On certain arithmetical functions

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1. Let $\sigma_s(n)$ denote the sum of the sth powers of the divisors of n (including 1 and n), and let

$$\sigma_s(0) = \tfrac{1}{2}\zeta(-s),$$

where $\zeta(s)$ is the Riemann Zeta-function. Further let

$$\sum_{r,s}(n) = \sigma_r(0)\sigma_s(n) + \sigma_r(1)\sigma_s(n-1) + \cdots + \sigma_r(n)\sigma_s(0). \tag{1}$$

In this paper I prove that

$$\sum_{r,s}(n) = \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\sigma_{r+s+1}(n) + \frac{\zeta(1-r)+\zeta(1-s)}{r+s}n\sigma_{r+s-1}(n) + O\{n^{\frac{2}{3}(r+s+1)}\}, \tag{2}$$

whenever r and s are positive odd integers. I also prove that there is no error term on the right-hand side of (2) in the following nine cases: $r=1, s=1; r=1, s=3; r=1, s=5; r=1, s=7; r=1, s=11, r=3, s=3; r=3, s=5; r=3,, s=9; r=5, s=7$. That is to say $\sum_{r,s}(n)$ has a finite expression in terms of $\sigma_{r+s+1}(n)$ and $\sigma_{r+s-1}(n)$ in these nine cases; but for other values of r and s it involves other arithmetical functions as well.
It appears probable, from the empirical results I obtain in §§ 18-23, that the error term on the right-hand side of (2) is of the form

$$O\{n^{\frac{1}{2}(r+s+1+\epsilon)}\}, \tag{3}$$

where ϵ is any positive number, and not of the form

$$o\{n^{\frac{1}{2}(r+s+1)}\}. \tag{4}$$

But all I can prove rigorously is (i) that the error is of the form

$$O\{n^{\frac{2}{3}(r+s+1)}\}$$

in all cases, (ii) that it is of the form

$$O\{n^{\frac{2}{3}(r+s+\frac{3}{4})}\} \tag{5}$$

if $r+s$ is of the form $6m$, (iii) that it is of the form

$$O\{n^{\frac{2}{3}(r+s+\frac{1}{2})}\} \tag{6}$$

if $r+s$ is of the form $6m+4$, and (iv) that it is not of the form

$$o\{n^{\frac{1}{2}(r+s)}\}. \tag{7}$$

It follows from (2) that, if r and s are positive odd integers, then

$$\sum_{r,s}(n) \sim \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\sigma_{r+s+1}(n). \tag{8}$$

It seems very likely that (8) is true for all positive values of r and s, but this I am at present unable to prove.

2. If $\sum_{r,s}(n)/\sigma_{r+s+1}(n)$ tends to a limit, then the limit must be

$$\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}.$$

For then

$$\begin{aligned}
\lim_{n\to\infty}\frac{\sum_{r,s}(n)}{\sigma_{r+s+1}(n)} &= \lim_{n\to\infty}\frac{\sum_{r,s}(1)+\sum_{r,s}(2)+\cdots+\sum_{r,s}(n)}{\sigma_{r+s+1}(1)+\sigma_{r+s+1}(2)+\cdots+\sigma_{r+s+1}(n)}\\
&= \lim_{x\to1}\frac{\sum_{r,s}(0)+\sum_{r,s}(1)x+\sum_{r,s}(2)x^2+\cdots}{\sigma_{r+s+1}(0)+\sigma_{r+s+1}(1)x+\sigma_{r+s+1}(2)x^2+\cdots}\\
&= \lim_{x\to1}\frac{S_rS_s}{S_{r+s+1}},
\end{aligned}$$

where

$$S_r = \tfrac{1}{2}\zeta(-r)+\frac{1^rx}{1-x}+\frac{2^rx^2}{1-x^2}+\frac{3^rx^3}{1-x^3}+\cdots \tag{9}$$

Now it is known that, if $r>0$, then

$$S_r \sim \frac{\Gamma(r+1)\zeta(r+1)}{(1-x)^{r+1}}, \tag{10}$$

as $x\to1$ *. Hence we obtain the result stated.

3. It is easy to see that

$$\begin{aligned}
&\sigma_r(1)+\sigma_r(2)+\sigma_r(3)+\cdots+\sigma_r(n)\\
&\quad = u_1+u_2+u_3+u_4+\cdots+u_n,
\end{aligned}$$

*Knopp, *Dissertation* (Berlin, 1907), p.34.

where

$$u_t = 1^r + 2^r + 3^r + \cdots + \left[\frac{n}{t}\right]^r.$$

From this it is easy to deduce that

$$\sigma_r(1) + \sigma_r(2) + \cdots + \sigma_r(n) \sim \frac{n^{r+1}}{r+1}\zeta(r+1)\ ^* \tag{11}$$

and

$$\sigma_r(1)(n-1)^s + \sigma_r(2)(n-2)^s + \cdots + \sigma_r(n-1)1^s \sim \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\zeta(r+1)n^{r+s+1},$$

provided $r > 0, s \geq 0$. Now

$$\sigma_s(n) > n^s,$$

and

$$\sigma_s(n) < n^s(1^{-s} + 2^{-s} + 3^{-s} + \cdots) = n^s\zeta(s).$$

From these inequalities and (1) it follows that

$$\varliminf \frac{\sum_{r,s}(n)}{n^{r+s+1}} \geq \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\zeta(r+1), \tag{12}$$

if $r > 0$ and $s \geq 0$; and

$$\varlimsup \frac{\sum_{r,s}(n)}{n^{r+s+1}} \leq \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\zeta(r+1)\zeta(s), \tag{13}$$

if $r > 0$ and $s > 1$. Thus $n^{-r-s-1}\sum_{r,s}(n)$ oscillates between limits included in the interval

$$\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\zeta(r+1),\ \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\zeta(r+1)\zeta(s).$$

On the other hand $n^{-r-s-1}\sigma_{r+s+1}(n)$ oscillates between 1 and $\zeta(r+s+1)$, assuming values as near as we please to either of these limits. The formula (8) shews that the actual limits of indetermination of $n^{-r-s-1}\sum_{r,s}(n)$ are

$$\begin{gathered}\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)},\\ \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)\zeta(r+s+1)}{\zeta(r+s+2)}.\end{gathered} \tag{14}$$

Naturally

$$\zeta(r+1) < \frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)} < \frac{\zeta(r+1)\zeta(s+1)\zeta(r+s+1)}{\zeta(r+s+2)} < \zeta(r+1)\zeta(s)\ .^\dagger$$

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*(10) follows from this as an immediate corollary.

What is remarkable about the formula (8) is that it shews the asymptotic equality of two functions neither of which itself increases in a regular manner.

4. It is easy to see that, if n is a positive integer, then

$$\cot \tfrac{1}{2}\theta \sin n\theta = 1 + 2\cos\theta + 2\cos 2\theta + \cdots + 2\cos(n-1)\theta + \cos n\theta.$$

Suppose now that

$$\begin{aligned} &\left(\tfrac{1}{4}\cot\tfrac{1}{2}\theta + \frac{x\sin\theta}{1-x} + \frac{x^2\sin 2\theta}{1-x^2} + \frac{x^3\sin 3\theta}{1-x^3} + \cdots\right)^2 \\ &= (\tfrac{1}{4}\cot\tfrac{1}{2}\theta)^2 + C_0 + C_1\cos\theta + C_2\cos 2\theta + C_3\cos 3\theta + \cdots, \end{aligned}$$

where C_n is independent of θ. Then we have

$$\begin{aligned} C_0 &= \tfrac{1}{2}\left(\frac{x}{1-x} + \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} + \cdots\right) \\ &+ \tfrac{1}{2}\left\{\left(\frac{x}{1-x}\right)^2 + \left(\frac{x^2}{1-x^2}\right)^2 + \left(\frac{x^3}{1-x^3}\right)^2 + \cdots\right\} \\ &= \tfrac{1}{2}\left\{\frac{x}{(1-x)^2} + \frac{x^2}{(1-x^2)^2} + \frac{x^3}{(1-x^3)^2} + \cdots\right\} \\ &= \tfrac{1}{2}\left\{\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \cdots\right\}. \end{aligned} \tag{15}$$

Again

$$\begin{aligned} C_n &= \tfrac{1}{2}\frac{x^n}{1-x^n} + \frac{x^{n+1}}{1-x^{n+1}} + \frac{x^{n+2}}{1-x^{n+2}} + \frac{x^{n+3}}{1-x^{n+3}} + \cdots \\ &+ \frac{x}{1-x}\cdot\frac{x^{n+1}}{1-x^{n+1}} + \frac{x^2}{1-x^2}\cdot\frac{x^{n+2}}{1-x^{n+2}} + \frac{x^3}{1-x^3}\cdot\frac{x^{n+3}}{1-x^{n+3}} + \cdots \\ &- \tfrac{1}{2}\left\{\frac{x}{1-x}\cdot\frac{x^{n-1}}{1-x^{n-1}} + \frac{x^2}{1-x^2}\cdot\frac{x^{n-2}}{1-x^{n-2}} + \cdots + \frac{x^{n-1}}{1-x^{n-1}}\cdot\frac{x}{1-x}\right\}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{C_n}{x^n}(1-x^n) &= \tfrac{1}{2} + \left(\frac{x}{1-x} - \frac{x^{n+1}}{1-x^{n+1}}\right) + \left(\frac{x^2}{1-x^2} - \frac{x^{n+2}}{1-x^{n+2}}\right) + \cdots \\ &- \tfrac{1}{2}\left\{\left(1 + \frac{x}{1-x} + \frac{x^{n-1}}{1-x^{n-1}}\right) + \left(1 + \frac{x^2}{1-x^2} + \frac{x^{n-2}}{1-x^{n-2}}\right)\right. \end{aligned}$$

[†]For example when $r = 1$ and $s = 9$ this inequality becomes $1.64493\ldots < 1.64616\ldots < 1.64697\ldots < 1.64823\ldots$.

$$+\cdots+\left(1+\frac{x^{n-1}}{1-x^{n-1}}+\frac{x}{1-x}\right)\Bigg\}$$

$$= \frac{1}{1-x^n}-\frac{n}{2}.$$

That is to say

$$C_n=\frac{x^n}{(1-x^n)^2}-\frac{nx^n}{2(1-x^n)}. \tag{16}$$

It follows that

$$\left(\tfrac{1}{4}\cot\tfrac{1}{2}\theta+\frac{x\sin\theta}{1-x}+\frac{x^2\sin 2\theta}{1-x^2}+\frac{x^3\sin 3\theta}{1-x^3}+\cdots\right)^2$$
$$=\left(\tfrac{1}{4}\cot\tfrac{1}{2}\theta\right)^2+\frac{x\cos\theta}{(1-x)^2}+\frac{x^2\cos 2\theta}{(1-x^2)^2}+\frac{x^3\cos 3\theta}{(1-x^3)^2}+\cdots$$
$$+\tfrac{1}{2}\left\{\frac{x}{1-x}(1-\cos\theta)+\frac{2x^2}{1-x^2}(1-\cos 2\theta)+\frac{3x^3}{1-x^3}(1-\cos 3\theta)+\cdots\right\}. \tag{17}$$

Similarly, using the equation

$$\cot^2\tfrac{1}{2}\theta(1-\cos n\theta)=$$
$$(2n-1)+4(n-1)\cos\theta+4(n-2)\cos 2\theta+\cdots+4\cos(n-1)\theta+\cos n\theta,$$

we can shew that

$$\Bigg\{\tfrac{1}{8}\cot^2\tfrac{1}{2}\theta+\tfrac{1}{12}+\frac{x}{1-x}(1-\cos\theta)+\frac{2x^2}{1-x^2}(1-\cos 2\theta)+$$
$$\frac{3x^3}{1-x^3}(1-\cos 3\theta)+\cdots\Bigg\}^2=\left(\tfrac{1}{8}\cot^2\tfrac{1}{2}\theta+\tfrac{1}{12}\right)^2$$
$$+\tfrac{1}{12}\left\{\frac{1^3x}{1-x}(5+\cos\theta)+\frac{2^3x^2}{1-x^2}(5+\cos 2\theta)+\frac{3^3x^3}{1-x^3}(5+\cos 3\theta)+\cdots\right\}. \tag{18}$$

For example, putting $\theta=\frac{2}{3}\pi$ and $\theta=\frac{1}{2}\pi$ in (17), we obtain

$$\left(\tfrac{1}{6}+\frac{x}{1-x}-\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}-\frac{x^5}{1-x^5}+\cdots\right)^2$$
$$=\tfrac{1}{36}+\tfrac{1}{3}\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{4x^4}{1-x^4}+\frac{5x^5}{1-x^5}+\cdots\right), \tag{19}$$

where $1,2,4,5,\ldots$ are the natural numbers without the multiples of 3; and

$$\left(\tfrac{1}{4}+\frac{x}{1-x}-\frac{x^3}{1-x^3}+\frac{x^5}{1-x^5}-\frac{x^7}{1-x^7}+\cdots\right)^2$$

$$= \tfrac{1}{16} + \tfrac{1}{2}\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{5x^5}{1-x^5} + \cdots\right), \tag{20}$$

where $1, 2, 3, 5, \ldots$ are the natural numbers without the multiples of 4.

5. It follows from (18) that

$$\left(\frac{1}{2\theta^2} + \frac{\theta^2}{2!}S_3 - \frac{\theta^4}{4!}S_5 + \frac{\theta^6}{6!}S_7 - \cdots\right)^2$$
$$= \frac{1}{4\theta^4} + \tfrac{1}{2}S_3 - \tfrac{1}{12}\left(\frac{\theta^2}{2!}S_5 - \frac{\theta^4}{4!}S_7 + \frac{\theta^6}{6!}S_9 - \cdots\right), \tag{21}$$

where S_r is the same as in (9). Equating the coefficients of θ^n in both sides in (21), we obtain

$$\frac{(n-2)(n+5)}{12(n+1)(n+2)}S_{n+3} = \binom{n}{2}S_3S_{n-1} + \binom{n}{4}S_5S_{n-8} +$$
$$\binom{n}{6}S_7S_{n-5} + \cdots + \binom{n}{n-2}S_{n-1}S_3, \tag{22}$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!},$$

if n is an even integer greater than 2.
Let us now suppose that

$$\Phi_{r,s}(x) = \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} m^r n^s x^{mn}, \tag{23}$$

so that

$$\Phi_{r,s}(x) = \Phi_{s,r}(x),$$

and

$$\left.\begin{aligned} \Phi_{0,s}(x) &= \frac{1^s x}{1-x} + \frac{2^s x^2}{1-x^2} + \frac{3^s x^3}{1-x^3} + \cdots = S_s - \tfrac{1}{2}\zeta(-s), \\ \Phi_{1,s}(x) &= \frac{1^s x}{(1-x)^2} + \frac{2^s x^2}{(1-x^2)^2} + \frac{3^s x^3}{(1-x^3)^2} + \cdots \end{aligned}\right\}. \tag{24}$$

Further let

$$\left.\begin{aligned} P &= -24S_1 = 1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \cdots\right)^*, \\ Q &= 240S_3 = 1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \cdots\right), \\ R &= -540S_5 = 1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \frac{3^5x^3}{1-x^3} + \cdots\right) \end{aligned}\right\}. \tag{25}$$

The putting $n = 4, 6, 8, \ldots$ in (22) we obtain the results contained in the following table.

TABLE 1

1. $1 - 24\Phi_{0,1}(x) = P.$
2. $1 + 240\Phi_{0,3}(x) = Q.$
3. $1 - 504\Phi_{0,5}(x) = R.$
4. $1 + 480\Phi_{0,7}(x) = Q^2.$
5. $1 - 264\Phi_{0,9}(x) = QR.$
6. $691 + 65520\Phi_{0,11}(x) = 441Q^3 + 250R^2.$
7. $1 - 24\Phi_{0,13}(x) = Q^2R.$
8. $3617 + 16320\Phi_{0,15}(x) = 1617Q^4 + 2000QR^2.$
9. $43867 - 28728\Phi_{0,17}(x) = 38367Q^3R + 5500R^3.$
10. $174611 + 13200\Phi_{0,19}(x) = 53361Q^5 + 121250Q^2R^2.$
11. $77683 - 552\Phi_{0,21}(x) = 57183Q^4R + 20500QR^3.$
12. $236364091 + 131040\Phi_{0,23}(x) = 49679091Q^6 + 176400000Q^3R^2$
    $+10285000R^4.$
13. $657931 - 24\Phi_{0,25}(x) = 392931Q^5R + 265000Q^2R^3.$
14. $3392780147 + 6960\Phi_{0,27}(x) = 489693897Q^7 + 2507636250Q^4R^2$
    $+395450000QR^4.$
15. $1723168255201 - 171864\Phi_{0,29}(x) = 815806500201Q^6R$
    $+881340705000Q^3R^3 + 26021050000R^5.$
16. $7709321041217 + 32640\Phi_{0,31}(x) = 764412173217Q^8$
    $+5323905468000Q^5R^2 + 1621003400000Q^2R^4.$

In general

$$\tfrac{1}{2}\zeta(-s) + \Phi_{0,s}(x) = \sum K_{m,n}Q^mR^n, \tag{26}$$

where $K_{m,n}$ is a constant and m and n are positive integers (including zero) satisfying the equation

$$4m + 6n = s + 1.$$

This is easily proved by induction, using (22).

*If $x = q^2$, then in the notation of elliptic functions

$$\begin{aligned} P &= \frac{12\eta\omega}{\pi^2} = \left(\frac{2K}{\pi}\right)^2\left(\frac{3E}{K} + k^2 - 2\right), \\ Q &= \frac{12g_2\omega^4}{\pi^4} = \left(\frac{2K}{\pi}\right)^4(1 - k^2 + k^4), \\ R &= \frac{216g_3\omega^6}{\pi^6} = \left(\frac{2K}{\pi}\right)^6(1 + k^2)(1 - 2k^2)(1 - \tfrac{1}{2}k^2). \end{aligned}$$

6. Again from (17) we have

$$\left(\frac{1}{2\theta}+\frac{\theta}{1!}S_1-\frac{\theta^3}{3!}S_3+\frac{\theta^5}{5!}S_5-\cdots\right)^2$$
$$=\frac{1}{4\theta^2}+S_1-\frac{\theta^2}{2!}\Phi_{1,2}(x)+\frac{\theta^4}{4!}\Phi_{1,4}(x)-\frac{\theta^6}{6!}\Phi_{1,6}(x)+\cdots$$
$$+\tfrac{1}{2}\left(\frac{\theta^2}{2!}S_3-\frac{\theta^4}{4!}S_5+\frac{\theta^6}{6!}S_7-\cdots\right). \tag{27}$$

Equating the coefficients of θ^n in both sides in (27) we obtain

$$\frac{n+3}{2(n+1)}S_{n+1}-\Phi_{1,n}(x)=\binom{n}{1}S_1S_{n-1}+\binom{n}{3}S_3S_{n-3}+$$
$$\binom{n}{5}S_5S_{n-5}+\cdots+\binom{n}{n-1}S_{n-1}S_1, \tag{28}$$

if n is a positive even integer. From this we deduce the results contained in Table II.

TABLE II

1. $288\Phi_{1,2}(x)=Q-P^2$.
2. $720\Phi_{1,4}(x)=PQ-R$.
3. $1008\Phi_{1,6}(x)=Q^2-PR$.
4. $720\Phi_{1,8}(x)=Q(PQ-R)$.
5. $1584\Phi_{1,10}(x)=3Q^3+2R^2-5PQR$.
6. $65520\Phi_{1,12}(x)=P(441Q^3+250R^2)-691Q^2R$.
7. $144\Phi_{1,14}(x)=Q(3Q^3+4R^2-7PQR)$.

In general

$$\Phi_{1,s}(x)=\sum K_{l,m,n}P^lQ^mR^n, \tag{29}$$

where $l\le 2$ and $2l+4m+6n=s+2$. This is easily proved by induction, using (28).

7. We have

$$\left.\begin{aligned} x\frac{dP}{dx} &= -\ 24\Phi_{1,2}(x)=\frac{P^2-Q}{12},\\ x\frac{dQ}{dx} &= \ 240\Phi_{1,4}(x)=\frac{PQ-R}{3},\\ x\frac{dR}{dx} &= -\ 504\Phi_{1,6}(x)=\frac{PR-Q^2}{2}\end{aligned}\right\} \tag{30}$$

Suppose now that $r<s$ and that $r+s$ is even. Then

$$\Phi_{r,s}(x)=\left(x\frac{d}{dx}\right)^r\Phi_{0,s-r}(x), \tag{31}$$

and $\Phi_{0,s-r}(x)$ is a polynomial in Q and R. Also

$$x\frac{dP}{dx},\ x\frac{dQ}{dx},\ x\frac{dR}{dx}$$

are polynomials in P, Q and R. Hence $\Phi_{r,s}(x)$ is a polynomial in P, Q and R. Thus we deduce the results contained in Table III.

TABLE III

1. $1728\Phi_{2,3}(x) = 3PQ - 2R - P^3$.
2. $1728\Phi_{2,5}(x) = P^2Q - 2PR + Q^2$.
3. $1728\Phi_{2,7}(x) = 2PQ^2 - P^2R - QR$.
4. $8640\Phi_{2,9}(x) = 9P^2Q^2 - 18PQR + 5Q^3 + 4R^2$.
5. $1728\Phi_{2,11}(x) = 6PQ^3 - 5P^2QR + 4PR^2 - 5Q^2R$.
6. $6912\Phi_{3,4}(x) = 6P^2Q - 8PR + 3Q^2 - P^4$.
7. $3456\Phi_{3,6}(x) = P^3Q - 3P^2R + 3PQ^2 - QR$.
8. $5184\Phi_{3,8}(x) = 6P^2Q^2 - 2P^3R - 6PQR + Q^3 + R^2$.
9. $20736\Phi_{4,5}(x) = 15PQ^2 - 20P^2R + 10P^3Q - 4QR - P^5$.
10. $41472\Phi_{4,7}(x) = 7(P^4Q - 4P^3R + 6P^2Q^2 - 4PQR) + 3Q^3 + 4R^2$.

In general

$$\Phi_{r,s}(x) = \sum K_{l,m,n} P^l Q^m R^n, \tag{32}$$

where $l - 1$ does not exceed the smaller of r and s and

$$2l + 4m + 6n = r + s + 1.$$

The results contained in these three tables are of course really results in the theory of elliptic functions. For example Q and R are substantially the invariants g_2 and g_3, and the formulæ of Table I are equivalent to the formulæ which express the coefficients in the series

$$\wp(u) = \frac{1}{u^2} + \frac{g_2u^2}{20} + \frac{g_3u^4}{28} + \frac{g_2^2u^6}{1200} + \frac{3g_2g_3u^8}{6160} + \cdots$$

in terms of g_2 and g_3. The elementary proof of these formulæ given in the preceding sections seems to be of some interest in itself.

8. In what follows we shall require to know the form of $\Phi_{1,s}(x)$ more precisely than is shewn by the formula (29).
We have

$$\tfrac{1}{2}\zeta(-s) + \Phi_{0,s}(x) = \sum K_{m,n} Q^m R^n, \tag{33}$$

where s is an odd integer greater than 1 and $4m + 6n = s + 1$. Also

$$x\frac{d}{dx}(Q^mR^n) = \left(\frac{m}{3} + \frac{n}{2}\right) PQ^mR^n - \left(\frac{m}{3}Q^{m-1}R^{n+1} + \frac{n}{3}Q^{m+2}R^{n-1}\right). \tag{34}$$

Differentiating (33) and using (34) we obtain

$$\Phi_{1,s+1}(x) = \tfrac{1}{12}(s+1)P\{\tfrac{1}{2}\zeta(-s) + \Phi_{0,s}(x)\} + \sum K_{m,n}Q^m R^n, \tag{35}$$

where s is an odd integer greater than 1 and $4m + 6n = s + 3$. But when $s = 1$ we have

$$\Phi_{1,2}(x) = \frac{Q - P^2}{288}. \tag{36}$$

9. Suppose now that

$$\begin{aligned} F_{r,s}(x) = \{\tfrac{1}{2}\zeta(-r) + \Phi_{0,r}(x)\}\{\tfrac{1}{2}\zeta(-s) + \Phi_{0,s}(x)\} \\ -\frac{\zeta(1-r)+\zeta(1-s)}{r+s}\Phi_{1,r+s}(x) - \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)} \\ \times\{\tfrac{1}{2}\zeta(-r-s-1) + \Phi_{0,r+s+1}(x)\}. \end{aligned} \tag{37}$$

Then it follows from (33), (35) and (36) that, if r and s are positive odd integers,

$$F_{r,s}(x) = \sum K_{m,n}Q^m R^n, \tag{38}$$

where

$$4m + 6n = r + s + 2.$$

But it is easy to see, from the functional equation satisfied by $\zeta(s)$, viz.

$$(2\pi)^{-s}\Gamma(s)\zeta(s)\cos\tfrac{1}{2}\pi s = \tfrac{1}{2}\zeta(1-s), \tag{39}$$

that

$$F_{r,s}(0) = 0. \tag{40}$$

Hence $Q^3 - R^2$ is a factor of the right-hand side in (38), that is to say

$$F_{r,s}(x) = (Q^3 - R^2)\sum K_{m,n}Q^m R^n, \tag{41}$$

where

$$4m + 6n = r + s - 10.$$

10. It is easy to deduce from (30) that

$$x\frac{d}{dx}\log(Q^3 - R^2) = P. \tag{42}$$

But it is obvious that

$$P = x\frac{d}{dx}\log[x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}]; \tag{43}$$

and the coefficient of x in $Q^3 - R^2 = 1728$. Hence

$$Q^3 - R^2 = 1728x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}. \tag{44}$$

But it is known that

$$\begin{aligned}&\{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots\}^3\\ &\quad = 1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \cdots\end{aligned} \tag{45}$$

Hence

$$Q^3 - R^2 = 1728x(1 - 3x + 5x^3 - 7x^6 + \cdots)^8. \tag{46}$$

The coefficient of $x^{\nu-1}$ in $1-3x+5x^3-\cdots$ is numerically less than $\sqrt{(8\nu)}$, and the coefficient of x^ν in $Q^3 - R^2$ is therefore numerically less than that of x^ν in

$$1728x\{\sqrt{(8\nu)}(1 + x + x^3 + x^6 + \cdots)\}^8.$$

But

$$x(1 + x + x^3 + x^6 + \cdots)^8 = \frac{1^3 x}{1-x^2} + \frac{2^3 x^2}{1-x^4} + \frac{3^3 x^3}{1-x^6} + \cdots, \tag{47}$$

and the coefficient of x^ν in the right-hand side is positive and less than

$$\nu^3\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \cdots\right).$$

Hence the coefficient of x^ν in $Q^3 - R^2$ is of the form

$$\nu^4 O(\nu^3) = O(\nu^7).$$

That is to say

$$Q^3 - R^2 = \sum O(\nu^7)x^\nu. \tag{48}$$

Differentiating (48) and using (42) we obtain

$$P(Q^3 - R^2) = \sum O(\nu^8)x^\nu. \tag{49}$$

Differentiating this again with respect to x we have

$$A(P^2 - Q)(Q^3 - R^2) + BQ(Q^3 - R^2) = \sum O(\nu^9)x^\nu,$$

where A and B are constants. But

$$P^2 - Q = -288\Phi_{1,2}(x) = -288\left\{\frac{1^2 x}{(1-x)^2} + \frac{2^2 x^2}{(1-x^2)^2} + \cdots\right\},$$

and the coefficient of x^ν in the right-hand side is a constant multiple of $\nu\sigma_1(\nu)$. Hence

$$\begin{aligned}(P^2-Q)(Q^3-R^2) &= \sum O\nu\sigma_1(\nu)x^\nu \sum O(\nu^7)x^\nu \\ &= \sum O(\nu^8)\{\sigma_1(1)+\sigma_1(2)+\cdots \\ &\quad +\sigma_1(\nu)\}x^\nu = \sum O(\nu^{10})x^\nu,\end{aligned}$$

and so

$$Q(Q^3-R^2) = \sum O(\nu^{10})x^\nu. \tag{50}$$

Differentiating this again with respect to x and using arguments similar to those used above, we deduce

$$R(Q^3-R^2) = \sum O(\nu^{12})x^\nu. \tag{51}$$

Suppose now that m and n are any two positive integers including zero, and that $m+n$ is not zero. Then

$$\begin{aligned}Q^m R^n(Q^3-R^2) &= Q(Q^3-R^2)Q^{m-1}R^n \\ &= \sum O(\nu^{10})x^\nu\{\sum O(\nu^3)x^\nu\}^{m-1}\{\sum O(\nu^5)x^\nu\}^n \\ &= \sum O(\nu^{10})x^\nu \sum O(\nu^{4m-5})x^\nu \sum O(\nu^{6n-1})x^\nu \\ &= \sum O(\nu^{4m+6n+6})x^\nu,\end{aligned}$$

If m is not zero, Similarly we can shew that

$$\begin{aligned}Q^m R^n(Q^3-R^2) &= R(Q^3-R^2)Q^m R^{n-1} \\ &= \sum O(\nu^{4m+6n+6})x^\nu,\end{aligned}$$

if n is not zero. Therefore in any case

$$(Q^3-R^2)Q^m R^n = \sum O(\nu^{4m+6n+6})x^\nu. \tag{52}$$

11. Now let r and s be any two positive odd integers including zero. Then, when $r+s$ is equal to 2,4,6,8 or 12, there are no values of m and n satisfying the relation

$$4m+6n = r+s-10$$

in (41); consequently in these cases

$$F_{r,s}(x) = 0. \tag{53}$$

When $r+s=10, m$ and n must both be zero, and this result does not apply; but it follows from (41) and (48) that

$$F_{r,s}(x)=\sum O(\nu^7)x^\nu. \tag{54}$$

And when $r+s\geq 14$ it follows from (52) that

$$F_{r,s}(x)=\sum O(\nu^{r+s-4})x^\nu. \tag{55}$$

Equating the coefficients of x^ν in both sides in (53), (54) and (55) we obtain

$$\begin{aligned}\sum_{r,s}(n)=&\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\sigma_{r+s+1}(n)\\&+\frac{\zeta(1-r)+\zeta(1-s)}{r+s}n\sigma_{r+s-1}(n)+E_{r,s}(n),\end{aligned} \tag{56}$$

where

$$\begin{aligned}E_{r,s}(n)&=0, &\quad r+s&=2,4,6,8,12;\\E_{r,s}(n)&=O(n^7), &\quad r+s&=10;\\E_{r,s}(n)&=O(n^{r+s-4}), &\quad r+s&\geq 14.\end{aligned}$$

Since $\sigma_{r+s+1}(n)$ is of order n^{r+s+1}, it follows that in all cases

$$\sum_{r,s}(n)\sim\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\sigma_{r+s+1}(n). \tag{57}$$

The following table gives the values of $\sum_{r,s}(n)$ when $r+s=2,4,6,8,12$.

TABLE IV

1. $\sum_{1,1}(n)=\frac{5\sigma_3(n)-6n\sigma_1(n)}{12}$.

2. $\sum_{1,3}(n)=\frac{7\sigma_5(n)-10n\sigma_3(n)}{80}$.

3. $\sum_{3,3}(n) = \frac{\sigma_7(n)}{120}$.

4. $\sum_{1,5}(n) = \frac{10\sigma_7(n)-21n\sigma_5(n)}{252}$.

5. $\sum_{3,5}(n) = \frac{11\sigma_9(n)}{5040}$.

6. $\sum_{1,7}(n) = \frac{11\sigma_9(n)-30n\sigma_7(n)}{480}$.

7. $\sum_{5,7}(n) = \frac{\sigma_{13}(n)}{10080}$.

8. $\sum_{3,9}(n) = \frac{\sigma_{13}(n)}{2640}$.

9. $\sum_{1,11}(n) = \frac{691\sigma_{13}(n)-2730n\sigma_{11}(n)}{65520}$.

12. In this connection it may be interesting to note that

$$\begin{aligned}\sigma_1(1)\sigma_3(n) + \sigma_1(3)\sigma_3(n-1) + \sigma_1(5)\sigma_3(n-2) + \cdots \\ +\sigma_1(2n+1)\sigma_3(0) = \tfrac{1}{240}\sigma_5(2n+1).\end{aligned} \tag{58}$$

This formula may be deduced from the identity

$$\begin{aligned}&\frac{1^5 x}{1-x} + \frac{3^5 x^2}{1-x^3} + \frac{5^5 x^3}{1-x^5} + \cdots \\ &= Q\left(\frac{x}{1-x} + \frac{3x^2}{1-x^3} + \frac{5x^3}{1-x^5} + \cdots\right),\end{aligned} \tag{59}$$

which can be proved by means of the theory of elliptic functions or by elementary methods.

13. More precise results concerning the order of $E_{r,s}(n)$ can be deduced from the theory of elliptic functions. Let

$$x = q^2.$$

Then we have

$$\left.\begin{aligned} Q &= \phi^8(q)\{1-(kk')^2\} \\ R &= \phi^{12}(q)(k'^2-k^2)\{1+\tfrac{1}{2}(kk')^2\} \\ &= \phi^{12}(q)\{1+\tfrac{1}{2}(kk')^2\}\sqrt{\{1-(2kk')^2\}} \end{aligned}\right\}, \tag{60}$$

where $\phi(q) = 1 + 2q + 2q^4 + 2q^9 + \cdots$
But, if

$$f(q) = q^{\frac{1}{24}}(1-q)(1-q^2)(1-q^3)\cdots,$$

then we know that

$$\left.\begin{aligned} 2^{\frac{1}{6}} f(q) &= k^{\frac{1}{12}} k'^{\frac{1}{3}} \phi(q) \\ 2^{\frac{1}{6}} f(-q) &= (kk')^{\frac{1}{12}} \phi(q) \\ 2^{\frac{1}{3}} f(q^2) &= (kk')^{\frac{1}{6}} \phi(q) \\ 2^{\frac{2}{3}} f(q^4) &= k^{\frac{1}{3}} k'^{\frac{1}{12}} \phi(q) \end{aligned}\right\} \tag{61}$$

It follows from (41), (60) and (61) that, if $r+s$ is of the form $4m+2$, but not equal to 2 or to 6, then

$$F_{r,s}(q^2) = \frac{f^{4(r+s-4)}(-q)}{f^{2(r+s-10)}(q^2)} \sum_{1}^{\frac{1}{4}(r+s-6)} K_n \frac{f^{24n}(q^2)}{f^{24n}(-q)}, \tag{62}$$

and if $r+s$ is of the form $4m$, but not equal to 4, 8 or 12, then

$$F_{r,s}(q^2) = \frac{f^{4(r+s-6)}(-q)}{f^{2(r+s-10)}(q^2)} \{f^8(q) - 16 f^8(q^4)\} \sum_{1}^{\frac{1}{4}(r+s-8)} K_n \frac{f^{24n}(q^2)}{f^{24n}(-q)}, \tag{63}$$

when K_n depends on r and s only. Hence it is easy to see that in all cases $F_{r,s}(q^2)$ can be expressed as

$$\begin{aligned} \sum K_{a,b,c,d,e,h,k} \{f^3(-q)\}^a \left\{\frac{f^5(-q)}{f^2(q^2)}\right\}^b \left\{\frac{f^5(q^2)}{f^2(-q)}\right\}^c \left\{\frac{f^5(q)}{f^2(q^2)} f^3(q)\right\}^d \\ \times \left\{\frac{f^5(q^4)}{f^2(q^2)} f^3(q^4)\right\}^e f^h(-q) f^k(q^2), \end{aligned} \tag{64}$$

where a, b, c, d, e, h, k are zero or positive integers such that

$$\begin{aligned} a + b + c + 2(d+e) &= [\tfrac{2}{3}(r+s+2)], \\ h + k &= 2(r+s+2) - 3[\tfrac{2}{3}(r+s+2)], \end{aligned}$$

and $[x]$ denotes as usual the greatest integer in x. But

$$\left.\begin{aligned} f(q) &= q^{\frac{1^2}{24}} - q^{\frac{5^2}{24}} - q^{\frac{7^2}{24}} + q^{\frac{11^2}{24}} + \cdots \\ f^3(q) &= q^{\frac{1^2}{8}} - 3q^{\frac{3^2}{8}} + 5q^{\frac{5^2}{8}} - 7q^{\frac{7^2}{8}} + \cdots \\ \frac{f^5(q)}{f^2(q^2)} &= q^{\frac{1^2}{24}} - 5q^{\frac{5^2}{24}} + 7q^{\frac{7^2}{24}} - 11q^{\frac{11^2}{24}} + \cdots \\ \frac{f^5(q^2)}{f^2(-q)} &= q^{\frac{1^2}{3}} - 2q^{\frac{2^2}{3}} + 4q^{\frac{4^2}{3}} - 5q^{\frac{5^2}{3}} + \cdots \end{aligned}\right\}, \tag{65}$$

where $1, 2, 4, 5, \cdots$ are the natural numbers without the multiples of 3, and $1, 5, 7, 11, \cdots$ are the natural odd numbers without the multiples of 3.
Hence it is easy to see that

$$n^{-\frac{1}{2}(a+b+c)-d-e}E_{r,s}(n)$$

is not of higher order than the coefficient of q^{2n} in

$$\phi^a(q^{\frac{1}{8}})\phi^b(q^{\frac{1}{24}})\phi^c(q^{\frac{1}{3}})\{\phi(q^{\frac{1}{24}})\phi(q^{\frac{1}{8}})\}^d\{\phi(q^{\frac{2}{3}})\phi(q^{\frac{1}{2}})\}^e\phi^h(q^{\frac{1}{24}})\phi^k(q^{\frac{1}{12}}),$$

or the coefficient of q^{48n} in

$$\phi^{a+d}(q^3)\phi^{b+d+h}(q)\phi^c(q^8)\phi^e(q^{16})\phi^e(q^{12})\phi^k(q^2).$$

But the coefficient of q^ν in $\phi^2(q^2)$ cannot exceed that of q^ν in $\phi^2(q)$, since

$$\phi^2(q) + \phi^2(-q) = 2\phi^2(q^2); \tag{66}$$

and it is evident that the coefficient of q^ν in $\phi(q^{4\lambda})$ cannot exceed that of q^ν in $\phi(q^\lambda)$. Hence it follows that

$$n^{-\frac{1}{2}[\frac{2}{3}(r+s+2)]}E_{r,s}(n)$$

is not of higher order than the coefficient of q^{48n} in

$$\phi^A(q)\phi^B(q^3)\phi^C(q^2),$$

where A, B, C are zero or positive integers such that

$$A + B + C = 2(r+s+2) - 2[\tfrac{2}{3}(r+s+2)],$$

and C is 0 or 1.
Now, if $r + s \geq 14$, we have

$$A + B + C \geq 12,$$

and so

$$A + B \geq 11.$$

Therefore one at least of A and B is greater than 5. But

$$\phi^6(q) = \sum_0^\infty O(\nu^2)q^\nu.^* \tag{67}$$

Hence it is easily deduced that

$$\phi^A(q)\phi^B(q^3)\phi^C(q^2) = \sum O\{\nu^{\frac{1}{2}(A+B+C)-1}\}q^\nu. \tag{68}$$

*See §§24–25.

It follows that

$$E_{r,s}(n) = O\{n^{r+s-\frac{1}{2}[\frac{2}{3}(r+s-1)]}\}, \tag{69}$$

If $r+s \geq 14$. We have already shewn in § 11 that, if $r+s=10$, then

$$E_{r,s}(n) = O(n^7). \tag{70}$$

This agrees with (69). Thus we see that in all cases

$$E_{r,s}(n) = O\{n^{\frac{2}{3}(r+s+1)}\}; \tag{71}$$

and that, if $r+s$ is of the form $6m$, then

$$E_{r,s}(n) = O\{n^{\frac{2}{3}(r+s+\frac{3}{4})}\}, \tag{72}$$

and if of the form $6m+4$, then

$$E_{r,s}(n) = O\{n^{\frac{2}{3}(r+s+\frac{1}{2})}\}. \tag{73}$$

14. I shall now prove that the order of $E_{r,s}(n)$ is not less than that of $n^{\frac{1}{2}(r+s)}$. In order to prove this result I shall follow the method used by Messrs Hardy and Littlewood in their paper "Some problems of Diophantine approximation" (II) *.
Let

$$q = e^{\pi i\tau}, q' = e^{\pi i T},$$

where

$$T = \frac{c+d\tau}{a+b\tau},$$

and

$$ad - bc = 1.$$

Also let

$$V = \frac{v}{a+b\tau}.$$

Then we have

$$\omega\sqrt{v}e^{\pi ibvV}\vartheta_1(v,\tau) = \sqrt{V}\vartheta_1(V,T), \tag{74}$$

where ω is an eighth root of unity and

$$\vartheta_1(v,\tau) = 2\sin\pi v \cdot q^{\frac{1}{4}}\Pi_1^{\infty}(1-q^{2n})(1-2q^{2n}\cos 2\pi v + q^{4n}). \tag{75}$$

* *Acta Mathematica*, Vol. XXXVII, pp. 193 – 238.

From (75) we have

$$\log \vartheta_1(v,\tau) = \log(2, \sin \pi v) + \tfrac{1}{4}\log q - \sum_1^\infty \frac{q^{2n}(1+2\cos 2n\pi v)}{n(1-q^{2n})}. \tag{76}$$

It follows from (74) and (76) that

$$\begin{aligned} &\log \sin \pi v + \tfrac{1}{2}\log v + \tfrac{1}{4}\log q + \log \omega - \sum_1^\infty \frac{q^{2n}(1+2\cos 2n\pi v)}{n(1-q^{2n})} \\ &= \log \sin \pi V + \tfrac{1}{2}\log V + \tfrac{1}{4}\log q' - \pi i b v V - \sum_1^\infty \frac{{q'}^{2n}(1+2\cos 2n\pi V)}{n(1-{q'}^{2n})}. \end{aligned} \tag{77}$$

Equating the coefficients of v^{8+1} on the two sides of (77), we obtain

$$\begin{aligned} &(a+b\tau)^{s+1}\left\{\tfrac{1}{2}\zeta(-s) + \frac{1^s q^2}{1-q^2} + \frac{2^s q^4}{1-q^4} + \frac{3^s q^6}{1-q^6} + \cdots\right\} \\ &= \tfrac{1}{2}\zeta(-s) + \frac{1^s q'^2}{1-q'^2} + \frac{2^s q'^4}{1-q'^4} + \frac{3^s q'^6}{1-q'^6} + \cdots, \end{aligned} \tag{78}$$

provided that s is an odd integer greater than 1. If, in particular,we put $s=3$ and $s=5$ in (78) we obtain

$$\begin{aligned} &(a+b\tau)^4\left\{1 + 240\left(\frac{1^3 q^2}{1-q^2} + \frac{2^3 q^4}{1-q^4} + \frac{3^3 q^6}{1-q^6} + \cdots\right)\right\} \\ &= \left\{1 + 240\left(\frac{1^3 q'^2}{1-q'^2} + \frac{2^3 q'^4}{1-q'^4} + \frac{3^3 q'^6}{1-q'^6} + \cdots\right)\right\}, \end{aligned} \tag{79}$$

and

$$\begin{aligned} &(a+b\tau)^6\left\{1 - 504\left(\frac{1^5 q^2}{1-q^2} + \frac{2^5 q^4}{1-q^4} + \frac{3^5 q^6}{1-q^6} + \cdots\right)\right\} \\ &= \left\{1 - 504\left(\frac{1^5 q'^2}{1-q'^2} + \frac{2^5 q'^4}{1-q'^4} + \frac{3^5 q'^6}{1-q'^6} + \cdots\right)\right\}. \end{aligned} \tag{80}$$

It follows from (38), (79) and (80) that

$$(a+b\tau)^{r+s+2} F_{r,s}(q^2) = F_{r,s}(q'^2). \tag{81}$$

It can easily be seen from (56) and (37) that

$$F_{r,s}(x) = \sum_1^\infty E_{r,s}(n) x^n. \tag{82}$$

Hence

$$(a+b\tau)^{r+s+2}\sum_{1}^{\infty}E_{r,s}(n)q^{2n}=\sum_{1}^{\infty}E_{r,s}(n)q'^{2n}. \tag{83}$$

It is important to observe that

$$\begin{aligned}E_{r,s}(1)=&\frac{\zeta(-r)+\zeta(-s)}{2}-\frac{\zeta(1-r)+\zeta(1-s)}{r+s}\\&-\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\neq 0,\end{aligned} \tag{84}$$

if $r+s$ is not equal to 2,4,6,8 or 12. This is easily proved by the help of the equation (39).

15. Now let

$$\tau=u+iy, t=e^{-\pi y}(u>0,y>0,0<t<1),$$

so that

$$q=e^{\pi iu-\pi y}=te^{\pi iu};$$

and let us suppose that p_n/q_n is a convergent to

$$u=\frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+\cdots},$$

so that

$$\eta_n=p_{n-1}q_n-p_nq_{n-1}=\pm 1.$$

Further, let us suppose that

$$a=p_n, \qquad b=-q_n,$$

$$c=\eta_n p_{n-1}, \quad d=-\eta_n q_{n-1},$$

so that

$$ad-bc=\eta_n^2=1.$$

Furthermore, let

$$y=1/(q_nq'_{n+1}),$$

where

$$q'_{n+1}=a'_{n+1}q_n+q_{n-1},$$

and a'_{n+1} is the complete quotient corresponding to a_{n+1}.
Then we have

$$|a+b\tau|=|p_n-q_nu-iq_ny|=\frac{|\pm 1-i|}{q'_{n+1}}=\frac{\sqrt{2}}{q'_{n+1}}, \tag{85}$$

and

$$|q'|=e^{-\pi\lambda},$$

where

$$\lambda = \mathbf{I}(T) = \mathbf{I}\left(\frac{c+d\tau}{a+b\tau}\right) + \mathbf{I}\left\{\frac{d}{b} - \frac{1}{b(a+b\tau)}\right\}$$
$$= \frac{y}{(1/q'_{n+1})^2 + qn^2y^2} = \frac{q'_{n+1}}{2q_n}, \tag{86}$$

and $\mathbf{I}(T)$ is the imaginary part of T. It follows from (83), (85) and (86) that

$$|\sum_1^\infty E_{r,s}(n)q^{2n}| = \left(\frac{q'_{n+1}}{\sqrt{2}}\right)^{r+s+2} |\sum_1^\infty E_{r,s}(n)q'^{2n}|$$
$$\geq \left(\frac{q'_{n+1}}{\sqrt{2}}\right)^{r+s+2} \{|E_{r,s}(1)|e^{-2\pi\lambda} - |E_{r,s}(2)e^{-4\pi\lambda} - |E_{r,s}(3)|e^{-6\pi\lambda} - \cdots\}. \tag{87}$$

We can choose a number λ_0, depending only on r and s, such that

$$|E_{r,s}(1)|e^{-2\pi\lambda} > 2\{|E_{r,s}(2)|e^{-4\pi\lambda} + |E_{r,s}(3)|e^{-6\pi\lambda} + \cdots\}$$

for $\lambda \geq \lambda_0$. Let us suppose $\lambda_0 > 10$. Let us also suppose that the continued fraction for u satisfies the condition

$$4\lambda_0 q_n > q'_{n+1} > 2\lambda_0 q_n \tag{88}$$

for an infinity of values of n. Then

$$\left|\sum_1^\infty E_{r,s}(n)q^{2n}\right| \geq \tfrac{1}{2}|E_{r,s}(1)| \left(\frac{q'_{n+1}}{\sqrt{2}}\right)^{r+s+2} e^{-4\pi\lambda_0} > K(q'_{n+1})^{r+s+2}, \tag{89}$$

where K depends on r and s only. Also

$$q_n q'_{n+1} = 1/y,$$

$$q'_{n+1} > \frac{1}{\sqrt{y}} = \sqrt{\left\{\frac{\pi}{\log(1/t)}\right\}} > \frac{K}{\sqrt{(1-t)}}.$$

It follows that, if u is an irrational number such that the condition (88) is satisfied for an infinity of values of n, then

$$\left|\sum_1^\infty E_{r,s}(n)q^{2n}\right| > K(1-t)^{-\frac{1}{2}(r+s+2)} \tag{90}$$

for an infinity of values of t tending to unity. But if we had

$$E_{r,s}(n) = o\{n^{\frac{1}{2}(r+s)}\}$$

then we should have

$$|\sum_{1}^{\infty} E_{r,s}(n)q^{2n}| = o\{(1-t)^{-\frac{1}{2}(r+s+2)}\},$$

which contradicts (90). It follows that the error term in $\sum_{r,s}(n)$ is not of the form

$$o\{n^{\frac{1}{2}(r+s)}\}. \tag{91}$$

The arithmetical function $\tau(n)$.

16. We have seen that

$$E_{r,s}(n) = 0,$$

if $r+s$ is equal to 2,4,6,8, or 12. In these cases $\sum_{r,s}(n)$ has a finite expression in terms of $\sigma_{r+s+1}(n)$ and $\sigma_{r+s-1}(n)$. In other cases $\sum_{r,s}(n)$ involves other arithmetical functions as well. The simplest of these is the function $\tau(n)$ defined by

$$\sum_{1}^{\infty} \tau(n)x^n = x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}. \tag{92}$$

These cases arise when $r+s$ has one of the values 10, 14, 16, 18, 20 or 24.
Suppose that $r+s$ has one of these values. Then

$$\frac{1728\sum_{1}^{\infty} E_{r,s}(n)x^n}{(Q^3-R^2)E_{r,s}(1)}$$

is, by (41) and (82), equal to the corresponding one of the functions

$$1, Q, R, Q^2, QR, Q^2R.$$

In other words

$$\sum_{1}^{\infty} E_{r,s}(n)x^n = E_{r,s}(1)\sum_{1}^{\infty} \tau(n)x^n$$
$$\left\{1 + \frac{2}{\zeta(11-r-s)}\sum_{1}^{\infty} n^{r+s-11}\frac{x^n}{1-x^n}\right\}. \tag{93}$$

We thus deduce the formulæ

$$E_{r,s}(n) = E_{r,s}(1)\tau(n), \tag{94}$$

if $r+s=10$; and

$$\sigma_{r+s-11}(0)E_{r,s}(n) = E_{r,s}(1)\{\sigma_{r+s-11}(0)\tau(n)$$

$$+\sigma_{r+s-11}(1)\tau(n-1)+\cdots+\sigma_{r+s-11}(n-1)\tau(1)\}, \tag{95}$$

if $r+s$ is equal to 14, 16, 18, 20 or 24. It follows from (94) and (95) that, if $r+s=r'+s'$, then

$$E_{r,s}(n)E_{r',s'}(1)=E_{r,s}(1)E_{r',s'}(n), \tag{96}$$

and in general

$$E_{r,s}(m)E_{r',s'}(n)=E_{r,s}(n)E_{r',s'}(m), \tag{97}$$

when $r+s$ has one of the values in question. The different cases in which $r+s$ has the same value are therefore not fundamentally distinct.

17. The values of $\tau(n)$ may be calculated as follows: differentiating (92) logarithmically with respect to x, we obtain

$$\sum_1^\infty n\tau(n)x^n = P\sum_1^\infty \tau(n)x^n. \tag{98}$$

Equating the coefficients of x^n in both sides in (98),we have

$$\tau(n)=\frac{24}{1-n}\{\sigma_1(1)\tau(n-1)+\sigma_1(2)\tau(n-2)+\cdots+\sigma_1(n-1)\tau(1)\}. \tag{99}$$

If, instead of starting with (92), we start with

$$\sum_1^\infty \tau(n)x^n = x(1-3x+5x^3-7x^6+\cdots)^8,$$

we can shew that

$$\begin{aligned}&(n-1)\tau(n)-3(n-10)\tau(n-1)+5(n-28)\tau(n-3)-7\\&(n-55)\tau(n-6)+\cdots \text{ to } [\tfrac{1}{2}\{1+\sqrt{(8n-7)}\}] \text{ terms } =0,\end{aligned} \tag{100}$$

where the rth term of the sequence 0,1,3,6, $\ldots$ is $\frac{1}{2}r(r-1)$, and the rth term of the sequence 1,10,28,55, $\ldots$ is $1+\frac{9}{2}r(r-1)$. We thus obtain the values of $\tau(n)$ in the following table.

TABLE V

n	$\tau(n)$	n	$\tau(n)$
1	$+1$	16	$+987136$
2	-24	17	-6905934
3	$+252$	18	$+2727432$
4	-1472	19	$+10661420$
5	$+4830$	20	-7109760
6	-6048	21	-4219488
7	-16744	22	-12830688
8	$+84480$	23	$+18643272$
9	-113643	24	$+21288960$
10	-115920	25	-25499225
11	$+534612$	26	$+13865712$
12	-370944	27	-73279080
13	-577738	28	$+24647168$
14	$+401856$	29	$+128406630$
15	$+1217160$	30	-29211840

18. Let us consider more particularly the case in which $r+s=10$. The order of $E_{r,s}(n)$ is then the same as that of $\tau(n)$. The determination of this order is a problem interesting in itself. We have proved that $E_{r,s}(n)$, and therefore $\tau(n)$, is of the form $O(n^7)$ and not of the form $o(n^5)$. There is reason for supposing that $\tau(n)$ is of the form $O(n^{\frac{11}{2}+\epsilon})$ and not of the form $o(n^{\frac{11}{2}})$. For it appears that

$$\sum_1^\infty \frac{\tau(n)}{n^t} = \prod_p \frac{1}{1-\tau(p)p^{-t}+p^{11-2t}}. \tag{101}$$

This assertion is equivalent to the assertion that, if

$$n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_r^{a_r},$$

where $p_1, p_2, \ldots, p_r$ are the prime divisors of n, then

$$n^{-\frac{11}{2}}\tau(n) = \frac{\sin(1+a_1)\theta_{p_1}}{\sin\theta_{p_1}} \frac{\sin(1+a_2)\theta_{p_2}}{\sin\theta_{p_2}} \cdots \frac{\sin(1+a_r)\theta_{p_r}}{\sin\theta_{p_r}}, \tag{102}$$

where

$$\cos\theta_p = \tfrac{1}{2}p^{-\frac{11}{2}}\tau(p).$$

It would follow that, if n and n' are prime to each other, we must have

$$\tau(nn') = \tau(n)\tau(n'). \tag{103}$$

Let us suppose that (102) is true, and also that (as appears to be highly probable)

$$\{2\tau(p)\}^2 \le p^{11}, \tag{104}$$

so that θ_p is real. Then it follows from (102) that

$$n^{-\frac{11}{2}}|\tau(n)| \leq (1+a_1)(1+a_2)\cdots(1+a_r),$$

that is to say

$$|\tau(n)| \leq n^{\frac{11}{2}} d(n), \tag{105}$$

where $d(n)$ denotes the number of divisors of n.
Now let us suppose that $n = p^a$, so that

$$n^{-\frac{11}{2}}\tau(n) = \frac{\sin(1+a)\theta_p}{\sin\theta_p}.$$

Then we can choose a as large as we please and such that

$$\left|\frac{\sin(1+a)\theta_p}{\sin\theta_p}\right| \geq 1.$$

Hence

$$|\tau(n)| \geq n^{\frac{11}{2}} \tag{106}$$

for an infinity of values of n.

19. It should be observed that precisely similar questions arise with regard to the arithmetical function $\Psi(n)$ defined by

$$\sum_0^\infty \Psi(n)x^n = f^{a_1}(x^{c_1})f^{a_2}(x^{c_2})\cdots f^{a_r}(x^{c_r}), \tag{107}$$

where

$$f(x) = x^{\frac{1}{24}}(1-x)(1-x^2)(1-x^3)\cdots,$$

the a's and c's are integers, the latter being positive,

$$\tfrac{1}{24}(a_1c_1 - a_2c_2 + \cdots + a_rc_r)$$

is equal to 0 or 1, and

$$l\left(\frac{a_1}{c_1} + \frac{a_2}{c_2} + \cdots + \frac{a_r}{c_r}\right),$$

where l is the least common multiple of $c_1, c_2, \ldots, c_r$, is equal to 0 or to a divisor of 24. The arithmetical functions $\chi(n), P(n), \chi_4(n), \Omega(n)$ and $\Theta(n)$, studied by Dr. Glaisher in the *Quarterly Journal*, Vols. XXXVI-XXXVIII, are of this type. Thus

$$\sum_1^\infty \chi(n)x^n \quad = \quad f^6(x^4),$$

$$\begin{aligned}
\sum_1^\infty P(n)x^n &= f^4(x^2)f^4(x^4),\\
\sum_1^\infty \chi_4(n)x^n &= f^4(x)f^2(x^2)f^4(x^4),\\
\sum_1^\infty \Omega(n)x^n &= f^{12}(x^2),\\
\sum_1^\infty \Theta(n)x^n &= f^8(x)f^8(x^2).
\end{aligned}$$

20. The results (101) and (104) may be written as

$$\sum_1^\infty \frac{E_{r,s}(n)}{n^t} = E_{r,s}(1)\prod_p \frac{1}{1-2c_p p^{-t}+p^{r+s+1-2t}}, \tag{108}$$

where

$$c_p^2 \le p^{r+s+1},$$

and

$$2c_pE_{r,s}(1) = E_{r,s}(p).$$

It seems probable that the result (108) is true not only for $r+s=10$ but also when $r+s$ is equal to 14, 16, 18, 20 or 24, and that

$$\left|\frac{E_{r,s}(n)}{E_{r,s}(1)}\right| \le n^{\frac{1}{2}(r+s+1)}d(n) \tag{109}$$

for all values of n, and

$$\left|\frac{E_{r,s}(n)}{E_{r,s}(1)}\right| \ge n^{\frac{1}{2}(r+s+1)} \tag{110}$$

for an infinity of values of n. If this be so, then

$$E_{r,s}(n) = O\{n^{\frac{1}{2}(r+s+1+\epsilon)}\}, E_{r,s}(n) \ne o\{n^{\frac{1}{2}(r+s+1)}\}. \tag{111}$$

And it seems very likely that these equations hold generally, whenever r and s are positive odd integers.

21. It is of some interest to see what confirmation of these conjectures can be found from a study of the coefficients in the expansion of

$$x\{(1-x^{24/\alpha})(1-x^{48/\alpha})(1-x^{72/\alpha})\cdots\}^a = \sum_1^\infty \Psi_\alpha(n)x^n,$$

where α is a divisor of 24. When $\alpha = 1$ and $\alpha = 3$ we know the actual value of $\Psi_\alpha(n)$. For we have

$$\sum_1^\infty \Psi_1(n)x^n = x^{1^2} - x^{5^2} - x^{7^2} + x^{11^2} + x^{13^2} - x^{17^2} - \cdots, \tag{112}$$

where 1, 5, 7, 11, ... the natural odd numbers without the multiples of 3; and

$$\sum_1^\infty \Psi_3(n)x^n = x^{1^2} - 3x^{3^2} + 5x^{5^2} - 7x^{7^2} + \cdots \tag{113}$$

The corresponding Dirichlet's series are

$$\sum_1^\infty \frac{\Psi_1(n)}{n^s} = \frac{1}{(1+5^{-2s})(1+7^{-2s})(1-11^{-2s})(1-13^{-2s})\cdots}, \tag{114}$$

where 5, 7, 11, 13, ... are the primes greater than 3, those of the form $12n \pm 5$ having the plus sign and those of the form $12n \pm 1$ the minus sign; and

$$\sum_1^\infty \frac{\Psi_3(n)}{n^s} = \frac{1}{(1+3^{1-2s})(1-5^{1-2s})(1+7^{1-2s})(1+11^{1-2s})\cdots} \tag{115}$$

where 3, 5, 7, 11, ... are the odd primes, those of the form $4n - 1$ having the plus sign and those of the form $4n + 1$ the minus sign.
It is easy to see that

$$|\Psi_1(n)| \le 1, \quad |\Psi_3(n)| \le \sqrt{n} \tag{116}$$

for all values of n, and

$$|\Psi_1(n)| = 1, \quad |\Psi_3(n)| = \sqrt{n} \tag{117}$$

for an infinity of values of n.
The next simplest case is that in which $\alpha = 2$. In this case it appears that

$$\sum_1^\infty \frac{\Psi_2(n)}{n^s} = \Pi_1\Pi_2, \tag{118}$$

where

$$\Pi_1 = \frac{1}{(1+5^{-2s})(1-7^{-2s})(1-11^{-2s})(1+17^{-2s})\cdots},$$

5, 7, 11, ... being the primes of the forms $12n - 1$ and $12n \pm 5$, those of the form $12n + 5$ having the plus sign and the rest the minus sign; and

$$\Pi_2 = \frac{1}{(1+13^{-s})^2(1-37^{-s})^2(1-61^{-s})^2(1+73^{-s})^2\cdots},$$

13, 37, 61, ... being the primes of the form $12n+1$, those of the form $m^2+(6n-3)^2$ having the plus sign and those of the form $m^2+(6n)^2$ the minus sign.
This is equivalent to the assertion that if

$$n = (5^{a_5}\cdot 7^{a_7}\cdot 11^{a_{11}}\cdot 17^{a_{17}}\cdots)^2 13^{a_{13}}\cdot 37^{a_{37}}\cdot 61^{a_{61}}\cdot 73^{a_{73}}\cdots,$$

where a_p is zero or a positive integer, then

$$\Psi_2(n) = (-1)^{a_5+a_{13}+a_{17}+a_{29}+a_{41}+\cdots}(1+a_{13})(1+a_{37})(1+a_{61})\cdots, \tag{119}$$

where 5, 13, 17, 29, ... are the primes of the form $4n+1$, excluding those of the form $m^2+(6n)^2$; and that otherwise

$$\Psi_2(n) = 0. \tag{120}$$

It follows that

$$|\Psi_2(n)| \leq d(n) \tag{121}$$

for all values of n, and

$$|\Psi_2(n)| \geq 1 \tag{122}$$

for an infinity of values of n. These results are easily proved to be actually true.

22. I have investigated also the cases in which α has one of the values 4, 6, 8 or 12. Thus for example, when $\alpha = 6$, I find

$$\sum_1^\infty \frac{\Psi_6(n)}{n^s} = \Pi_1\Pi_2,^* \tag{123}$$

where

$$\Pi_1 = \frac{1}{(1-3^{2-2s})(1-7^{2-2s})(1-11^{2-2s})\cdots},$$

3, 7, 11, ... being the primes of the form $4n-1$; and

$$\Pi_2 = \frac{1}{(1-2c_5\cdot 5^{-s}+5^{2-2s})(1-2c_{13}\cdot 13^{-s}+13^{2-2s})\cdots},$$

5, 13, 17, ... being the primes of the form $4n+1$, and $c_p = u^2-(2v)^2$, where u and v are the unique pair of positive integers for which $p = u^2+(2v)^2$. This is equivalent to the assertion that if

$$n = (3^{a_3}\cdot 7^{a_7}\cdot 11^{a_{11}}\cdots)^2\cdot 5^{a_5}\cdot 13^{a_{13}}\cdot 17^{a_{17}}\cdots,$$

$^*\Psi_6(n)$ is Dr. Glaisher's $\lambda(n)$.

then

$$\frac{\Psi_6(n)}{n} = \frac{\sin(1+a_5)\theta_5}{\sin\theta_5} \cdot \frac{\sin(1+a_{13})\theta_{13}}{\sin\theta_{13}} \cdot \frac{\sin(1+a_{17})\theta_{17}}{\sin\theta_{17}} \cdots, \tag{124}$$

where

$$\tan\tfrac{1}{2}\theta_p = \frac{u}{2v} \quad (0 < \theta_p < \pi),$$

and that otherwise $\Psi_6(n) = 0$. From these results it would follow that

$$|\Psi_6(n)| \leq nd(n) \tag{125}$$

for all values of n, and

$$|\Psi_6(n)| \geq n \tag{126}$$

for an infinity of values of n. What can actually be proved to be true is that

$$|\Psi_6(n) < 2nd(n)$$

for all values of n, and

$$|\Psi_6(n)| \geq n$$

for an infinity of values of n.

23. In the case in which $\alpha = 4$ I find that, if

$$n = (5^{a_5} \cdot 11^{a_{11}} \cdot 17^{a_{17}} \cdots)^2 \cdot 7^{a_7} \cdot 13^{a_{13}} \cdot 19^{a_{19}} \cdots,$$

where 5, 11, 17, ... are the primes of the form $6m-1$ and 7, 13, 19, ... are those of the form $6m+1$, then

$$\frac{\Psi_4(n)}{\sqrt{n}} = (-1)^{a_5+a_{11}+a_{17}+\cdots} \frac{\sin(1+a_7)\theta_7}{\sin\theta_7} \cdot \frac{\sin(1+a_{13})\theta_{13}}{\sin\theta_{13}} \cdots, \tag{127}$$

where

$$\tan\theta_p = \frac{u\sqrt{3}}{1 \pm 3v} \quad (0 < \theta_p < \pi),$$

and u and v are the unique pair of positive integers for which $p = 3u^2 + (1 \pm 3v)^2$; and that $\Psi_4(n) = 0$ for other values.
In the case in which $\alpha = 8$ I find that, if

$$n = (2^{a_2} \cdot 5^{a_5} \cdot 11^{a_{11}} \cdots)^2 \cdot 7^{a_7} \cdot 13^{a_{13}} \cdot 19^{a_{19}} \cdots,$$

where 2, 5, 11, ... are the primes of the form $3m-1$ and 7, 13, 19, ... are those of the form $6m+1$, then

$$\frac{\Psi_8(n)}{n\sqrt{n}} = (-1)^{a_2+a_5+a_{11}+\cdots} \frac{\sin 3(1+a_7)\theta_7}{\sin 3\theta_7} \cdot \frac{\sin 3(1+a_{13})\theta_{13}}{\sin 3\theta_{13}} \cdots, \tag{128}$$

where θ_p is the same as in (127); and that $\Psi_8(n) = 0$ for other values.
The case in which $\alpha = 12$ will be considered in § 28.
In short, such evidence as I have been able to find, while not conclusive, points to the truth of the results conjectured in § 18.

24. Analysis similar to that of the preceding sections may be applied to some interesting arithmetical functions of a different kind. Let

$$\phi^s(q) = 1 + 2\sum_1^\infty r_s(n)q^n, \tag{129}$$

where

$$\phi(q) = 1 + 2q + 2q^4 + 2q^9 + \cdots,$$

so that $r_s(n)$ is the number of representations of n as the sum of s squares. Further let

$$\begin{aligned}\sum_1^\infty \delta_2(n)q^n &= 2\left(\frac{q}{1-q} - \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} - \cdots\right)\\ &= 2\left(\frac{q}{1+q^2} + \frac{q^2}{1+q^4} + \frac{q^3}{1+q^6} + \cdots\right);\end{aligned} \tag{130}$$

$$(2^s-1)B_s\sum_1^\infty \delta_{2s}(n)q^n = s\left(\frac{1^{s-1}q}{1+q} + \frac{2^{s-1}q^2}{1-q^2} + \frac{3^{s-1}q^3}{1+q^3} + \cdots\right), \tag{131}$$

when s is a multiple of 4;

$$(2^s-1)B_s\sum_1^\infty \delta_{2s}(n)q^n = s\left(\frac{1^{s-1}q}{1-q} + \frac{2^{s-1}q^2}{1+q^2} + \frac{3^{s-1}q^3}{1-q^3} + \cdots\right), \tag{132}$$

when $s+2$ is a multiple of 4;

$$\begin{aligned}E_s\sum_1^\infty \delta_{2s}(n)q^n = 2^s&\left(\frac{1^{s-1}q}{1+q^2} + \frac{2^{s-1}q^2}{1+q^4} + \frac{3^{s-1}q^3}{1+q^6} + \cdots\right)\\ &+2\left(\frac{1^{s-1}q}{1-q}\frac{3^{s-1}q^3}{1-q^3} + \frac{5^{s-1}q^5}{1-q^5} - \cdots\right),\end{aligned} \tag{133}$$

when $s-1$ is a multiple of 4;

$$\begin{aligned}E_s\sum_1^\infty \delta_{2s}(n)q^n = 2^s&\left(\frac{1^{s-1}q}{1+q^2} + \frac{2^{s-1}q^2}{1+q^4} + \frac{3^{s-1}q^3}{1+q^6} + \cdots\right)\\ &-2\left(\frac{1^{s-1}q}{1-q} - \frac{3^{s-1}q^3}{1-q^3} + \frac{5^{s-1}q^5}{1-q^5} - \cdots\right),\end{aligned} \tag{134}$$

when $s+1$ is a multiple of 4. In these formulæ

$$B_2 = \tfrac{1}{6}, B_4 = \tfrac{1}{30}, B_6 = \tfrac{1}{42}, B_8 = \tfrac{1}{30}, B_{10} = \tfrac{5}{66}, \dots$$

are Bernoulli's numbers, and

$$E_1 = 1, E_3 = 1, E_5 = 5, E_7 = 61, E_9 = 1385, \dots$$

are Euler's numbers. Then $\delta_{2s}(n)$ is in all cases an arithmetical function depending on the real divisors of n; thus, for example, when $s+2$ is a multiple of 4, we have

$$(2^s - 1)B_s\delta_{2s}(n) = s\{\sigma_{s-1}(n) - 2^s\sigma_{s-1}(\tfrac{1}{4}n)\}, \tag{135}$$

where $\sigma_s(x)$ should be considered as equal to zero if x is not an integer.
Now let

$$r_{2s}(n) = \delta_{2s}(n) + e_{2s}(n). \tag{136}$$

Then I can prove (see § 26) that

$$e_{2s}(n) = 0 \tag{137}$$

if $s = 1, 2, 3, 4$ and that

$$e_{2s}(n) = O(n^{s-1-\frac{1}{2}[\frac{2}{3}s]+\epsilon}) \tag{138}$$

for all positive integral values of s. But it is easy to see that, if $s \geq 3$, then

$$Hn^{s-1} < \delta_{2s}(n) < Kn^{s-1}, \tag{139}$$

where H and K are positive constants. It follows that

$$r_{2s}(n) \sim \delta_{2s}(n) \tag{140}$$

for all positive integral values of s.
It appears probable, from the empirical results I obtain at the end of this paper, that

$$e_{2s}(n) = O\{n^{\frac{1}{2}(s-1)+\epsilon}\} \tag{141}$$

for all positive integral values of s; and that

$$e_{2s}(n) \neq o\{n^{\frac{1}{2}(s-1)}\} \tag{142}$$

if $s \geq 5$. But all that I can actually prove is that

$$e_{2s}(n) = O(n^{s-1-\frac{1}{2}[\frac{2}{3}s]}) \tag{143}$$

if $s \geq 9$ and that

$$e_{2s}(n) \neq o(n^{\frac{1}{2}s-1}) \tag{144}$$

if $s \geq 5$.

25. Let

$$f_{2s}(q) = \sum_1^\infty e_{2s}(n)q^n = \sum_1^\infty \{r_{2s}(n) - \delta_{2s}(n)\}q^n. \tag{145}$$

Then it can be shewn by the theory of elliptic functions that

$$f_{2s}(q) = \phi^{2s}(q) \sum_{1 \leq n \leq \frac{1}{4}(s-1)} K_n (kk')^{2n}, \tag{146}$$

that is to say that

$$f_{2s}(q) = \frac{f^{4s}(-q)}{f^{2s}(q^2)} \sum_{1 \leq n \leq \frac{1}{4}(s-1)} K_n \frac{f^{24n}(q^2)}{f^{24n}(-q)}, \tag{147}$$

where $\phi(q)$ and $f(q)$ are the same as in § 13. We thus obtain the results contained in the following table.

TABLE VI

1. $f_2(q) = 0, \quad f_4(q) = 0, \quad f_6(q) = 0, \quad f_8(q) = 0.$
2. $5f_{10}(q) = 16\frac{f^{14}(q^2)}{f^4(-q)}, \quad f_{12}(q) = 8f^{12}(q^2).$
3. $61f_{14}(q) = 728f^4(-q)f^{10}(q^2), \quad 17f_{16}(q) = 256f^8(-q)f^8(q^2).$
4. $1385f_{18}(q) = 24416f^{12}(-q)f^6(q^2) - 256\frac{f^{30}(q^2)}{f^{12}(-q)}.$
5. $31f_{20}(q) = 616f^{16}(-q)f^4(q^2) - 128\frac{f^{28}(q^2)}{f^8(-q)}.$
6. $50521f_{22}(q) = 1103272f^{20}(-q)f^2(q^2) - 821888\frac{f^{26}(q^2)}{f^4(-q)}.$
7. $691f_{24}(q) = 16576f^{24}(-q) - 32768f^{24}(q^2).$

It follows from the last formula of Table VI that

$$\tfrac{691}{64}e_{24}(n) = (-1)^{n-1}259\tau(n) - 512\tau(\tfrac{1}{2}n), \tag{148}$$

where $\tau(n)$ is the same as in § 16, and $\tau(x)$ should be considered as equal to zero if x is not an integer.

Results equivalent to 1,2,3,4 of Table VI were given by Dr. Glaisher in the *Quarterly Journal*, Vol. XXXVIII. The arithmetical functions called by him

$$\chi_4(n),\ \Omega(n),\ W(n),\ \Theta(n),\ U(n)$$

are the coefficients of q^n in

$$\frac{f^{14}(q^2)}{f^4(-q)}, f^{12}(q^2), f^4(-q)f^{10}(q^2), f^8(q)f^8(q^2), f^{12}(-q)f^6(q^2).$$

He gave reduction formulæ for these functions and observed how the functions which I call $e_{10}(n), e_{12}(n)$ and $e_{16}(n)$ can be defined by means of the complex divisors of n. It is very likely that $\tau(n)$ is also capable of such a definition.

26. Now let us consider the order of $e_{2s}(n)$. It is easy to see from (147) that $f_{2s}(q)$ can be expressed in the form

$$\sum K_{a,b,c,h,k}\{f^3(-q)\}^a \left\{\frac{f^5(-q)}{f^2(q^2)}\right\}^b \left\{\frac{f^5(q^2)}{f^2(-q)}\right\}^c f^h(-q)f^k(q^2), \tag{149}$$

where a, b, c, h, k are zero or positive integers, such that

$$a+b+c = [\tfrac{2}{3}s],\ \ h+k = 2s-3[\tfrac{2}{3}s].$$

Proceeding as in § 13 we can easily shew that

$$n^{-\frac{1}{2}[\frac{2}{3}s]}e_{2s}(n)$$

cannot be of higher order than the coefficient of q^{24n} in

$$\phi^A(q)\phi^B(q^3)\phi^C(q^2), \tag{150}$$

where C is 0 or 1 and

$$A+B+C = 2s-2[\tfrac{2}{3}s].$$

Now, if $s \geq 5, A+B+C \geq 4$; and so $A+B \geq 3$. Hence one at least of A and B is greater than 1. But we know that

$$\phi^2(q) = \sum O(\nu^\epsilon)q^\nu.$$

It follows that the coefficient of q^{24n} in (150) is of order not exceeding

$$n^{\frac{1}{2}(A+B+C)-1+\epsilon}.$$

Thus

$$e_{2s}(n) = O(n^{s-1-\frac{1}{2}[\frac{2}{3}s]+\epsilon}) \tag{151}$$

for all positive integral values of s.

27. When $s \geq 9$ we can obtain a slightly more precise result.
If $s \geq 16$ we have $A+B+C \geq 12$; and so $A+B \geq 11$. Hence one at least of A and B is greater than 5. But

$$\phi^6(q) = \sum O(\nu^2) q^\nu.$$

It follows that the coefficient of q^{24n} in (150) is of order not exceeding

$$n^{\frac{1}{2}(A+B+C)-1},$$

or that

$$e_{2s}(n) = O(n^{s-1-\frac{1}{2}[\frac{2}{3}s]}), \tag{152}$$

if $s \geq 16$. We can easily shew that (152) is true when $9 \leq s \leq 16$ considering all the cases separately, using the identities.

$$\begin{aligned}
&f^{12}(-q) f^6(q^2) = \{f^3(-q)\}^4, \{f^3(q^2)\}^2, \\
&\frac{f^{30}(q^2)}{f^{12}(-q)} = \left\{\frac{f^5(q^2)}{f^2(-q)}\right\}^6, \\
&f^{16}(-q) f^4(q^2) = \left\{\frac{f^5(-q)}{f^2(q^2)}\right\}^4 \left\{\frac{f^5(q^2)}{f^2(-q)}\right\}^2 f^2(q^2), \\
&\frac{f^{28}(q^2)}{f^8(-q)} = \left\{\frac{f^5(q^2)}{f^2(-q)}\right\}^4 \{f^3(q^2)\}^2 f^2(q^2), \cdots,
\end{aligned}$$

and proceeding as in the previous two sections.
The argument of §§ 14-15 may also be applied to the function $e_{2s}(n)$. We find that

$$e_{2s}(n) \neq o(n^{\frac{1}{2}s-1}). \tag{153}$$

I leave the proof to the reader.

28. There is reason to suppose that

$$\left.\begin{aligned}
e_{2s}(n) &= O\{n^{\frac{1}{2}(s-1+\epsilon)}\} \\
e_{2s}(n) &\neq o\{n^{\frac{1}{2}(s-1)}\}
\end{aligned}\right\}, \tag{154}$$

if $s \geq 5$. I find, for example, that

$$\sum_1^\infty \frac{e_{10}(n)}{n^s} = \frac{e_{10}(1)}{1+2^{2-s}} \Pi_1 \Pi_2, \tag{155}$$

where

$$\Pi_1 = \frac{1}{(1-3^{4-2s})(1-7^{4-2s})(1-11^{4-2s})\cdots},$$

3, 7, 11, ... being the primes of the form $4n-1$, and

$$\Pi_2 = \frac{1}{(1-2c_5 \cdot 5^{-s} + 5^{4-2s})(1-2c_{13} \cdot 13^{-s} + 13^{4-2s})\cdots},$$

5, 13, 17, ... being the primes of the form $4n+1$, and

$$c_p = u^2 - (4v)^2,$$

where u and v are the unique pair of positive integers satisfying the equation

$$u^2 + (4v)^2 = p^2.$$

The equation (155) is equivalent to the assertion that, if

$$n = (3^{a_3} \cdot 7^{a_7} \cdot 11^{a_{11}} \cdots)^2 \cdot 2^{a_2} \cdot 5^{a_5} \cdot 13^{a_{13}} \cdots,$$

where a_p is zero or a positive integer, then

$$\frac{e_{10}(n)}{n^2 e_{10}(1)} = (-1)^{a_2} \frac{\sin 4(1+a_5)\theta_5}{\sin 4\theta_5} \cdot \frac{\sin 4(1+a_{13})\theta_{13}}{\sin 4\theta_{13}} \cdots, \tag{156}$$

where

$$\tan\theta_p = \frac{u}{v} \quad (0 < \theta_p < \tfrac{1}{2}\pi),$$

u and v being integers satisfying the equation $u^2 + v^2 = p$; and $e_{10}(n) = 0$ otherwise. If this is true then we should have

$$\left|\frac{e_{10}(n)}{e_{10}(1)}\right| \le n^2 d(n) \tag{157}$$

for all values of n, and

$$\left|\frac{e_{10}(n)}{e_{10}(1)}\right| \ge n^2 \tag{158}$$

for an infinity of values of n. In this case we can prove that, if n is the square of a prime of the form $4m-1$, then

$$\left|\frac{e_{10}(n)}{e_{10}(1)}\right| = n^2.$$

Similarly I find that

$$\sum_1^\infty \frac{e_{12}(n)}{n^s} = e_{12}(1) \prod_p \left(\frac{1}{1 + 2c_p \cdot p^{-s} + p^{5-2s}}\right), \tag{159}$$

p being an odd prime and $c_p^2 \leq p^5$. From this it would follow that

$$\left|\frac{e_{12}(n)}{e_{12}(1)}\right| \leq n^{\frac{5}{2}} d(n) \tag{160}$$

for all values of n, and

$$\left|\frac{e_{12}(n)}{e_{12}(1)}\right| \geq n^{\frac{5}{2}} \tag{161}$$

for an infinity of values of n.
Finally I find that

$$\sum_1^{\infty} \frac{e_{16}(n)}{n^s} = \frac{e_{16}(1)}{1+2^{3-s}} \prod_p \left(\frac{1}{1+2c_p \cdot p^{-s} + p^{7-2s}}\right), \tag{162}$$

p being an odd prime and $c_p^2 \leq p^7$. From this it would follow that

$$\left|\frac{e_{16}(n)}{e_{16}(1)}\right| \leq n^{\frac{7}{2}} d(n) \tag{163}$$

for all values of n, and

$$\left|\frac{e_{16}(n)}{e_{16}(1)}\right| \geq n^{\frac{7}{2}} \tag{164}$$

for an infinity of values of n.
In the case in which $2s = 24$ we have

$$\tfrac{691}{64} e_{24}(n) = (-1)^{n-1} 259\tau(n) - 512\tau(\tfrac{1}{2}n).$$

I have already stated the reasons for supposing that

$$|\tau(n)| \leq n^{\frac{11}{2}} d(n)$$

for all values of n, and

$$|\tau(n)| \geq n^{\frac{11}{2}}$$

for an infinity of values of n.