A. Proof of Lemma 7

The equality follows from the symmetry in HD. To prove the upper bound, observe that

$$\mathbb{E}\left[(Z_i^{\max})^2\right] = \mathrm{Var}\left(Z_i^{\max}\right) + \left(\mathbb{E}\left[Z_i^{\max}\right]\right)^2.$$

Let $D(j)$ be the j^{th} diagonal entry of D. To bound the first term observe that $Z_i^{\max}$ is a function of d independent random variables $D(1), D(2), \ldots D(d)$. Changing $D(j)$ changes the $Z_i^{\max}$ by at most $\frac{2X_i(j)}{\sqrt{d}}$. Hence, applying Efron-Stein variance bound (Efron & Stein, 1981) yields

$$\mathrm{Var}\left(Z_i^{\max}\right) \leq \sum_{j=1}^{d} \frac{4X_i^2(j)}{2d} = \frac{2\left|\left|X_i\right|\right|_2^2}{d}.$$

To bound the second term, observe that for every $\beta > 0$,

$$\beta Z_i^{\max} = \log \exp\left(\beta Z_i^{\max}\right) \leq \log\left(\sum_{j=1}^{d} e^{\beta Z_i(j)}\right).$$

Note that $Z_i(k) = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} D(j)H(k,j)X_i(j)$. Since the $D(j)$'s are Radamacher random variables and $|H(k,j)| = 1$ for all k, j, the distributions of $Z_i(k)$ is same for all k. Hence by Jensen's inequality,

$$\mathbb{E}\left[Z_i^{\max}\right] \leq \frac{1}{\beta}\mathbb{E}\left[\log\left(\sum_{j=1}^{d} e^{\beta Z_i(j)}\right)\right]$$
$$\leq \frac{1}{\beta}\log\left(\sum_{j=1}^{d} \mathbb{E}[e^{\beta Z_i(j)}]\right) = \frac{1}{\beta}\log\left(d\mathbb{E}[e^{\beta Z_i(1)}]\right).$$

Since $Z_i(1) = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} D(j)X_i(j)$,

$$\mathbb{E}[e^{\beta Z_i(1)}] = \mathbb{E}\left[e^{\frac{\beta \sum_j D(j)X_i(j)}{\sqrt{d}}}\right] \overset{(a)}{=} \prod_{j=1}^{d} \mathbb{E}\left[e^{\frac{\beta D(j)X_i(j)}{\sqrt{d}}}\right]$$
$$= \prod_{j=1}^{d} \frac{e^{-\beta X_i(j)/\sqrt{d}} + e^{\beta X_i(j)/\sqrt{d}}}{2}$$
$$\overset{(b)}{\leq} \prod_{j=1}^{d} e^{\beta^2 X^2(j)/2d} = e^{\beta^2 ||X_i||_2^2/2d},$$

where (a) follows from the fact that the $D(i)$'s are independent and (b) follows from the fact that $e^a + e^{-a} \leq 2e^{a^2/2}$ for any a. Hence,

$$\mathbb{E}[Z_i^{\max}] \leq \min_{\beta \geq 0} \frac{\log d}{\beta} + \frac{\beta\left|\left|X_i\right|\right|_2^2}{2d} \leq \frac{2\left|\left|X_i\right|\right|_2 \sqrt{\log d}}{\sqrt{2d}}.$$