1}\left(\frac{|c|}{4\kappa}\right)^{n-1} = \frac{2|c|}{3(4\kappa-|c|)}, \quad (\kappa > \frac{|c|}{4}).\end{aligned} \tag{2.13}$$

Furthermore, if we use reverse triangle inequality $|z_1 - z_2| \geq ||z_1| - |z_2||$ and the inequalities $\left(\frac{3}{2}\right)_n \geq \left(\frac{3}{2}\right)^n$, $(\kappa)_n \geq \kappa^n$ $(n \in \mathbb{N})$, then we get

$$\begin{aligned}\left|\frac{g_p(z)}{z}\right| &= \left|1 + \sum_{n\geq 1} \frac{(-c/4)^n}{(3/2)_n(\kappa)_n} z^n\right| \geq 1 - \sum_{n\geq 1}\left(\frac{|c|}{6\kappa}\right)^n \\ &= 1 - \frac{|c|}{6\kappa}\sum_{n\geq 1}\left(\frac{|c|}{6\kappa}\right)^{n-1} = \frac{6\kappa - 2|c|}{6\kappa - |c|}, \quad (\kappa > \frac{|c|}{6})\end{aligned} \tag{2.14}$$

which is positive. Next, by combining the inequalities (2.13) with (2.14), we immediately see that

$$\left|\frac{zg_p'(z)}{g_p(z)}-1\right| \leq \frac{|c|\,(6\kappa-|c|)}{3(4\kappa-|c|)(3\kappa-|c|)}. \tag{2.15}$$

So, for $\kappa > \frac{7M+2+\sqrt{M^2+12M+4}}{24M}\,|c|$, we obtain

$$\left|\frac{zg_p'(z)}{g_p(z)}-1\right| < M, \tag{2.16}$$

where M is the solution of the equation $\cos M = M$. From Theorem 1.4, we get $\Re\left\{g_p'(z)\right\} > 0$ for all $z \in \mathcal{U}$.

(iii) Suppose that $\kappa > \frac{9+\sqrt{17}}{24}\,|c|$, from the inequality (2.15), we have

$$\left|\frac{zg_p'(z)}{g_p(z)}-1\right| < 1 \tag{2.17}$$

which shows $g_p(z)$ is starlike in $\mathcal{U}$.

(iv) By using the well-known triangle inequality and the inequalities $(\frac{3}{2})_n > \frac{n(n+1)}{2}$, $(\kappa)_n \geq \kappa^n$ $(n \in \mathbb{N})$, we arrive at the following

$$\begin{aligned}\left|zg_p''(z)\right| &= \left|\sum_{n\geq 1}\frac{n(n+1)(-c/4)^n}{(3/2)_n\,(\kappa)_n}z^n\right| \leq 2\frac{|c|}{4\kappa}\sum_{n\geq 1}\left(\frac{|c|}{4\kappa}\right)^{n-1} \\ &= \frac{2\,|c|}{4\kappa-|c|}, \quad \left(\kappa > \frac{|c|}{4}\right).\end{aligned} \tag{2.18}$$

Furthermore, if we use reverse triangle inequality and the inequalities $\left(\frac{3}{2}\right)_n \geq \frac{3(n+1)}{4}$, $(\kappa)_n \geq \kappa^n$ $(n \in \mathbb{N})$, we have

$$\begin{aligned}\left|g_p'(z)\right| &= \left|1+\sum_{n\geq 1}\frac{(n+1)(-c/4)^n}{(3/2)_n\,(\kappa)_n}z^n\right| \geq 1-\frac{4}{3}\frac{|c|}{4\kappa}\sum_{n\geq 1}\left(\frac{|c|}{4\kappa}\right)^{n-1} \\ &= \frac{12\kappa-7\,|c|}{3(4\kappa-|c|)}, \quad \left(\kappa > \frac{|c|}{4}\right)\end{aligned} \tag{2.19}$$

which is positive. Next, by combining the inequalities (2.18) with (2.19), we immediately deduce that

$$\left|\frac{zg_p''(z)}{g_p'(z)}\right| \leq \frac{6\,|c|}{12\kappa-7\,|c|}. \tag{2.20}$$

So, for $\kappa > \frac{13}{12}\,|c|$ we have

$$\left|\frac{zg_p''(z)}{g_p'(z)}\right| < 1. \tag{2.21}$$

This shows $g_p(z)$ is convex in $\mathcal{U}$.

(v) Suppose that $\kappa > \frac{1}{3}|c|$, by using well-known triangle inequality and the inequalities $\left(\frac{3}{2}\right)_n \geq \left(\frac{3}{2}\right)^n$, $(\kappa)_n \geq \kappa^n \quad (n \in \mathbb{N})$, we get

$$\left|\frac{g_p(z)}{z} - 1\right| = \left|\sum_{n\geq 1} \frac{(-c/4)^n}{(3/2)_n (\kappa)_n} z^n\right| \leq \sum_{n\geq 1} \left(\frac{|c|}{6\kappa}\right)^n \tag{2.22}$$
$$= \frac{|c|}{6\kappa} \sum_{n\geq 1} \left(\frac{|c|}{6\kappa}\right)^{n-1} = \frac{|c|}{6\kappa - |c|} < 1.$$

So, from Theorem 1.5, $g_p(z)$ is starlike for $|z| < \frac{1}{2}$.

(vi) Suppose that $\kappa > \frac{7}{12}|c|$, by using well-known triangle inequality and the inequalities $\left(\frac{3}{2}\right)_n \geq \frac{3(n+1)}{4}$, $(\kappa)_n \geq \kappa^n \ (n \in \mathbb{N})$, we obtain

$$\left|g_p'(z)\right| = \left|\sum_{n\geq 1} \frac{(n+1)(-c/4)^n}{(3/2)_n (\kappa)_n} z^n\right| \leq \frac{4}{3}\frac{|c|}{4\kappa} \sum_{n\geq 1} \left(\frac{|c|}{4\kappa}\right)^{n-1} \tag{2.23}$$
$$= \frac{4|c|}{3(4\kappa - |c|)} < 1.$$

So, from Theorem 1.6, $g_p(z)$ is convex for $|z| < \frac{1}{2}$. $\square$

Struve functions. Choosing $b = c = 1$, we obtain the differential equation (2.1) and the Struve function of order p, defined by (2.2) satisfies this equation. In particular, the results of Theorem 2.2 become:

Corollary 2.3 *Let $\mathcal{H}_p : \mathcal{U} \longrightarrow \mathbb{C}$ be defined by*

$$\mathcal{H}_p(z) = 2^p \sqrt{\pi} \Gamma\left(p + \frac{3}{2}\right) z^{-p-1} H_p(z).$$

Then the following assertions are true:

(i) If $p \geq \frac{-7+\sqrt{3}}{8}$, then $\Re\left[\mathcal{H}_p\left(z^{1/2}\right)\right] > 0$ for all $z \in \mathcal{U}$;
(ii) If $p > \frac{-29M+\sqrt{M^2+12M+4}}{24M}$, where M is the solution of the equation $\cos M = M$, then $\Re\left[z\mathcal{H}_p\left(z^{1/2}\right)\right]' > 0$, for all $z \in \mathcal{U}$ and hence $\mathcal{H}_p\left(z^{1/2}\right)$ is univalent in $\mathcal{U}$;
(iii) If $p > \frac{-27+\sqrt{17}}{24}$, then $z\mathcal{H}_p\left(z^{1/2}\right)$ is starlike in $\mathcal{U}$;
(iv) If $p > -\frac{5}{12}$, then $z\mathcal{H}_p\left(z^{1/2}\right)$ is convex in $\mathcal{U}$;
(v) If $p > -\frac{7}{6}$, then $z\mathcal{H}_p\left(z^{1/2}\right)$ is starlike for $|z| < \frac{1}{2}$;
(vi) If $p > -\frac{11}{12}$, then $z\mathcal{H}_p\left(z^{1/2}\right)$ is convex for $|z| < \frac{1}{2}$.

Modified Struve functions. Choosing $b = 1$, $c = -1$, we obtain the differential equation (2.3) and the modified Struve function of order p, defined by (2.4). For the function $\mathcal{L}_p : \mathcal{U} \longrightarrow \mathbb{C}$ be defined by $\mathcal{L}_p(z) = 2^p \sqrt{\pi} \Gamma\left(p + \frac{3}{2}\right) z^{-p-1} L_p(z)$, the properties are same like for function $\mathcal{H}_p$, because we have $|c| = 1$. More precisely, we have the following results.

Corollary 2.4 *The following assertions are true:*

(*i*) *If* $p \geq \frac{-7+\sqrt{3}}{8}$, *then* $\Re\left[\mathcal{L}_p\left(z^{1/2}\right)\right] > 0$ *for all* $z \in \mathcal{U}$;

(*ii*) *If* $p > \frac{-29M+\sqrt{M^2+12M+4}}{24M}$, *where* M *is the solution of the equation* $\cos M = M$, *then* $\Re\left[z\mathcal{L}_p\left(z^{1/2}\right)\right]' > 0$ *for all* $z \in \mathcal{U}$ *and hence* $\mathcal{L}_p\left(z^{1/2}\right)$ *is univalent in* $\mathcal{U}$;

(*iii*) *If* $p > \frac{-27+\sqrt{17}}{24}$, *then* $z\mathcal{L}_p\left(z^{1/2}\right)$ *is starlike in* $\mathcal{U}$;

(*iv*) *If* $p > -\frac{5}{12}$, *then* $z\mathcal{L}_p\left(z^{1/2}\right)$ *is convex in* $\mathcal{U}$;

(*v*) *If* $p > -\frac{7}{6}$, *then* $z\mathcal{L}_p\left(z^{1/2}\right)$ *is starlike for* $|z| < \frac{1}{2}$;

(*vi*) *If* $p > -\frac{11}{12}$, *then* $z\mathcal{L}_p\left(z^{1/2}\right)$ *is convex for* $|z| < \frac{1}{2}$.

Example 2.1 For $p = -\frac{1}{2} > \frac{-7+\sqrt{3}}{8}$, we obtain

$$\mathcal{H}_{-1/2}\left(z^{1/2}\right) = 2^{-1/2}\sqrt{\pi}z^{-1/4}H_{-1/2}(z^{1/2}) = \frac{\sin\sqrt{z}}{\sqrt{z}}.$$

From part (i) of Corollary 2.3, we have $\Re\left[\mathcal{H}_{-1/2}\left(z^{1/2}\right)\right] = \Re\left[\frac{\sin\sqrt{z}}{\sqrt{z}}\right] > 0.$

3 Convexity and starlikeness of order $\alpha$of the generalized Struve functions

The following results contain conditions for the functions u_p, g_p and w_p to be convex and starlike of order α in the unit disk.

Theorem 3.1 *If* $b, c, p \in \mathbb{R}$, $\kappa = p + (b+2)/2$ *and* $\alpha \in [0, 2-\sqrt{2})$ *then* u_p *satisfy the following property:*

If $\kappa \geq \frac{3\alpha^2-10\alpha+5+(1-\alpha)\sqrt{(1-\alpha)^2+c^2(\alpha^2-4\alpha+2)}}{4(\alpha^2-4\alpha+2)}$, *then* $\Re\{u_p(z)\} > \alpha$, *for all* $z \in \mathcal{U}$.

Proof. First suppose that $c = 0$ we have $u_p(z) \equiv 1$, thus $\Re\{u_p(z)\} > \alpha$ for all $z \in \mathcal{U}$. Now suppose that $\kappa \geq \frac{3\alpha^2-10\alpha+5+(1-\alpha)\sqrt{(1-\alpha)^2+c^2(\alpha^2-4\alpha+2)}}{4(\alpha^2-4\alpha+2)}$ and $c \neq 0$. Define the function $h : \mathcal{U} \longrightarrow \mathbb{C}$ by $h(z) = \frac{u_p(z)-\alpha}{1-\alpha}$. Since u_p satisfies (2.11), h will satisfy the following differential equation:

$$\begin{aligned} &4z^2h''(z) + 2(2p+b+3)zh'(z) \\ &+ (cz+2p+b)\left(h(z) + \frac{\alpha}{1-\alpha}\right) - \left(\frac{2p+b}{1-\alpha}\right) = 0. \end{aligned} \tag{3.1}$$

If we use $\psi(r,s,t;z) = 4t+2(2p+b+3)s+(cz+2p+b)(r+\frac{\alpha}{1-\alpha})-(\frac{2p+b}{1-\alpha})$ and $\mathbb{E} = \{0\}$, we see that equation (3.1) implies $\psi(h(z), zh'(z), z^2h''(z); z) \in \mathbb{E}$ for all $z \in \mathcal{U}$. Now we use Lemma 1.1 to prove that $\Re\{u_p(z)\} > 0$ for all $z \in \mathcal{U}$. If we put $z = x+iy$, where $x, y \in \mathbb{R}$, then $\Re\{\psi(\rho i, \sigma, \mu+vi; x+iy)\} = 4(\mu+\sigma)+2(2\kappa-1)\sigma-c\rho y+(cx+2\kappa-2)\frac{\alpha}{1-\alpha}-\frac{2(\kappa-1)}{1-\alpha}$, for all $\rho, \sigma, \mu, v \in \mathbb{R}$. Let $\rho, \sigma, \mu, v \in \mathbb{R}$ satisfy $\mu + \sigma \leq 0$ and $\sigma \leq -(1+\rho^2)/2$. Since $\kappa > 1/2$, we have $\Re\{\psi(\rho i, \sigma, \mu+vi; x+iy)\} \leq -(2\kappa-1)\rho^2 - c\rho y - (2\kappa-1) + (cx+2\kappa-2)\frac{\alpha}{1-\alpha} - \frac{2(\kappa-1)}{1-\alpha}$. Set $Q_1(p) = -(2\kappa-1)\rho^2 - c\rho y - (2\kappa-1) + (cx+2\kappa-2)\frac{\alpha}{1-\alpha} - \frac{2(\kappa-1)}{1-\alpha}$.

This value will be strictly negative for all real ρ, because the discriminant Δ_1 of $Q_1(p)$ satisfies

$$\begin{aligned}\Delta_1 &= c^2y^2 + 4(2\kappa-1)\left(-(2\kappa-1)+(cx+2\kappa-2)\frac{\alpha}{1-\alpha}-\frac{2(\kappa-1)}{1-\alpha}\right)\\ &< c^2(1-x^2)-4(2\kappa-1)^2+4(2\kappa-1)(cx+2\kappa-2)\frac{\alpha}{1-\alpha}\\ &-\frac{8(2\kappa-1)(\kappa-1)}{1-\alpha} =: Q_1(x)\le 0,\end{aligned}$$

whenever $x^2+y^2<1$ and the discriminant Δ_2 of $Q_2(x)$ is negative satisfies Δ_2 has the form $\Delta_2 = 4c^2[4(2\kappa-1)^2\frac{\alpha^2}{(1-\alpha)^2} - 4(2\kappa-1)(4\kappa-3)+c^2]$ and this is negative if and only if we have $\kappa \ge \frac{3\alpha^2-10\alpha+5+(1-\alpha)\sqrt{(1-\alpha)^2+c^2(\alpha^2-4\alpha+2)}}{4(\alpha^2-4\alpha+2)}$. Hence by Lemma 1.1 we conclude that $\Re\{h(z)\} = \Re[\frac{1}{1-\alpha}(u_p(z)-\alpha)] > 0$, for all $z\in\mathcal{U}$, and this implies that $\Re\{u_p(z)\} > \alpha$ for all $z\in\mathcal{U}$, as we required. □

Remark 3.1 If we choose $\alpha = 0$ in Theorem 3.1 then we get part (i) of Theorem 2.2.

Theorem 3.2 *If $b, c, p \in \mathbb{R}$, $\kappa = p + (b+2)/2$ and $\alpha \in [0,1)$ then g_p and w_p have the following properties:*

(i) If $\kappa > \frac{9-\alpha+\sqrt{\alpha^2-14\alpha+7}}{24(1-\alpha)}|c|$ then $g_p(z)\in\mathcal{S}^*(\alpha)$;

(ii) If $\kappa > \frac{13-7\alpha}{12(1-\alpha)}|c|$ then $g_p(z)\in\mathcal{C}(\alpha)$;

(iii) If $\kappa > \frac{9-\alpha+\sqrt{\alpha^2-14\alpha+7}}{24(1-\alpha)}|c|$ and $\alpha\ne 0$, then $z\to z^{(1-2\alpha-p)/[2\alpha]}w_p(z^{1/[2\alpha]})$ is starlike in $\mathcal{U}$.

Proof. (i) Assume that $\kappa > \frac{13-7\alpha}{12(1-\alpha)}|c|$. Then from the inequality (2.15), we get

$$\left|\frac{zg_p'(z)}{g_p(z)}-1\right| \le \frac{|c|\,(6\kappa-|c|)}{3(4\kappa-|c|)(3\kappa-|c|)} < 1-\alpha. \tag{3.2}$$

This shows that $g_p(z)\in\mathcal{S}^*(\alpha)$.

(ii) If $\kappa > \frac{13-7\alpha}{12(1-\alpha)}|c|$ then from the inequality (2.20), we have

$$\left|\frac{zg_p''(z)}{g_p'(z)}\right| \le \frac{6\,|c|}{12\kappa-7\,|c|} < 1-\alpha. \tag{3.3}$$

This shows $g_p(z)\in\mathcal{C}(\alpha)$.

(iii) Define the function $h_p:\mathcal{U}\to\mathbb{C}$ by $h_p(z) = z^{(1-2\alpha-p)/[2\alpha]}w_p\left(z^{1/[2\alpha]}\right)$. Since $h_p(z) = \frac{1}{2^p\sqrt{\pi}\Gamma(\kappa)}z^{(1-\alpha)/\alpha}u_p\left(z^{1/\alpha}\right)$, it follows that

$$\frac{zh_p'(z)}{h_p(z)} = \frac{1}{\alpha}\left[\frac{z^{1/\alpha}g_p'(z^{1/\alpha})}{g_p(z^{1/\alpha})}-\alpha\right].$$

Finally, because g_p is starlike of order α, we deduce that h_p is starlike in $\mathcal{U}$. □

Remark 3.2 If we choose $\alpha = 0$ in parts (i) and (ii) of Theorem 3.2 then we get parts (iii) and (iv) of Theorem 2.2, respectively.

4 Close-to-convexity of the generalized Struve functions

Motivated by the paper of Baricz [2], we discuss in this section a few conditions concerning the parameters of u_p, which guarantee the close-to-convexity with respect to the convex functions $f_1,\ f_2 : \mathcal{U} \longrightarrow \mathbb{C}$, defined by $f_1(z) := -\log(1-z)$ and $f_2(z) := \frac{1}{2}\log\frac{1+z}{1-z}$.

Theorem 4.1 *If $b, c, p \in \mathbb{R}$, $\kappa = p + (b+2)/2$ and $\alpha \in [0,1)$ then the function g_p satisfy the following property:*
If $\kappa > \frac{\alpha+9}{4(1-\alpha)}|c|$ then $g_p(z) \in \mathcal{K}(\frac{1+\alpha}{2})$ and $\Re\left\{g_p'(z)\right\} > \frac{1+\alpha}{2}$, for all $z \in \mathcal{U}$.

Proof. If $\kappa > \frac{\alpha+9}{4(1-\alpha)}|c|$ then from the inequality (2.18), and Theorem 1.7, we have $|zg_p''(z)| = \frac{2|c|}{4\kappa-|c|} < \frac{1-\alpha}{4}$, $(z \in \mathcal{U}; 0 \le \alpha < 1)$. So $\Re\left\{g_p'(z)\right\} > \frac{1+\alpha}{2}$. This shows that $g_p(z) \in \mathcal{K}(\frac{1+\alpha}{2})$. □

Theorem 4.2 *If $c < 0$ and $b, p \in \mathbb{R}$, then the following assertions are true*:

(*i*) *If $\kappa \ge -c/3$ then $g_p(z)$ is close-to-convex with respect to the function f_1 and hence univalent in $\mathcal{U}$.*
(*ii*) *If $\kappa \ge -c/2$ then $z \longrightarrow zu_p(z^2)$ is close-to-convex with respect to the function f_2 and hence univalent in $\mathcal{U}$.*

Proof. (i) From (2.10) we have $g_p(z) = zu_p(z) = z + b_1z^2 + b_2z^3 + ... + b_{n-1}z^n + ...$, where b_n is defined by (2.9). Clearly we have $b_{n-1} > 0$ for all $n \ge 2$ and $2b_1 = -c/(3\kappa) \le 1$. From the definition of the ascending factorial notation we observe that (we use the formula $(\kappa)_n = (\kappa+n-1)(\kappa)_{n-1}$)

$$b_n = -\frac{c}{2(2n+1)(\kappa+n-1)}b_{n-1}.$$

We use Lemma 1.2 to prove that $g_p(z)$ is close-to-convex with respect to the function $f_1(z) = -\log(1-z)$. Therefore, we need to show that $\{nb_{n-1}\}_{n\ge1}$ is a decreasing sequence. By a short computation we obtain

$$\begin{aligned} nb_{n-1} - (n+1)b_n &= b_{n-1}\left[n + \frac{c(n+1)}{2(2n+1)(\kappa+n-1)}\right] \\ &= \frac{b_{n-1}U_1(n)}{2(2n+1)(\kappa+n-1)}, \end{aligned}$$

where $U_1(n) = 4n^3 + 2(2\kappa-1)n^2 + (2\kappa-2+c)n + c$. Using the inequalities $n^3 \ge 3n^2 - 3n + 1$ and $n^2 \ge 2n-1$, we get $U_1(n) \ge (4\kappa+10)n^2 + (2\kappa-14+c)n + c + 4 \ge (10\kappa+6+c)n - 4\kappa + c - 6 \ge U_1(1) = 6\kappa + 2c \ge 0$ because $4\kappa + 10 > 0$ and $10\kappa + 6 + c > 0$ by the assumptions. This implies that $nb_{n-1} - (n+1)b_n \ge 0$, for all $n \ge 1$, thus $\{nb_{n-1}\}_{n\ge1}$ is a decreasing sequence. By Lemma 1.2 it follows that $g_p(z)$ is close-to-convex with respect to the convex function $-\log(1-z)$.

(ii) We have $zu_p(z^2) = z + b_1z^3 + b_2z^5 + ... + b_{n-1}z^{2n-1} + ...$, where b_n is defined by (2.9). Therefore we have $3b_1 = -c/(2\kappa) \le 1$ and $b_{n-1} > 0$ for all $n \ge 2$. We want to show that $\{(2n-1)b_{n-1}\}_{n\geq 2}$ is a decreasing sequence. Fix $n \geq 2$. Then we have

$$\begin{aligned}(2n-1)b_{n-1}-(2n+1)b_n &= b_{n-1}\left[2n-1+\frac{c(2n+1)}{2(2n+1)\,(\kappa+n-1)}\right]\\ &= \frac{b_{n-1}U_2(n)}{2(2n+1)\,(\kappa+n-1)},\end{aligned}$$

where $U_2(n) = 8n^3+8(\kappa-1)n^2+2\,(c-1)\,n-2(\kappa-1)+c$. Using the inequalities $n^3 \geq 3n^2-3n+1$ and $n^2 \geq 2n-1$, we obtain $U_2(n) \geq 8(\kappa+2)n^2+2\,(c-13)\,n-2(\kappa-5)+c \geq 2(8\kappa+c+13)n-2(5\kappa+3)+c \geq 3(2\kappa+c) \geq 0$. Hence $\{(2n-1)b_{n-1}\}_{n\geq 2}$ is a decreasing sequence. By applying Lemma 1.2 we get the desired conclusion. □

Remark 4.1 Observe that choosing $c=-1$ and $b=1$ in Theorem 4.2 we obtain the following sufficient conditions of close-to-convexity:

(i) If $p \geq -7/6$ then $z\mathcal{L}_p\left(z^{1/2}\right)$ is close-to-convex with respect to the function f_1.
(ii) If $p \geq -1$ then $z\mathcal{L}_p\,(z)$ is close-to-convex with respect to the function f_2.

Let $f \in \mathcal{A}$. The Alexander transform $\mathbb{A}[f] : \mathcal{U} \longrightarrow \mathbb{C}$ of f is defined by $\mathbb{A}[f](z) = \int_0^z \frac{f(t)}{t}dt = z+\sum_{n\geq 2}\frac{a_n}{n}z^n$. The following theorem contains some properties of the Alexander transform of the function $g_p(z)$.

Theorem 4.3 *Let $b, p \in \mathbb{R}$ and $c < \frac{91-\sqrt{11353}}{8} \approx -1.9438$. If $\kappa \geq \frac{-(20c+29)+\sqrt{160c^2-3640c-6839}}{120}$, then the function $\mathbb{A}[g_p]$ is close-to-convex with respect to the function $-\log(1-z)$ and it is starlike in $\mathcal{U}$. Moreover, we have that $\Re\{u_p(z)\} > 1/2$ holds for all $z \in \mathcal{U}$.*

Proof. From (2.10) we have

$$g_p(z) = zu_p(z) = z+\sum_{n\geq 2} b_{n-1}z^n = z+\sum_{n\geq 2}\frac{(-c/4)^{n-1}}{(3/2)_{n-1}\,(\kappa)_{n-1}}z^n.$$

So, the Alexander transform of the function $g_p(z)$ takes the form

$$\mathbb{A}[g_p](z) = \sum_{n\geq 1} A_n z^n, \text{ where } A_n = \frac{b_{n-1}}{n} = \frac{(-c/4)^{n-1}}{n\,(3/2)_{n-1}\,(\kappa)_{n-1}}, \quad \text{for all } n \geq 1.$$

Obviously we have $A_1 = 1$. Because c is negative and $\kappa \geq \lambda(c) \geq -c/6 > 0$, we also have $A_n > 0$ for all $n \geq 2$, where

$$\lambda(c) := \frac{-(20c+29)+\sqrt{160c^2-3640c-6839}}{120}. \tag{4.1}$$

Next we prove that the sequence $\{nA_n\}_{n\geq 1}$ is decreasing. Fix any $n \geq 1$. From the definiton of the Pochhammer symbol it follows

$$(n+1)A_{n+1} = -\frac{cn}{2(2n+1)\,(\kappa+n-1)}A_n. \tag{4.2}$$

Using (4.2) we have

$$nA_n - (n+1)A_{n+1} = \frac{nU_1(n)}{2(2n+1)\,(\kappa+n-1)}A_n, \tag{4.3}$$

where $U_1(n) = 4n^2 + 2(2\kappa-1)n + 2\kappa + c - 2$. Since $n^2 \geq 2n-1$ and $6\kappa > -c$, we have $U_1(n) \geq (4\kappa+6)n + 2\kappa - 6 + c \geq U_1(1) = 6\kappa + c > 0$. Consequently, (4.3) yields $nA_n > (n+1)A_{n+1}$. This shows that the sequence $\{nA_n\}_{n\geq 1}$ is strictly decreasing.

Next, we show that the sequence $\{nA_n - (n+1)A_{n+1}\}_{n\geq 1}$ is also decreasing. For convenience we denote $B_n = nA_n - (n+1)A_{n+1}$ for each $n \geq 1$. Fix any $n \geq 1$. Using (4.3), we find that

$$B_n - B_{n+1} = \frac{nU_2(n)A_n}{4(2n+1)(2n+3)\,(\kappa+n)\,(\kappa+n-1)},$$

where $U_2(n) = 16n^4 + (\kappa+16)\,n^3 + D_1n^2 + D_2n + D_3$,

$$D_1 = 16\kappa^2 + 48\kappa + 8c - 20, \quad D_2 = 32\kappa^2 + 8\kappa c - 8\kappa + 12c - 12,$$
$$D_3 = 12\kappa^2 + 12\kappa c - 12\kappa + c^2 + 32.$$

Our aim is to show that $U_2(n) > 0$. First we observe that the inequality $n^4 \geq 4n^3 - 6n^2 + 4n - 1$ holds. By using this inequality we obtain $U_2(n) \geq V(n)$, where $V(n) = (\kappa+80)\,n^3 + (D_1-96)n^2 + (D_2+64)n + D_3 - 16$. Clearly, the coefficient of n^3 in the above expression is nonnegative, since $\kappa > 0$. Therefore using that $n^3 \geq 3n^2 - 3n + 1$, we obtain $V(n) \geq W(n)$, where

$$W(n) = D_4n^2 + D_5n + D_6,$$

$$D_4 = 16\kappa^2 + 51\kappa + 8c + 124, \quad D_5 = 32\kappa^2 + 8\kappa c - 11\kappa + 12c - 188,$$
$$D_6 = 12\kappa^2 + 12\kappa c - 11\kappa + c^2 + 96.$$

Now, we observe that D_4 is also nonnegative, because

$$\kappa \geq \lambda(c) > \left[-51 + \sqrt{-5335 - 512c}\right]/32,$$

where the value $\left[-51 + \sqrt{-5335 - 512c}\right]/32$ is the greatest root of the equation $D_4 = 0$. Similarly $n^2 \geq 2n-1$, therefore $W(n) \geq X(n)$, where $X(n) = D_7n + D_8$, $D_7 = 2D_4 + D_5$ and $D_8 = D_6 - D_4$. Analogously, by the hypothesis, we can deduce easily that $D_7 = 64\kappa^2 + (91+8c)\,\kappa + 28c + 60$. Indeed, the relation

$$\kappa \geq \lambda(c) > \frac{\left[-(8c+91) + \sqrt{64c^2 - 5712c - 7079}\right]}{128} =: \kappa_c$$

(here κ_c is the greatest root of the equation $D_7 = 0$) implies that D_7 is nonnegative, and leads to $X(n) \geq X(1)$. In this case $X(1) = D_4 + D_5 + D_6 = 60\kappa^2 + (20c+29)\,\kappa + c^2 + 20c + 32$ is also positive, because $\kappa \geq \lambda(c) > 0$. Thus, we proved a chain of inequalities $U_2(n) \geq V(n) \geq W(n) \geq X(n) \geq X(1) > 0$, which implies $B_n - B_{n+1} > 0$. Hence the sequence $\{nA_n - (n+1)A_{n+1}\}_{n\geq 1}$ is strictly decreasing. By Lemma 1.3 we deduce that $\mathbb{A}[g_p]$ is starlike in $\mathcal{U}$.

The sequence $\{nA_n\}_{n\geq 1}$ is strictly decreasing and $2A_2 = b_1 = -c/(6\kappa) < 1$. Thus it follows by Lemma 1.2 that $\mathbb{A}[g_p]$ is close-to-convex with respect to the function $-\log(1-z)$. Now, we apply Lemma 1.3 to prove that $\Re\{u_p(z)\} > 1/2$ for all $z \in \mathcal{U}$. For this consider $g = u_p$. Therefore we have $C_n = b_{n-1} = nA_n$ for all $n \geq 1$ and thus, the sequence $\{C_n\}_{n\geq 1}$ is strictly decreasing. In addition we have $C_n - 2C_{n+1} + C_{n+2} = B_n - B_{n+1} > 0$ for all $n \geq 1$. Hence, Lemma 1.3 yields the asserted property, which completes the proof. □

Corollary 4.4 *If $b, p \in \mathbb{R}$ and $c < \frac{91-\sqrt{11353}}{8} \approx -1.9438$ such that $\kappa \geq \lambda(c) - 1$ then the function*

$$f_p(z) = \int_0^z \frac{1 - u_p(t) - 2tu_p'(t)}{t} dt \tag{4.4}$$

is univalent in $\mathcal{U}$, where $\lambda(c)$ is given in (4.1).

Proof. By the proof of Theorem 4.3 the Alexander transform

$$\int_0^z u_{p+1}(t)dt$$

is close-to-convex with respect to the function $-\log(1-z)$ if $\kappa \geq \lambda(c)-1$, and therefore, in particular, it is univalent. Using part (v) of Proposition 2.1, we have

$$\int_0^z u_{p+1}(t)dt = \frac{2\kappa}{c} \int_0^z \frac{1 - u_p(t) - 2tu_p'(t)}{t} dt = \frac{2\kappa}{c} f_p(z).$$

Consequently the function $\frac{2\kappa}{c} f_p(z)$ is univalent in $\mathcal{U}$. Since the addition of a constant and the multiplication by a nonzero quantity do not disturb the univalence, we immediately deduce that f_p is univalent in $\mathcal{U}$. This completes the proof. □

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