# Supplementary Material: Scalable Bayesian Low-Rank Decomposition of Incomplete Multiway Tensors

In our MGP based CP decomposition, the tensor element $x_{\boldsymbol{i}}$, with $\boldsymbol{i} = [i_1, i_2, \cdots, i_K]$ its $K$-dimensional index vector, can be concisely represented as:

$$x_{\boldsymbol{i}} = \sum_{r=1}^{R} \lambda_r \prod_{k=1}^{K} u_{i_k r}^{(k)}$$

where

$$\lambda_r \sim \mathcal{N}(0, \tau_r^{-1}), \quad 1 \le r \le R$$

$$\tau_r = \prod_{l=1}^{r} \delta_l, \quad \delta_l \sim Ga(a_c, 1) \quad a_c > 1 \quad 1 < r \le R$$

$$\boldsymbol{u}_r^{(k)} \sim N(\boldsymbol{\mu}^{(k)}, \Sigma^{(k)}),\ 1 < r \le R,\ 1 < k \le K$$

**Theorem 1.** *With $a_c > 1$, the sequence $\sum_{r=1}^{R} \lambda_r \prod_{k=1}^{K} u_{i_k r}^{(k)}$ converges in $\ell_2$ as $R \to \infty$.*

**Proof:** To prove Theorem 1, we will make use of the following Lemma.

**Lemma 1.** *A sequence of random variables $\{X_n\}_{n\ge 1}$ converges to a random variable $X$ ($\mathbb{E}(X^2) < \infty$) in $\ell_2$ iff $\mathbb{E}(X_{n+k} - X_n)^2 \to 0$ as $n, k \to \infty$.*

Based on the Lemma 1, to prove Theorem 1, we need to show that $\sum_{r=R+1}^{\infty} \mathbb{E}(\lambda_r \prod_{k=1}^{K} u_{i_k r}^{(k)})^2 = 0$.

Note that, $\forall r$, in our MGP-CP construction $\lambda_r$ and $\{u_{i_k r}^{(k)}\}_{k=1}^{K}$ are independent of each other, and $\mathbb{E}(\lambda_r) = 0$. Therefore we have

$$\sum_{r=R+1}^{\infty} \mathbb{E}(\lambda_r \prod_{k=1}^{K} u_{i_k r}^{(k)})^2 = \sum_{r=R+1}^{\infty} \mathbb{E}(\lambda_r)^2 \prod_{k=1}^{K} \mathbb{E}(u_{i_k r}^{(k)})^2 \tag{1}$$

where $\mathbb{E}(\lambda_r)^2 = \mathbb{E}(\mathbb{E}(\lambda_r)^2 | \tau_r)) = \mathbb{E}(\frac{1}{\prod_{l=1}^{r} \delta_l}) = \frac{1}{a_c^r}$.

Under the assumption that the vectors $u_{i_k}^{(k)}$ have bounded variances, $\mathbb{E}(u_{i_k r}^{(k)})^2$ will be bounded by $\zeta_k < \infty$. Substituting these in Equation 1, we have

$$\sum_{r=R+1}^{\infty} \mathbb{E}(\lambda_r \prod_{k=1}^{K} u_{i_k r}^{(k)})^2 < \prod_{k=1}^{K} \zeta_k \sum_{r=R+1}^{\infty} \frac{1}{a_c^r} = \prod_{k=1}^{K} \zeta_k \frac{1}{a_c^R (a_c - 1)}$$

Taking the limit $R \to \infty$

$$\lim_{R\to\infty} \sum_{r=R+1}^{\infty} \mathbb{E}(\lambda_r \prod_{k=1}^{K} u_{i_k r}^{(k)})^2 = \lim_{R\to\infty} \frac{1}{a_c^R (a_c - 1)} \prod_{k=1}^{K} \zeta_k = 0$$

Thus the sequence $\sum_{r=1}^{R} \lambda_r \prod_{k=1}^{K} u_{i_k r}^{(k)}$ converges in $\ell_2$ as $R \to \infty$, which completes the proof of Theorem 1.

Although the MGP construction is originally unbounded [1], a consequence of Theorem 1 is that even when a finite truncation level $R$ is used, the approximation error due truncation $R$ decreases exponentially fast with increasing $R$, as given by the following:

**Theorem 2.** *Denote the residual by* $M^R_{i_1 i_2 \dots i_k} = \sum_{r=R+1}^{\infty} \lambda_r \prod_{k=1}^{K} u^{(k)}_{i_k r}$. *Then* $\forall \epsilon > 0$ *we have* $P\{(M^R_{i_1 i_2 \dots i_k})^2 > \epsilon\} < \frac{1}{\epsilon a_c^R (a_c - 1)} \prod_{k=1}^{K} \zeta_k$

**Proof:** We have $P\{(M^R_{i_1 i_2 \dots i_k})^2 \leq \epsilon\} = \mathbb{E}(P\{(M^R_{i_1 i_2 \dots i_k})^2 \leq \epsilon | \tau\}) = 1 - \mathbb{E}(P\{(M^R_{i_1 i_2 \dots i_k})^2 > \epsilon | \tau\}) = 1 - \frac{\mathbb{E}(\mathbb{E}((M^R_{i_1 i_2 \dots i_k})^2 | \tau))}{\epsilon}$ where $\mathbb{E}(\mathbb{E}((M^R_{i_1 i_2 \dots i_k})^2 | \tau)) < \frac{1}{a_c^R (a_c - 1)} \prod_{k=1}^{K} \zeta_k$ as shown in the proof of Theorem 1. Thus we have $P\{(M^R_{i_1 i_2 \dots i_k})^2 > \epsilon\} < \frac{1}{\epsilon a_c^R (a_c - 1)} \prod_{k=1}^{K} \zeta_k$.

# References


[1] A. Bhattacharya and D. Dunson. Sparse bayesian infinite factor models. *Biometrika*, 2011.