Reasoning with Inconsistency-tolerant Fuzzy Description Logics

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Abstract: An inconsistency-tolerant fuzzy description logic is introduced and a translation from this logic to a standard fuzzy description logic is constructed. A theorem for embedding the proposed inconsistency-tolerant fuzzy description logic into the standard fuzzy description logic is proven via this translation. A relative decidability theorem for the inconsistency-tolerant fuzzy description logic w.r.t. the standard fuzzy description logic is also proven using this embedding theorem. These proposed logic and translation are intended to effectively handle inconsistent fuzzy knowledge bases. By using the translation, the previously developed algorithms and methods for the standard fuzzy description logic can be re-purposed for appropriately handling inconsistent fuzzy knowledge bases that are described by the proposed logic. Furthermore, an inconsistency-tolerant fuzzy temporal next-time description logic is obtained from the inconsistency-tolerant fuzzy description logic by adding a temporal next-time operator. Similar results as those for the inconsistency-tolerant fuzzy description logic are also obtained for this temporal extension.

1 INTRODUCTION

Even though handling fuzzy (vague or imprecise) concepts is well-known to be a significant issue in knowledge representation (KR) in AI, inconsistency handling is of growing importance in KR because inconsistencies can frequently occur in the real world. Thus, combining these issues is also regarded as a significant issue in KR, especially for realizing smart knowledge-based systems. Knowledge-based systems would be smarter, more robust, and more fine-grained if they were capable of handling inconsistent fuzzy knowledge bases. To effectively handle inconsistent fuzzy knowledge bases, this paper introduces an *inconsistency-tolerant fuzzy description logic*, if-$\mathcal{ALC}$. This proposed logic is regarded as an extension of the standard *fuzzy description logic* f-$\mathcal{ALC}$ originally introduced by Straccia in (Straccia, 2001), although f-$\mathcal{ALC}$ was, however, not referred to as f-$\mathcal{ALC}$ in the original paper. A translation from if-$\mathcal{ALC}$ to f-$\mathcal{ALC}$ is defined and a theorem for embedding if-$\mathcal{ALC}$ into f-$\mathcal{ALC}$ is proven using this translation. A relative decidability theorem for if-$\mathcal{ALC}$ w.r.t. f-$\mathcal{ALC}$ is proven via this embedding theorem. This relative decidability theorem shows that if a decision problem for f-$\mathcal{ALC}$ is decidable, then the counterpart decision problem for if-$\mathcal{ALC}$ is also decidable. Based on the proposed translation, we can reuse the previously developed algorithms and methods for f-$\mathcal{ALC}$ in suitably handling inconsistent fuzzy knowledge bases that are represented by if-$\mathcal{ALC}$. Furthermore, in this study, an *inconsistency-tolerant fuzzy temporal next-time description logic*, itf-$\mathcal{ALC}$, is obtained from if-$\mathcal{ALC}$ by adding the temporal next-time operator that was originally introduced by Prior in (Prior, A.N., 1957; Prior, A.N., 1967). Similar results as those for if-$\mathcal{ALC}$ are also obtained for itf-$\mathcal{ALC}$. Thus, we can also reuse the previously developed algorithms and methods for f-$\mathcal{ALC}$ in appropriately handling inconsistent temporal fuzzy knowledge bases that are described by itf-$\mathcal{ALC}$.

As argued by Straccia in (Straccia, 1997a), combining an *inconsistency-tolerant (or paraconsistent) logic* with a *fuzzy logic* is important for handling inconsistent vague information. For this purpose, a *four-valued (inconsistency-tolerant) fuzzy propositional logic*, which is a combination of a *four-valued logic* and a fuzzy propositional logic, was introduced by Straccia in (Straccia, 1997a). It was shown in (Straccia, 1997a) that this logic can effectively handle both inconsistent and vague predicates with suitable computational properties. Furthermore, in another paper (Straccia, 2000), a *four-valued (inconsistency-tolerant) fuzzy description logic* was introduced by

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Straccia to obtain a logical framework for multimedia information retrieval. This four-valued fuzzy description logic is essentially an extension of both the fuzzy description logic f-$\mathcal{ALC}$ and a *four-valued description logic* that was also developed by Straccia in another paper (Straccia, 1997b). In (Straccia, 2000), a sequent calculus for the four-valued fuzzy description logic was introduced, and the completeness theorem with respect to this sequent calculus was proved. The validity problem for the four-valued fuzzy description logic was also shown to be decidable in (Straccia, 2000). The aim of this study is to advance, from a purely logical point of view, the ideas proposed by Straccia for combining an inconsistency-tolerant logic with a fuzzy description logic. The proposed logic if-$\mathcal{ALC}$ is a combination and extension of f-$\mathcal{ALC}$ and an *inconsistency-tolerant (or paraconsistent) description logic*, $\mathcal{SALC}$, which was introduced in (Kamide, 2013).

The difference between the proposed logic if-$\mathcal{ALC}$ and the four-valued fuzzy description logic proposed by Straccia (Straccia, 2000) is explained as follows. Although these logics are essentially equivalent, the main formal difference is that if-$\mathcal{ALC}$ has a paraconsistent negation connective, but Straccia's logic has no such paraconsistent negation connective. Using the paraconsistent negation connective in if-$\mathcal{ALC}$ entails the following merits: (1) the inconsistency in if-$\mathcal{ALC}$ can be handled explicitly by using the paraconsistent negation connective (i.e., the notion of paraconsistency is formally defined with respect to the paraconsistent negation connective) and (2) it is useful for handling some practical applications (e.g., as presented in (Wagner, 1991), a database needs two kinds of negations including a paraconsistent one). Another formal difference is that Straccia's logic uses two kinds of fuzzy valuations denoted as $|\cdot|^t$ and $|\cdot|^f$, but if-$\mathcal{ALC}$ has a single fuzzy valuation that just coincides with a fuzzy interpretation function. This simplification of the fuzzy interpretation function in if-$\mathcal{ALC}$ entails the following theoretical merits: (1) we can show a theorem for embedding if-$\mathcal{ALC}$ into f-$\mathcal{ALC}$, (2) we can show the relative decidability of if-$\mathcal{ALC}$ with respect to f-$\mathcal{ALC}$, and (3) we can obtain the temporal extension itf-$\mathcal{ALC}$ of if-$\mathcal{ALC}$. These theoretical results, which were not obtained for Straccia's logic, are the main contribution of this study.

To clarify the construction of our proposed logic if-$\mathcal{ALC}$, we address a brief survey of closely related description logics as follows. *Description logics* (Baader et al., 2003) are well-known to be a family of logic-based knowledge representation formalisms with applications in various fields, such as the field of developing web ontology languages. A number of useful description logics including a standard description logic, $\mathcal{ALC}$ (Schmidt-Schauss and Smolka, 1991), have been studied by many researchers. Inconsistency-tolerant (or paraconsistent) description logics and their neighbors have been studied by several researchers ((Ma et al., 2007; Ma et al., 2008; Meghini and Straccia, 1996; Meghini et al., 1998; Odintsov, S.P. and Wansing, 2003; Odintsov, S.P. and Wansing, 2008; Patel-Schneider, Peter F., 1989; Straccia, 1997b; Kaneiwa, 2007; Zhang and Lin, 2008; Zhang et al., 2009; Kamide, 2011; Kamide, 2012; Kamide, 2013; Kamide, 2020a)) to cope with inconsistencies that frequently occur in the real world. The inconsistency-tolerant description logic $\mathcal{SALC}$ (Kamide, 2013), which is a base logic for if-$\mathcal{ALC}$, is an extension and combination of both $\mathcal{ALC}$ and *Belnap and Dunn's four-valued logic* (also referred to as a *first degree entailment logic*) (Belnap, 1977b; Belnap, 1977a; Dunn J.M., 1976). For a survey of inconsistency-tolerant description logics, see (Kamide, 2013). On the other hand, fuzzy description logics and their applications have extensively been studied by many researchers (e.g., (Yen, 1991; Tresp and Molitor, 2018; Straccia, 1997a; Hájek, 2005; Stoilos et al., 2007; Jiang et al., 2010; Straccia, 2015; Baader et al., 2015; Baader et al., 2017; Bobillo and Straccia, 2017; Borgwardt and Penaloza, 2017; Kamide, 2020b)) to deal with fuzzy knowledge bases. The fuzzy description logic f-$\mathcal{ALC}$ introduced in (Straccia, 2001), which was, however, not referred to as f-$\mathcal{ALC}$ in the original paper, is regarded as a result of combining (or integrating) the standard description logic $\mathcal{ALC}$ with *Zadeh fuzzy logic*, which is based on the idea of *fuzzy sets* by Zadeh (Zadeh, L.A., 1965). For a comprehensive survey of fuzzy description logics and their applications, see (Straccia, 2015; Borgwardt and Penaloza, 2017).

The contents of this paper are presented as follows. In Section 2, f-$\mathcal{ALC}$ (Straccia, 2001) is addressed to develop if-$\mathcal{ALC}$. In Section 3, if-$\mathcal{ALC}$ is obtained from f-$\mathcal{ALC}$ by adding the paraconsistent negation connective $\sim$ and a translation from if-$\mathcal{ALC}$ to f-$\mathcal{ALC}$ is defined. A theorem for embedding if-$\mathcal{ALC}$ into f-$\mathcal{ALC}$ is proven via this translation. A relative decidability theorem for if-$\mathcal{ALC}$ w.r.t. f-$\mathcal{ALC}$ is also proven via this embedding theorem. Furthermore, we show a relative complexity theorem for if-$\mathcal{ALC}$ w.r.t. f-$\mathcal{ALC}$. This relative complexity theorem shows that if the underlying decision problem for if-$\mathcal{ALC}$ is decidable, then the complexity of the decision procedure of the problem for if-$\mathcal{ALC}$ is the same as the complexity of the decision procedure of the counterpart problem for f-$\mathcal{ALC}$. In Section 4, itf-$\mathcal{ALC}$ is obtained from if-$\mathcal{ALC}$ by adding the temporal next-time operator X and a translation from itf-$\mathcal{ALC}$ to if-$\mathcal{ALC}$ is defined. A theorem for embedding itf-$\mathcal{ALC}$ into if-$\mathcal{ALC}$ is proven via this translation. The relative decidability and complexity theorems for itf-$\mathcal{ALC}$ w.r.t. if-$\mathcal{ALC}$ are proven via this embedding theorem. Similarly, a theorem for embedding itf-$\mathcal{ALC}$ into f-$\mathcal{ALC}$ and the relative decidability and complexity theorems for itf-$\mathcal{ALC}$ w.r.t. f-$\mathcal{ALC}$ are proven. In Section 5, this paper is concluded with some remarks.

2 PRELIMINARIES: FUZZY DESCRIPTION LOGIC

We introduce the fuzzy description logic f-$\mathcal{ALC}$. The concepts of f-$\mathcal{ALC}$ are constructed from atomic concepts and roles by $\sqcap$ (intersection), $\sqcup$ (union), $\neg$ (complement), $\forall R$ (universal concept quantification), and $\exists R$ (existential concept quantification). We use the letter A to denote atomic concepts, the letter R to denote roles, the letters C and D to denote concepts, and the letters a and b to denote individual names. We use the symbol $\equiv$ to represent the equality of symbols.

The definition of fuzzy sets is as follows.

Definition 2.1. *A fuzzy set S w.r.t. a universe U is characterized by a mapping (referred to as* membership function*) μ_S: $U \rightarrow [0,1]$ where $[0,1]$ is the closed real unit interval. The membership function satisfies the following restrictions: For any $u \in U$ and any fuzzy sets S_1 and S_2 with respect to U:*

1. $\mu_{\overline{S_1}}(u) := 1 - \mu_{S_1}(u)$,
2. $\mu_{S_1 \cap S_2}(u) := min\{\mu_{S_1}(u), \mu_{S_2}(u)\}$,
3. $\mu_{S_1 \cup S_2}(u) := max\{\mu_{S_1}(u), \mu_{S_2}(u)\}$.

where $\overline{S_1}$ is a fuzzy complement of S_1 in U, $S_1 \cap S_2$ denotes a fuzzy set intersection of S_1 and S_2, and $S_1 \cup S_2$ denotes a fuzzy set union of S_1 and S_2.

The definition of f-$\mathcal{ALC}$ is as follows.

Definition 2.2. *A* fuzzy interpretation *(or* fuzzy model*) I for* f-$\mathcal{ALC}$ *is a pair $\langle \Delta^I, \cdot^I \rangle$ such that*

1. *Δ^I is a non-empty set as for the crisp case,*
2. *$\cdot^I$ is a* fuzzy interpretation function *such that*
   (a) *for any individual names a and b, we have $a^I, b^I \in \Delta^I$ such that $a^I \neq b^I$ if $a \neq b$,*
   (b) *for any atomic concepts A, we have $A^I : \Delta^I \rightarrow [0,1]$,*
   (c) *for any roles R, we have $R^I : \Delta^I \times \Delta^I \rightarrow [0,1]$.*

The fuzzy interpretation function is inductively extended to concepts by the following clauses: for any $d \in \Delta^I$,

1. $(\neg C)^I(d) := 1 - C^I(d)$,
2. $(C \sqcap D)^I(d) := min\{C^I(d), D^I(d)\}$,
3. $(C \sqcup D)^I(d) := max\{C^I(d), D^I(d)\}$,
4. $(\forall R.C)^I(d) := inf_{d' \in \Delta^I}\{max\{1 - R^I(d,d'), C^I(d')\}\}$,
5. $(\exists R.C)^I(d) := sup_{d' \in \Delta^I}\{min\{R^I(d,d'), C^I(d')\}\}$.

Remark 2.3. *The fuzzy interpretation defined in Definition 2.2 is constructed on the basis of* Zadeh logic. *We can consider other fuzzy interpretation functions that are based on the following fuzzy logics:* Gödel logic*,* Łukasiewicz logic*, and* product logic*. But, we do not discuss such interpretations in this study.*

The definition of fuzzy assertion is as follows.

Definition 2.4. *A* crisp assertion *(denoted by α) is an expression of the form $a : C$ or $(a,b) : R$ where a and b are individual names, C is a concept, and R is a role. A* crisp primitive assertion *is either a crisp assertion of the form $a : A$ where A is an atomic concept, or a crisp assertion of the form $(a,b) : R$ where R is a role. A* fuzzy assertion *(denoted as ψ) is an expression of the form $\langle \alpha \geq n \rangle$ or $\langle \alpha \leq m \rangle$ where α is a crisp assertion, $n \in (0,1]$, and $m \in [0,1)$. A* fuzzy ABox *is a finite set of fuzzy assertions. We frequently use some similar expressions of the form $\langle \alpha > n \rangle$ or $\langle \alpha < n \rangle$. A fuzzy interpretation I satisfies a fuzzy assertion $\langle a : C \geq n \rangle$ or $\langle (a,b) : R \geq n \rangle$ iff $C^I(a^I) \geq n$ or $R^I(a^I, b^I) \geq n$, respectively. Similarly, a fuzzy interpretation I satisfies a fuzzy assertion $\langle a : C \leq m \rangle$ or $\langle (a,b) : R \leq m \rangle$ iff $C^I(a^I) \leq m$ or $R^I(a^I, b^I) \leq m$, respectively. A* fuzzy primitive assertion *is a fuzzy assertion involving a primitive assertion.*

We use an expression $I \models \psi$ to denote the fact that a fuzzy interpretation I satisfies a fuzzy assertion ψ.

The definition of fuzzy terminological axiom is as follows.

Definition 2.5. *A* fuzzy general concept inclusion *is an expression of the form $C \prec D$ where C and D are concepts. A fuzzy interpretation I satisfies a fuzzy general concept inclusion $C \prec D$ iff $\forall d \in \Delta^I [C^I(d) \leq D^I(d)]$. A* fuzzy concept specialization *is a restricted fuzzy general concept inclusion of the form $A \prec C$ where A is an atomic concept and C is a concept. A* fuzzy concept equivalence *is an expression of the form $C :\approx D$ where C and C are concepts. A fuzzy interpretation I satisfies a fuzzy concept equivalence $C :\approx D$ iff $\forall d \in \Delta^I [C^I(d) = D^I(d)]$. A* fuzzy concept definition *is a restricted fuzzy concept equivalence of the form $A :\approx C$ where A is an atomic concept and C is a concept. A* fuzzy terminological axiom *(denoted by υ) is either a fuzzy general concept inclusion or a fuzzy concept equivalence. A* fuzzy TBox *is a finite set of fuzzy terminological axioms.*

We use an expression $I \models \upsilon$ to denote the fact that a fuzzy interpretation I satisfies a fuzzy terminological axiom υ.

Remark 2.6. *It was assumed in* (Straccia, 2001) *that there is no cycle definition in a fuzzy TBox and there is no fuzzy general concept inclusion in a fuzzy TBox (i.e., atomic concept appears more than once on the left hand side of a fuzzy terminological axiom).*

The definitions of fuzzy knowledge base, fuzzy entailment, and fuzzy subsumption are as follows.

Definition 2.7. *A* fuzzy knowledge base *is a pair of a fuzzy ABox and a fuzzy TBox. Let Σ be a fuzzy knowledge base. We use an expression Σ_A to denote the set of assertions in Σ and we use an expression Σ_T to denote the set of terminological axions in Σ. A fuzzy interpretation I satisfies a fuzzy knowledge base Σ iff I satisfies each element of Σ. A fuzzy knowledge base Σ* fuzzy entails *a fuzzy assertion ψ (denoted by $\Sigma \models \psi$) iff every fuzzy interpretation of Σ also satisfies ψ. A concept D* fuzzy subsumes *a concept C with respect to a set Σ_T of terminological axioms (denoted by $C \prec_{\Sigma_T} D$) iff for every fuzzy interpretation I of Σ_T, $\forall d \in \Delta^I [C^I(d) \leq D^I(d)]$ holds.*

We use an expression $I \models \Sigma$ to denote the fact that a fuzzy interpretation I satisfies a fuzzy knowledge base Σ.

Definition 2.8. *A fuzzy assertion ψ is called* satisfiable *in* f-$\mathcal{ALC}$ *iff it has a fuzzy interpretation I such that $I \models \psi$. A fuzzy terminological axiom υ is called* satisfiable *in* f-$\mathcal{ALC}$ *iff it has a fuzzy interpretation I such that $I \models \upsilon$. A fuzzy knowledge base Σ is called* satisfiable *in* f-$\mathcal{ALC}$ *iff it has a fuzzy interpretation I such that $I \models \gamma$ for every element γ of Σ.*

3 INCONSISTENCY-TOLERANT FUZZY DESCRIPTION LOGIC

We introduce an inconsistency-tolerant fuzzy description logic, if-$\mathcal{ALC}$. We also use the same notions and terminologies for if-$\mathcal{ALC}$ as those for f-$\mathcal{ALC}$. The concepts of if-$\mathcal{ALC}$ are obtained from the concepts of f-$\mathcal{ALC}$ by adding $\sim$ (paraconsistent negation).

The definition of if-$\mathcal{ALC}$ is as follows.

Definition 3.1. *An* inconsistency-tolerant fuzzy interpretation *(or* inconsistency-tolerant fuzzy model*) II for* if-$\mathcal{ALC}$ *is a pair $\langle \Delta^{II}, \cdot^{II} \rangle$ where*

1. *Δ^{II} is a non-empty set as for the crisp case,*
2. *$\cdot^{II}$ is an* inconsistency-tolerant fuzzy interpretation function *where*
   - *(a) for any individual names a and b, we have $a^{II}, b^{II} \in \Delta^{II}$ such that $a^{II} \neq b^{II}$ if $a \neq b$,*
   - *(b) for any atomic concepts A, we have $A^{II} : \Delta^{II} \to [0,1]$,*
   - *(c) for any negated atomic concepts $\sim A$, we have $(\sim A)^{II} : \Delta^{II} \to [0,1]$,*
   - *(d) for any roles R, we have $R^{II} : \Delta^{II} \times \Delta^{II} \to [0,1]$.*

The inconsistency-tolerant fuzzy interpretation function is inductively extended to concepts by the following clauses: For any $d \in \Delta^{II}$,

1. $(\neg C)^{II}(d) := 1 - C^{II}(d)$,
2. $(C \sqcap D)^{II}(d) := min\{C^{II}(d), D^{II}(d)\}$,
3. $(C \sqcup D)^{II}(d) := max\{C^{II}(d), D^{II}(d)\}$,
4. $(\forall R.C)^{II}(d) := inf_{d' \in \Delta^{II}}\{max\{1 - R^{II}(d,d'), C^{II}(d')\}\}$,
5. $(\exists R.C)^{II}(d) := sup_{d' \in \Delta^{II}}\{min\{R^{II}(d,d'), C^{II}(d')\}\}$,
6. $(\sim\sim C)^{II}(d) := C^{II}(d)$,
7. $(\sim\neg C)^{II}(d) := 1 - (\sim C)^{II}(d)$,
8. $(\sim(C \sqcap D))^{II}(d) := max\{(\sim C)^{II}(d), (\sim D)^{II}(d)\}$,
9. $(\sim(C \sqcup D))^{II}(d) := min\{(\sim C)^{II}(d), (\sim D)^{II}(d)\}$,
10. $(\sim\forall R.C)^{II}(d) := sup_{d' \in \Delta^{II}}\{min\{R^{II}(d,d'), (\sim C)^{II}(d')\}\}$,
11. $(\sim\exists R.C)^{II}(d) := inf_{d' \in \Delta^{II}}\{max\{1 - R^{II}(d,d'), (\sim C)^{II}(d')\}\}$.

The following proposition shows that some properties concerning $\sim$ hold for if-$\mathcal{ALC}$.

Proposition 3.2. *For any concepts C and D and any inconsistency-tolerant fuzzy interpretation function $\cdot^{II}$ on* if-$\mathcal{ALC}$*, we can obtain the following conditions w.r.t. $\sim$:*

1. $(\sim\sim C)^{II}(d) = C^{II}(d)$,
2. $(\sim\neg C)^{II}(d) = (\neg\sim C)^{II}(d)$,
3. $(\sim(C \sqcap D))^{II}(d) = ((\sim C) \sqcup (\sim D))^{II}(d)$,
4. $(\sim(C \sqcup D))^{II}(d) = ((\sim C) \sqcap (\sim D))^{II}(d)$,
5. $(\sim\forall R.C)^{II}(d) = (\exists R.\sim C)^{II}(d)$,
6. $(\sim\exists R.C)^{II}(d) = (\forall R.\sim C)^{II}(d)$.

Proof. Straightforward by Definition 3.1. ■

Remark 3.3. *We make the following remarks.*

1. *Intuitively speaking,* if-$\mathcal{ALC}$ *is extended based on the following additional axiom schemes for $\sim$:*
   - *(a) $\sim\sim C \leftrightarrow C$,*
   - *(b) $\sim\neg C \leftrightarrow \neg\sim C$,*
   - *(c) $\sim(C \sqcap D) \leftrightarrow \sim C \sqcup \sim D$,*
   - *(d) $\sim(C \sqcup D) \leftrightarrow \sim C \sqcap \sim D$,*
   - *(e) $\sim(\forall R.C) \leftrightarrow \exists R.\sim C$,*
   - *(f) $\sim(\exists R.C) \leftrightarrow \forall R.\sim C$.*
2. *In general, a semantic consequence relation $\models$ is said to be* paraconsistent with respect to a negation connective $\sim$ *if there are formulas α and β such that $\{\alpha, \sim\alpha\} \not\models \beta$. We now consider*

the case of if-$\mathcal{ALC}$*. Assume an inconsistency-tolerant fuzzy interpretation* $II = \langle \Delta^{II}, \cdot^{II} \rangle$ *such that for a pair of distinct atomic concepts* A *and* B, $A^{II}(d) = m$, $({\sim}A)^{II}(d) = m$, *and* $B^{II}(d) = n$ *where* $d \in \Delta^{II}$ *and* $m > n$ *with* $m, n \in [0,1]$*. Then,* $(A \sqcap {\sim}A)^{II}(d) \leq B^{II}(d)$ *does not hold. Thus,* if-$\mathcal{ALC}$ *is regarded as paraconsistent with respect to* $\sim$*. Note that* if-$\mathcal{ALC}$ *is not paraconsistent with respect to* $\neg$*.*

Some notions on if-$\mathcal{ALC}$ are defined as follows.

Definition 3.4. *The notions of* inconsistency-tolerant fuzzy assertion*,* inconsistency-tolerant fuzzy primitive assertion*,* inconsistency-tolerant fuzzy general concept inclusion*,* inconsistency-tolerant fuzzy concept specialization*,* inconsistency-tolerant fuzzy concept equivalence*,* inconsistency-tolerant fuzzy concept definition*,* inconsistency-tolerant fuzzy ABox*,* inconsistency-tolerant fuzzy terminological axiom*,* inconsistency-tolerant fuzzy TBox *and* inconsistency-tolerant fuzzy knowledge base *are defined in the same manner as those in Definitions 2.4, 2.5, and 2.7. The notion of the satisfiability of inconsistency-tolerant fuzzy knowledge base is defined in the same manner as that of defined in Definition 2.8.*

The same notations that are introduced in Definitions 2.4, 2.5, and 2.7 are also used for those of if-$\mathcal{ALC}$.

Next, we introduce a translation from if-$\mathcal{ALC}$ into f-$\mathcal{ALC}$. Using this translation, we prove a theorem for embedding if-$\mathcal{ALC}$ into f-$\mathcal{ALC}$. Using this embedding theorem, we prove a relative decidability theorem for if-$\mathcal{ALC}$ with respect to f-$\mathcal{ALC}$.

We use the symbol N_C to be a non-empty set of atomic concepts, the symbol N'_C to be the set $\{A' \mid A \in N_C\}$ of atomic concepts, the symbol N_R to be a non-empty set of roles, and the symbol N_I to be a non-empty set of individual names.

The definition of the translation form if-$\mathcal{ALC}$ to f-$\mathcal{ALC}$ is as follows.

Definition 3.5. *The language* $\mathcal{L}^n$ *of* if-$\mathcal{ALC}$ *is defined using* N_C, N_R, N_I, $\sim$, $\neg, \sqcap, \sqcup$, $\forall R$, *and* $\exists R$*. The language* $\mathcal{L}$ *of* f-$\mathcal{ALC}$ *is obtained from* $\mathcal{L}^n$ *by adding* N'_C *and deleting* $\sim$*. A mapping* f *from* $\mathcal{L}^n$ *to* $\mathcal{L}$ *is defined inductively by:*

1. *for any* $R \in N_R$ *and any* $a \in N_I$, $f(R) := R$ *and* $f(a) := a$,
2. *for any* $A \in N_C$, $f(A) := A$ *and* $f({\sim}A) := A' \in N'_C$,
3. $f(\neg C) := \neg f(C)$,
4. $f(C \sharp D) := f(C) \sharp f(D)$ *where* $\sharp \in \{\sqcap, \sqcup\}$,
5. $f(\sharp R.C) := \sharp f(R).f(C) = \sharp R.f(C)$ *where* $\sharp \in \{\forall, \exists\}$,
6. $f({\sim}{\sim}C) := f(C)$,
7. $f({\sim}\neg C) := \neg f({\sim}C)$,
8. $f({\sim}(C \sqcap D)) := f({\sim}C) \sqcup f({\sim}D)$,
9. $f({\sim}(C \sqcup D)) := f({\sim}C) \sqcap f({\sim}D)$,
10. $f({\sim}\forall R.C) := \exists f(R).f({\sim}C) = \exists R.f({\sim}C)$,
11. $f({\sim}\exists R.C) := \forall f(R).f({\sim}C) = \forall R.f({\sim}C)$.

Analogous notations and expressions like $f(\psi)$, $f(\tau)$, and $f(\Sigma)$ are also adopted for an inconsistency-tolerant fuzzy assertion ψ, an inconsistency-tolerant fuzzy terminological axiom τ, and an inconsistency-tolerant fuzzy knowledge base Σ, respectively. For example, we use an expression $f(\Sigma)$ to denote the result of replacing every occurrence of a concept (or a role) β in Σ by an occurrence of $f(\beta)$. We also use analogous notations for the other mappings discussed later.

Remark 3.6. *The translation function introduced above is a modification of the translation function introduced by Ma et al.* (Ma et al., 2007) *to embed* $\mathcal{ALC}4$ *into* $\mathcal{ALC}$*. A similar translation function has been used by Gurevich* (Gurevich, 1977)*, Rautenberg* (Rautenberg, 1979) *and Vorob'ev* (Vorob'ev, N.N, 1952) *to embed Nelson's constructive logic* (Almukdad and Nelson, 1984; Nelson, 1949) *into intuitionistic logic. Some similar translations have also been used, for example, in* (Kamide and Shramko, 2017; Kamide and Zohar, 2019) *to embed some paraconsistent logics into classical logic.*

Prior to show the embedding theorem, we need to prove some lemmas.

Lemma 3.7. *Let* f *be the mapping defined in Definition 3.5. For any inconsistency-tolerant fuzzy interpretation* $II := \langle \Delta^{II}, \cdot^{II} \rangle$ *of* if-$\mathcal{ALC}$*, we can construct a fuzzy interpretation* $I := \langle \Delta^I, \cdot^I \rangle$ *of* f-$\mathcal{ALC}$ *such that for any concept* C *in* $\mathcal{L}^n$, $C^{II} = f(C)^I$*.*

Proof. Let II be an inconsistency-tolerant fuzzy interpretation $\langle \Delta^{II}, \cdot^{II} \rangle$. We now construct a fuzzy interpretation $I := \langle \Delta^I, \cdot^I \rangle$ such that

1. Δ^I is a non-empty set such that $\Delta^I = \Delta^{II}$,
2. $\cdot^I$ is a fuzzy interpretation function where
   (a) $R^I = R^{II}$ and $a^I = a^{II}$ for any $R \in N_R$ and any $a \in N_I$.
   (b) $A^I = A^{II}$ for any atomic concept $A \in N_C$.
   (c) $(A')^I = ({\sim}A)^{II}$ for any atomic concept $A' \in N'_C$.

This lemma is then proved by induction on the complexity of C.

• Base step:

1. Case $C \equiv A \in N_C$: We obtain: $A^{II} = A^I = f(A)^I$ (by the definition of f).
2. Case $C \equiv {\sim}A$ with $A \in N_C$: We obtain: $({\sim}A)^{II} = (A')^I = f({\sim}A)^I$ (by the definition of f).

• Induction step: We show some cases.

1. Case $C \equiv \neg D$: We obtain:
   $(\neg D)^{II}(d)$
   $= 1 - D^{II}(d)$
   $= 1 - f(D)^{I}(d)$ (by induction hypothesis)
   $= (\neg f(D))^{I}(d)$
   $= f(\neg D)^{I}(d)$ (by the definition of f).
2. Case $C \equiv C_1 \sqcap C_2$: We obtain:
   $(C_1 \sqcap C_2)^{II}(d)$
   $= min\{C_1^{II}(d), C_2^{II}(d)\}$
   $= min\{f(C_1)^{I}(d), f(C_2)^{I}(d)\}$ (by induction hypothesis)
   $= (f(C_1) \sqcap f(C_2))^{I}(d)$
   $= f(C_1 \sqcap C_2)^{I}(d)$ (by the definition of f).
3. Case $C \equiv \forall R.D$: We obtain:
   $(\forall R.D)^{II}(d)$
   $= inf_{d' \in \Delta^{II}}\{max\{1 - R^{II}(d,d'), D^{II}(d')\}\}$
   $= inf_{d' \in \Delta^{II}}\{max\{1 - R^{II}(d,d'), f(D)^{I}(d')\}\}$ (by induction hypothesis)
   $= inf_{d' \in \Delta^{I}}\{max\{1 - R^{I}(d,d'), f(D)^{I}(d')\}\}$ (by $\Delta^{II} = \Delta^{I}$ and $R^{II} = R^{I}$)
   $= (\forall R.f(D))^{I}(d)$
   $= f(\forall R.D)^{I}(d)$ (by the definition of f).
4. Case $C \equiv \sim\sim D$: We obtain:
   $(\sim\sim D)^{II}(d)$
   $= D^{II}(d)$
   $= f(D)^{I}(d)$ (by induction hypothesis)
   $= f(\sim\sim D)^{I}(d)$ (by the definition of f).
5. Case $C \equiv \sim\neg D$: We obtain:
   $(\sim\neg D)^{II}(d)$
   $= 1 - (\sim D)^{II}(d)$
   $= 1 - f(\sim D)^{I}(d)$ (by induction hypothesis)
   $= (\neg f(\sim D))^{I}(d)$
   $= f(\sim\neg D)^{I}(d)$ (by the definition of f).
6. Case $C \equiv \sim(C_1 \sqcap C_2)$: We obtain:
   $(\sim(C_1 \sqcap C_2))^{II}(d)$
   $= max\{(\sim C_1)^{II}(d), (\sim C_2)^{II}(d)\}$
   $= max\{f(\sim C_1)^{I}(d), f(\sim C_2)^{I}(d)\}$ (by induction hypothesis)
   $= (f(\sim C_1) \sqcup f(\sim C_2))^{I}(d)$
   $= f(\sim(C_1 \sqcap C_2))^{I}(d)$ (by the definition of f).
7. Case $C \equiv \sim\forall R.D$: We obtain:
   $(\sim\forall R.D)^{II}(d)$
   $= sup_{d' \in \Delta^{II}}\{min\{R^{II}(d,d'), (\sim D)^{II}(d')\}\}$
   $= sup_{d' \in \Delta^{II}}\{min\{R^{II}(d,d'), f(\sim D)^{I}(d')\}\}$ (by induction hypothesis)
   $= sup_{d' \in \Delta^{I}}\{min\{R^{I}(d,d'), f(\sim D)^{I}(d')\}\}$ (by $\Delta^{II} = \Delta^{I}$ and $R^{II} = R^{I}$)
   $= (\exists R.f(\sim D))^{I}(d)$
   $= f(\sim\forall R.D)^{I}(d)$ (by the definition of f). ■

Lemma 3.8. *Let f be the mapping defined in Definition 3.5. For any inconsistency-tolerant fuzzy interpretation $II := \langle \Delta^{II}, \cdot^{II} \rangle$ of* if-$\mathcal{ALC}$*, we can construct a fuzzy interpretation $I := \langle \Delta^{I}, \cdot^{I} \rangle$ of* f-$\mathcal{ALC}$ *such that for any inconsistency-tolerant fuzzy assertion ψ in $\mathcal{L}^n$, $II \models \psi$ iff $I \models f(\psi)$.*

Proof. By using Lemma 3.7. ■

Lemma 3.9. *Let f be the mapping defined in Definition 3.5. For any fuzzy interpretation $I := \langle \Delta^{I}, \cdot^{I} \rangle$ of* f-$\mathcal{ALC}$*, we can construct an inconsistency-tolerant fuzzy interpretation $II := \langle \Delta^{II}, \cdot^{II} \rangle$ of* if-$\mathcal{ALC}$ *such that for any inconsistency-tolerant fuzzy assertion ψ in $\mathcal{L}^n$, $II \models \psi$ iff $I \models f(\psi)$.*

Proof. Similar to the proof of Lemma 3.8. ■

The theorem for embedding if-$\mathcal{ALC}$ into f-$\mathcal{ALC}$ is presented as follows.

Theorem 3.10. *Let f be the mapping defined in Definition 3.5. For any inconsistency-tolerant fuzzy assertion ψ, we have: ψ is satisfiable in* if-$\mathcal{ALC}$ *iff $f(\psi)$ is satisfiable in* f-$\mathcal{ALC}$*.*

Proof. By using Lemmas 3.8 and 3.9. ■

The theorem for relative decidability of if-$\mathcal{ALC}$ w.r.t. f-$\mathcal{ALC}$ is presented as follows.

Theorem 3.11. *If the satisfiability problem of a fuzzy knowledge base (with or without some modifications or restrictions) for* f-$\mathcal{ALC}$ *is decidable, then the satisfiability problem of the counterpart inconsistency-tolerant fuzzy knowledge base for* if-$\mathcal{ALC}$ *is also decidable. Similarly, if the fuzzy entailment problem for* f-$\mathcal{ALC}$ *and the fuzzy subsumption problem for* f-$\mathcal{ALC}$ *are decidable, then the counterpart inconsistency-tolerant entailment problem for* if-$\mathcal{ALC}$ *and the counterpart inconsistency-tolerant subsumption problem for* if-$\mathcal{ALC}$ *are also decidable.*

Proof. Using Definition 3.5 and Theorem 3.10, we can transform the problems of if-$\mathcal{ALC}$ into those of f-$\mathcal{ALC}$. Thus, if the problems of f-$\mathcal{ALC}$ are decidable, then those of if-$\mathcal{ALC}$ are also decidable. ■

The theorem for relative complexity of if-$\mathcal{ALC}$ w.r.t. f-$\mathcal{ALC}$ is presented as follows.

Theorem 3.12. *If a decision problem for* if-$\mathcal{ALC}$ *(e.g., one of the decision problems that are addressed in Theorem 3.11) is decidable, then the complexity of the decision problem of* if-$\mathcal{ALC}$ *is the same as that of the counterpart decision problem of* f-$\mathcal{ALC}$*.*

Proof. By using Definition 3.5, Theorem 3.10, and the fact that the mapping f defined in Definition 3.5 is a polynomial-time reduction. ■

Remark 3.13. *It is known that some significant decision problems for fuzzy description logics (especially with some general concept inclusions) have not yet been solved. It is also known that if some general concept inclusions are available in some fuzzy description logics, then the corresponding decision problems for the fuzzy description logics are undecidable. For more information on the undecidability, decidability, and complexity of fuzzy description logics, see, for example,* (Baader et al., 2015; Baader et al., 2017) *and the references therein. If some open decision problems for fuzzy description logics will be solved, then Theorems 3.11 and 3.12 will be useful.*

4 INCONSISTENCY-TOLERANT TEMPORAL NEXT-TIME FUZZY DESCRIPTION LOGIC

We introduce an inconsistency-tolerant temporal next-time fuzzy description logic, itf-$\mathcal{ALC}$. For itf-$\mathcal{ALC}$, we use the same terminologies and notions as those for f-$\mathcal{ALC}$ and if-$\mathcal{ALC}$. The concepts of itf-$\mathcal{ALC}$ are obtained from the concepts of if-$\mathcal{ALC}$ by adding X (next-time operator). We use the symbol ω to represent the set of natural numbers. We inductively define an expression $\mathrm{X}^n C$ with $n \in \omega$ by $\mathrm{X}^0 C := C$ and $\mathrm{X}^{n+1} C := \mathrm{X}\mathrm{X}^n C$.

The definition of itf-$\mathcal{ALC}$ is as follows.

Definition 4.1. *An* inconsistency-tolerant temporal next-time fuzzy interpretation *(or* inconsistency-tolerant temporal next-time fuzzy model*)* $\mathcal{ITI}$ *for* itf-$\mathcal{ALC}$ *is a pair* $\langle \Delta^{\mathcal{ITI}}, \{\cdot^{I^i}\}_{i\in\omega} \rangle$ *where*

1. $\Delta^{\mathcal{ITI}}$ *is a non-empty set as for the crisp case,*
2. *for any* $i \in \omega$, $\cdot^{I^i}$ *is an* inconsistency-tolerant temporal next-time fuzzy interpretation function *such that*
   (a) *for any individual names a and b, we have* $a^{I^i}, b^{I^i} \in \Delta^{\mathcal{ITI}}$ *such that* $a^{I^i} \neq b^{I^i}$ *if* $a \neq b$,
   (b) *for any atomic concepts A, we have* $A^{I^i} : \Delta^{\mathcal{ITI}} \to [0,1]$,
   (c) *for any negated atomic concepts* $\sim A$, *we have* $(\sim A)^{I^i} : \Delta^{\mathcal{ITI}} \to [0,1]$,
   (d) *for any roles R, we have* $R^{I^i} : \Delta^{\mathcal{ITI}} \times \Delta^{\mathcal{ITI}} \to [0,1]$,
3. *for any individual name a, any role R, and any* $j,k \in \omega$, *we have* $a^{I^j} = a^{I^k}$ *and* $R^{I^j} = R^{I^k}$.

The inconsistency-tolerant fuzzy temporal next-time interpretation functions are inductively extended to concepts by the following clauses: For any $d \in \Delta^{\mathcal{ITI}}$ *and any* $i \in \omega$,

1. $(\mathrm{X}C)^{I^i}(d) := C^{I^{i+1}}(d)$,
2. $(\neg C)^{I^i}(d) := 1 - C^{I^i}(d)$,
3. $(C \sqcap D)^{I^i}(d) := min\{C^{I^i}(d), D^{I^i}(d)\}$,
4. $(C \sqcup D)^{I^i}(d) := max\{C^{I^i}(d), D^{I^i}(d)\}$,
5. $(\forall R.C)^{I^i}(d) := inf_{d' \in \Delta^{\mathcal{ITI}}}\{max\{1 - R^{I^i}(d,d'), C^{I^i}(d')\}\}$,
6. $(\exists R.C)^{I^i}(d) := sup_{d' \in \Delta^{\mathcal{ITI}}}\{min\{R^{I^i}(d,d'), C^{I^i}(d')\}\}$,
7. $(\sim \mathrm{X}C)^{I^i}(d) := (\sim C)^{I^{i+1}}(d)$,
8. $(\sim\sim C)^{I^i}(d) := C^{I^i}(d)$,
9. $(\sim\neg C)^{I^i}(d) := 1 - (\sim C)^{I^i}(d)$,
10. $(\sim(C \sqcap D))^{I^i}(d) := max\{(\sim C)^{I^i}(d), (\sim D)^{I^i}(d)\}$,
11. $(\sim(C \sqcup D))^{I^i}(d) := min\{(\sim C)^{I^i}(d), (\sim D)^{I^i}(d)\}$,
12. $(\sim\forall R.C)^{I^i}(d) := sup_{d' \in \Delta^{\mathcal{ITI}}}\{min\{R^{I^i}(d,d'), (\sim C)^{I^i}(d')\}\}$,
13. $(\sim\exists R.C)^{I^i}(d) := inf_{d' \in \Delta^{\mathcal{ITI}}}\{max\{1 - R^{I^i}(d,d'), (\sim C)^{I^i}(d')\}\}$.

The following proposition shows that some clauses concerning X hold for itf-$\mathcal{ALC}$.

Proposition 4.2. *For any concepts C and D and any inconsistency-tolerant temporal next-time fuzzy interpretation function* $\cdot^{\mathcal{ITI}}$ *on* itf-$\mathcal{ALC}$, *we can obtain the following clauses with respect to* X*:*

1. $(\mathrm{X}\sharp C)^{\mathcal{ITI}}(d) = (\sharp \mathrm{X}C)^{\mathcal{ITI}}(d)$ *where* $\sharp \in \{\sim, \neg\}$,
2. $(\mathrm{X}(C \sharp D))^{\mathcal{ITI}}(d) = ((\mathrm{X}C) \sharp (\mathrm{X}D))^{\mathcal{ITI}}(d)$ *where* $\sharp \in \{\sqcap, \sqcup\}$,
3. $(\mathrm{X}\sharp R.C)^{\mathcal{ITI}}(d) = (\sharp R.\mathrm{X}C)^{\mathcal{ITI}}(d)$ *where* $\sharp \in \{\forall, \exists\}$.

Proof. Straightforward by Definition 4.1. ■

Remark 4.3. *We make the following remarks.*

1. *Intuitively speaking,* itf-$\mathcal{ALC}$ *is extended based on the following additional axiom schemes for* X*:*
   (a) $\mathrm{X}\sharp C \leftrightarrow \sharp \mathrm{X}C$ *where* $\sharp \in \{\sim, \neg\}$,
   (b) $\mathrm{X}(C \sharp D) \leftrightarrow (\mathrm{X}C)\sharp(\mathrm{X}D)$ *where* $\sharp \in \{\sqcap, \sqcup\}$,
   (c) $\mathrm{X}(\sharp R.C) \leftrightarrow \sharp R.(\mathrm{X}C)$ *where* $\sharp \in \{\forall, \exists\}$.
2. itf-$\mathcal{ALC}$ *is an extension of* if-$\mathcal{ALC}$, *because the zero interpretation function* $\cdot^{I^0}$ *includes* $\cdot^{II}$.
3. itf-$\mathcal{ALC}$ *uses the following* constant domain assumption: *It has the single common domain* $\Delta^{\mathcal{ITI}}$.
4. itf-$\mathcal{ALC}$ *uses the following* rigid role and name assumption: *For any role R, any individual name a and any* $i, j \in \omega$, *we have* $R^{I^i} = R^{I^j}$ *and* $a^{I^i} = a^{I^j}$.
5. *The temporal fragment* tf-$\mathcal{ALC}$ *of* itf-$\mathcal{ALC}$ *can be obtained from* itf-$\mathcal{ALC}$ *by deleting all the parts concerning* $\sim$.

Similar notions and notations as those defined in Definitions 2.4, 2.5, and 2.7 are also used for those of itf-$\mathcal{ALC}$. However, the notions of satisfiability need some specific definitions. These notions are defined as follows.

Definition 4.4. *An inconsistency-tolerant temporal next-time fuzzy interpretation function* I^i *satisfies an inconsistency-tolerant temporal next-time fuzzy assertion* $\langle a : C \geq n\rangle$ *or* $\langle (a,b) : R \geq n\rangle$ *iff* $C^{I^i}(a^{I^i}) \geq n$ *or* $R^{I^i}(a^{I^i}, b^{I^i}) \geq n$*, respectively. Similarly, an inconsistency-tolerant temporal next-time fuzzy interpretation function* I^i *satisfies an inconsistency-tolerant temporal next-time fuzzy assertion* $\langle a : C \leq m\rangle$ *or* $\langle (a,b) : R \leq m\rangle$ *iff* $C^{I^i}(a^{I^i}) \leq m$ *or* $R^{I^i}(a^{I^i}, b^{I^i}) \leq m$*, respectively. Moreover, an inconsistency-tolerant temporal next-time fuzzy interpretation* $I\mathcal{T}I$ *satisfies an inconsistency-tolerant temporal next-time fuzzy assertion* $\langle a : C \geq n\rangle$ *or* $\langle (a,b) : R \geq n\rangle$ *iff* $C^{I^0}(a^{I^0}) \geq n$ *or* $R^{I^0}(a^{I^0}, b^{I^0}) \geq n$*, respectively. Similarly, an inconsistency-tolerant temporal next-time fuzzy interpretation* $I\mathcal{T}I$ *satisfies an inconsistency-tolerant temporal next-time fuzzy assertion* $\langle a : C \leq m\rangle$ *or* $\langle (a,b) : R \leq m\rangle$ *iff* $C^{I^0}(a^{I^0}) \leq m$ *or* $R^{I^0}(a^{I^0}, b^{I^0}) \leq m$*, respectively.*

We use an expression $I^i \models \psi$ to denote the fact that an inconsistency-tolerant temporal next-time fuzzy interpretation function I^i satisfies an inconsistency-tolerant temporal next-time fuzzy assertion ψ. We also use an expression $I\mathcal{T}I \models \psi$ to denote the fact that an inconsistency-tolerant temporal next-time fuzzy interpretation $I\mathcal{T}I$ satisfies an inconsistency-tolerant temporal next-time fuzzy assertion ψ. Note that $I\mathcal{T}I \models \psi$ is equivalent to $I^0 \models \psi$.

Next, we introduce a translation from itf-$\mathcal{ALC}$ to if-$\mathcal{ALC}$. Using this translation, we prove a theorem for embedding itf-$\mathcal{ALC}$ into if-$\mathcal{ALC}$. Using this embedding theorem, we prove a relative decidability theorem for itf-$\mathcal{ALC}$ w.r.t. if-$\mathcal{ALC}$.

We use the same symbols N_C, N_R, and N_I as those used for if-$\mathcal{ALC}$. Furthermore, we use the new symbol N_C^i to denote the set $\{A^i \mid A \in N_C\}$ of atomic concepts where $A^0 = A$ (i.e., $N_C^0 = N_C$).

The definition of the translation from itf-$\mathcal{ALC}$ to if-$\mathcal{ALC}$ is as follows.

Definition 4.5. *The language* $\mathcal{L}^t$ *of* itf-$\mathcal{ALC}$ *is defined using* N_C, N_R, N_I, $\sim$, $\neg$, $\sqcap$, $\sqcup$, $\forall R$, $\exists R$, *and* X. *The language* $\mathcal{L}^n$ *of* if-$\mathcal{ALC}$ *is obtained from* $\mathcal{L}^t$ *by adding* $\bigcup_{i\in\omega} N_C^i$ *and deleting* X. *A mapping g from* $\mathcal{L}^t$ *to* $\mathcal{L}^n$ *is defined inductively by:*

1. *for any* $R \in N_R$ *and any* $a \in N_I$, $g(R) := R$ *and* $g(a) := a$,
2. *for any* $A \in N_C$, $g(\mathrm{X}^i A) := A^i \in N_C^i$ *(esp.* $g(A) := A$*),*
3. $g(\mathrm{X}^i \sharp C) := \sharp g(\mathrm{X}^i C)$ *where* $\sharp \in \{\neg, \sim\}$,
4. $g(\mathrm{X}^i(C \sharp D)) := g(\mathrm{X}^i C) \sharp g(\mathrm{X}^i D)$ *where* $\sharp \in \{\sqcap, \sqcup\}$,
5. $g(\mathrm{X}^i \sharp R.C) := \sharp R.g(\mathrm{X}^i C)$ *where* $\sharp \in \{\forall, \exists\}$.

Prior to prove the embedding theorem, we need to show some lemmas.

Lemma 4.6. *Let g be the mapping defined in Definition 4.5. For any inconsistency-tolerant temporal next-time fuzzy interpretation* $I\mathcal{T}I := \langle \Delta^{I\mathcal{T}I}, \{\cdot^{I^i}\}_{i\in\omega}\rangle$ *of* itf-$\mathcal{ALC}$*, we can construct an inconsistency-tolerant fuzzy interpretation* $II := \langle \Delta^{II}, \cdot^{II}\rangle$ *of* if-$\mathcal{ALC}$ *such that for any concept C in* $\mathcal{L}^t$ *and any* $i \in \omega$, $C^{I^i} = g(\mathrm{X}^i C)^{II}$.

Proof. Suppose that $I\mathcal{T}I$ is an inconsistency-tolerant temporal next-time fuzzy interpretation $\langle \Delta^{I\mathcal{T}I}, \{\cdot^{I^i}\}_{i\in\omega}\rangle$. Then, we construct an inconsistency-tolerant fuzzy interpretation $II := \langle \Delta^{II}, \cdot^{II}\rangle$ where

1. Δ^{II} is a non-empty set such that $\Delta^{II} = \Delta^{I\mathcal{T}I}$,
2. $\cdot^{II}$ is an inconsistency-tolerant fuzzy interpretation function such that
   (a) $R^{II} = R^{I^i}$ and $a^{II} = a^{I^i}$ for any $R \in N_R$ and any $a \in N_I$.
   (b) $(A^i)^{II} = A^{I^i}$ (esp. $A^{II} = A^{I^0}$) for any atomic concept $A^i \in N_C^i$.
   (c) $(\sim A^i)^{II} = (\sim A)^{I^i}$ (esp. $(\sim A)^{II} = (\sim A)^{I^0}$) for any negated atomic concept $\sim A^i$ with $A^i \in N_C^i$.

Then, this lemma is proved by induction on the complexity of C.

• Base step:

1. Case $C \equiv A \in N_C$: We obtain: $(A)^{I^i} = (A^i)^{II} = g(\mathrm{X}^i A)^{II}$ (by the definition of g).
2. Case $C \equiv \sim A$ with $A \in N_C$: We obtain: $(\sim A)^{I^i} = (\sim A^i)^{II} = (\sim g(\mathrm{X}^i A))^{II}$ (by the definition of g) $= g(\mathrm{X}^i \sim A)^{II}$ (by the definition of g).

• Induction step: We show some cases.

1. Case $C \equiv \mathrm{X}D$: We obtain:
   $(\mathrm{X}D)^{I^i}(d)$
   $= D^{I^{i+1}}(d)$
   $= g(\mathrm{X}^{i+1}D)^{II}(d)$ (by induction hypothesis)
   $= g(\mathrm{X}^i \mathrm{X}D)^{II}(d)$.
2. Case $C \equiv \neg D$: We obtain:
   $(\neg D)^{I^i}(d)$
   $= 1 - D^{I^i}(d)$
   $= 1 - g(\mathrm{X}^i D)^{II}(d)$ (by induction hypothesis)
   $= (\neg g(\mathrm{X}^i D))^{II}(d)$
   $= g(\mathrm{X}^i \neg D)^{II}(d)$ (by the definition of g).
3. Case $C \equiv C_1 \sqcap C_2$: We obtain:
   $(C_1 \sqcap C_2)^{I^i}(d)$
   $= min\{C_1^{I^i}(d), C_2^{I^i}(d)\}$
   $= min\{g(\mathrm{X}^i C_1)^{II}(d), g(\mathrm{X}^i C_2)^{II}(d)\}$ (by induction hypothesis)

$= (g(X^iC_1) \sqcap g(X^iC_2))^{II}(d)$
$= g(X^i(C_1 \sqcap C_2))^{II}(d)$ (by the definition of g).

4. Case $C \equiv \forall R.D$: We obtain:
$(\forall R.D)^{I^i}(d)$
$= inf_{d' \in \Delta^{ITI}}\{max\{1 - R^{I^i}(d,d'), D^{I^i}(d')\}\}$
$= inf_{d' \in \Delta^{ITI}}\{max\{1 - R^{I^i}(d,d'), g(X^iD)^{II}(d')\}\}$ (by induction hypothesis)
$= inf_{d' \in \Delta^{II}}\{max\{1 - R^{II}(d,d'), g(X^iD)^{II}(d')\}\}$ (by $\Delta^{ITI} = \Delta^{II}$ and $R^{I^i} = R^{II}$)
$= (\forall R.g(X^iD))^{II}(d)$
$= g(X^i\forall R.D)^{II}(d)$ (by the definition of g).

5. Case $C \equiv \sim XD$: We obtain:
$(\sim XD)^{I^i}(d)$
$= (\sim D)^{I^{i+1}}(d)$
$= g(X^{i+1}\sim D)^{II}(d)$ (by induction hypothesis)
$= (\sim g(X^{i+1}D))^{II}(d)$ (by the definition of g)
$= (\sim g(X^iXD))^{II}(d)$
$= g(X^i\sim XD)^{II}(d)$ (by the definition of g).

6. Case $C \equiv \sim\sim D$: We obtain:
$(\sim\sim D)^{I^i}(d)$
$= D^{I^i}(d)$
$= g(X^iD)^{II}(d)$ (by induction hypothesis)
$= (\sim\sim g(X^iD))^{II}(d)$
$= g(X^i\sim\sim D)^{II}(d)$ (by the definition of g).

7. Case $C \equiv \sim\neg D$: We obtain:
$(\sim\neg D)^{I^i}(d)$
$= 1 - (\sim D)^{I^i}(d)$
$= 1 - g(X^i\sim D)^{II}(d)$ (by induction hypothesis)
$= (\neg g(X^i\sim D))^{II}(d)$
$= (\neg\sim g(X^iD))^{II}(d)$
$= (\sim\neg g(X^iD))^{II}(d)$
$= g(X^i\sim\neg D)^{II}(d)$ (by the definition of g).

8. Case $C \equiv \sim(C_1 \sqcap C_2)$: We obtain:
$(\sim(C_1 \sqcap C_2))^{I^i}(d)$
$= max\{(\sim C_1)^{I^i}(d), (\sim C_2)^{II^i}(d)\}$
$= max\{g(X^i\sim C_1)^{II}(d), g(X^i\sim C_2)^{II}(d)\}$ (by induction hypothesis)
$= max\{(\sim g(X^iC_1))^{II}(d), (\sim g(X^iC_2))^{II}(d)\}$ (by the definition of g)
$= (\sim(g(X^iC_1) \sqcap g(X^iC_2)))^{II}(d)$
$= (\sim g(X^i(C_1 \sqcap C_2)))^{II}(d)$ (by the definition of g)
$= g(X^i\sim(C_1 \sqcap C_2))^{II}(d)$ (by the definition of g).

9. Case $C \equiv \sim\forall R.D$: We obtain:
$(X^i\sim\forall R.D)^{I^i}(d)$
$= sup_{d' \in \Delta^{ITI}}\{min\{R^{I^i}(d,d'), (\sim D)^{I^i}(d')\}\}$
$= sup_{d' \in \Delta^{ITI}}\{min\{R^{I^i}(d,d'), g(X^i\sim D)^{II}(d')\}\}$ (by induction hypothesis)
$= sup_{d' \in \Delta^{II}}\{min\{R^{II}(d,d'), g(X^i\sim D)^{II}(d')\}\}$ (by $\Delta^{ITI} = \Delta^{II}$ and $R^{I^i} = R^{II}$)
$= sup_{d' \in \Delta^{II}}\{min\{R^{II}(d,d'), (\sim g(X^iD))^{II}(d')\}\}$ (by the definition of g).
$= (\sim\forall R.g(X^iD))^{II}(d)$
$= (\sim g(X^i\forall R.D))^{II}(d)$ (by the definition of g)
$= g(X^i\sim\forall R.D)^{II}(d)$ (by the definition of g). ■

We use an expression $X^i\psi$ to denote the fact that the concept C appearing in an inconsistency-tolerant temporal next-time fuzzy assertion ψ is replaced with X^iC.

Lemma 4.7. *Let g be the mapping defined in Definition 4.5. For any inconsistency-tolerant temporal next-time fuzzy interpretation $ITI := \langle \Delta^{ITI}, \{\cdot^{I^i}\}_{i \in \omega}\rangle$ of* itf-$\mathcal{ALC}$*, we can construct an inconsistency-tolerant fuzzy interpretation $II := \langle \Delta^{II}, \cdot^{II}\rangle$ of* if-$\mathcal{ALC}$ *such that for any inconsistency-tolerant temporal next-time fuzzy assertion ψ in $\mathcal{L}^t$ and any $i \in \omega$, $I^i \models \psi$ iff $II \models g(X^i\psi)$.*

Proof. By using Lemma 4.6. ■

Lemma 4.8. *Let g be the mapping defined in Definition 4.5. For any inconsistency-tolerant fuzzy interpretation $II := \langle \Delta^{II}, \cdot^{II}\rangle$ of* if-$\mathcal{ALC}$*, we can construct an inconsistency-tolerant temporal next-time fuzzy interpretation $ITI := \langle \Delta^{ITI}, \{\cdot^{I^i}\}_{i \in \omega}\rangle$ of* itf-$\mathcal{ALC}$ *such that for any inconsistency-tolerant temporal next-time fuzzy assertion ψ in $\mathcal{L}^t$ and any $i \in \omega$, $I^i \models \psi$ iff $II \models g(X^i\psi)$.*

Proof. Similar to the proof of Lemma 4.7. ■

The theorem for embedding itf-$\mathcal{ALC}$ into if-$\mathcal{ALC}$ is presented as follows.

Theorem 4.9. *Let g be the mapping defined in Definition 4.5. For any inconsistency-tolerant temporal next-time fuzzy assertion ψ, ψ is satisfiable in* itf-$\mathcal{ALC}$ *iff $g(\psi)$ is satisfiable in* if-$\mathcal{ALC}$.

Proof. By using Lemmas 4.7 and 4.8. ■

The theorem for relative decidability of itf-$\mathcal{ALC}$ w.r.t. if-$\mathcal{ALC}$ is presented as follows.

Theorem 4.10. *If the satisfiability problem of an inconsistency-tolerant fuzzy knowledge base (with or without some modifications or restrictions) for* if-$\mathcal{ALC}$ *is decidable, then the satisfiability problem of the counterpart inconsistency-tolerant temporal next-time fuzzy knowledge base for* itf-$\mathcal{ALC}$ *is also decidable. Similarly, if an inconsistency-tolerant fuzzy entailment problem for* if-$\mathcal{ALC}$ *and* *an inconsistency-tolerant fuzzy subsumption problem for* if-$\mathcal{ALC}$ *are decidable, then the counterpart inconsistency-tolerant temporal next-time fuzzy entailment problem for* itf-$\mathcal{ALC}$ *and the counterpart inconsistency-tolerant temporal next-time fuzzy subsumption problem for* itf-$\mathcal{ALC}$ *are also decidable.*

Proof. By Definition 4.5 and Theorem 4.9. ■

The theorem for relative complexity of itf-$\mathcal{ALC}$ w.r.t. if-$\mathcal{ALC}$ is presented as follows.

Theorem 4.11. *If a decision problem for* itf-$\mathcal{ALC}$ *is decidable, then the complexity of the decision problem for* itf-$\mathcal{ALC}$ *is the same as that of the counterpart decision problem for* if-$\mathcal{ALC}$*.*

Proof. By using Definition 4.5, Theorem 4.9, and the fact that the mapping g defined in Definition 4.5 is a polynomial-time reduction. ■

The theorem for embedding itf-$\mathcal{ALC}$ into f-$\mathcal{ALC}$ is presented as follows.

Theorem 4.12. *Let h be the composition gf of the mappings f and g defined in Definitions 3.5 and 4.5. For any inconsistency-tolerant temporal next-time fuzzy assertion ψ, ψ is satisfiable in* itf-$\mathcal{ALC}$ *iff $h(\psi)$ is satisfiable in* f-$\mathcal{ALC}$*.*

Proof. By combining Theorems 3.10 and 4.9. ■

The theorem for relative decidability of itf-$\mathcal{ALC}$ w.r.t. f-$\mathcal{ALC}$ is presented as follows.

Theorem 4.13. *If the satisfiability problem of a fuzzy knowledge base (with or without some modifications or restrictions) for* f-$\mathcal{ALC}$ *is decidable, then the satisfiability problem of the counterpart inconsistency-tolerant temporal next-time fuzzy knowledge base for* itf-$\mathcal{ALC}$ *is also decidable. Similarly, if a fuzzy entailment problem for* f-$\mathcal{ALC}$ *and a fuzzy subsumption problem for* f-$\mathcal{ALC}$ *are decidable, then the counterpart inconsistency-tolerant temporal next-time fuzzy entailment problem for* itf-$\mathcal{ALC}$ *and the counterpart inconsistency-tolerant temporal next-time fuzzy subsumption problem for* itf-$\mathcal{ALC}$ *are also decidable.*

Proof. By combining Theorems 3.11 and 4.10. ■

The theorem for relative complexity of itf-$\mathcal{ALC}$ w.r.t. f-$\mathcal{ALC}$ is presented as follows.

Theorem 4.14. *If a decision problem for* itf-$\mathcal{ALC}$ *is decidable, then the complexity of the decision problem for* itf-$\mathcal{ALC}$ *is the same as that of the counterpart decision problem for* f-$\mathcal{ALC}$*.*

Proof. By combining Theorems 3.12 and 4.11. ■

Remark 4.15. *We make the following remarks.*

1. *We can introduce a translation function from* tf-$\mathcal{ALC}$ *to* f-$\mathcal{ALC}$ *and a translation function from* itf-$\mathcal{ALC}$ *to* tf-$\mathcal{ALC}$*. By using these translation functions, we can show the theorem for embedding* tf-$\mathcal{ALC}$ *into* f-$\mathcal{ALC}$ *and the theorem for embedding* itf-$\mathcal{ALC}$ *into* tf-$\mathcal{ALC}$*.*
2. *Similarly, we can show the relative decidability and complexity theorems for* tf-$\mathcal{ALC}$ *w.r.t.* f-$\mathcal{ALC}$ *and the relative decidability and complexity theorems for* itf-$\mathcal{ALC}$ *w.r.t.* tf-$\mathcal{ALC}$*.*

5 CONCLUDING REMARKS

In this study, we introduced the inconsistency-tolerant fuzzy description logic if-$\mathcal{ALC}$, which is an extension of the standard fuzzy description logic f-$\mathcal{ALC}$ originally developed in (Straccia, 2001). A translation from if-$\mathcal{ALC}$ to f-$\mathcal{ALC}$ was defined and a theorem for embedding if-$\mathcal{ALC}$ into f-$\mathcal{ALC}$ was proven using this translation. The relative decidability and complexity theorems for if-$\mathcal{ALC}$ with respect to f-$\mathcal{ALC}$ were also proven using this embedding theorem. The relative decidability theorem states that if a decision problem for f-$\mathcal{ALC}$ is decidable, then the counterpart decision problem for if-$\mathcal{ALC}$ is also decidable. The relative complexity theorem states that if the underlying decision problem for if-$\mathcal{ALC}$ is decidable, then the complexity of the decision procedure of the underlying problem for if-$\mathcal{ALC}$ is the same as the complexity of the decision procedure of the counterpart problem for f-$\mathcal{ALC}$. It was thus shown in this study that by using the embedding theorem, the previously developed algorithms and methods for f-$\mathcal{ALC}$ can be re-purposed for appropriately handling inconsistent fuzzy knowledge-bases that are described by if-$\mathcal{ALC}$. Furthermore, in this study, the inconsistency-tolerant fuzzy temporal next-time description logic itf-$\mathcal{ALC}$ was obtained from if-$\mathcal{ALC}$ by adding the temporal next-time operator X. Similar results as those for if-$\mathcal{ALC}$ were also obtained for itf-$\mathcal{ALC}$. It was thus also shown in this study that the previously developed algorithms and methods for f-$\mathcal{ALC}$ can be re-purposed for appropriately handling inconsistent temporal fuzzy knowledge bases that are described by itf-$\mathcal{ALC}$.

Finally, we address a brief survey of inconsistency-tolerant description logics, because inconsistency-tolerant description logics are not very popular. An *inconsistency-tolerant four-valued terminological logic* was introduced by Patel-Schneider (Patel-Schneider, Peter F., 1989). A sequent calculus for reasoning in four-valued description logics was developed by Straccia (Straccia, 1997b). Application of a four-valued description logic to information retrieval was proposed by Meghini et al. (Meghini and Straccia, 1996; Meghini et al., 1998). Three kinds of *inconsistency-tolerant constructive description logics* were introduced by Odintsov and Wansing (Odintsov, S.P. and Wansing, 2003; Odintsov, S.P. and Wansing, 2008). These inconsistency-tolerant constructive description logics are based on single-interpretation semantics, which can be seen as a description logic version of the single-consequence Kripke semantics for *Nelson's paraconsistent four-valued logic* N4 (Almukdad and Nelson, 1984; Nelson, 1949). Several *paraconsistent four-valued description logics* including $\mathcal{ALC}4$ were developed by Ma et al. (Ma et al., 2007; Ma et al., 2008). The logic $\mathcal{ALC}4$ is based on *four-valued semantics*, and can be effectively translated to $\mathcal{ALC}$. By using this translation, the satisfiability problem for $\mathcal{ALC}4$ was shown to be decidable. However, $\mathcal{ALC}4$ and its variants have no classical negation (or complement). Several *quasi-classical description logics* were developed by Zhang et al. (Zhang and Lin, 2008; Zhang et al., 2009). These quasi-classical description logics are based on *quasi-classical semantics* and have the classical negation. However, translations of the quasi-classical description logics to the counterpart standard description logics were not proposed. An inconsistency-tolerant description logic, $\mathcal{PALC}$, was introduced by Kamide (Kamide, 2012) based on the idea of a multiple-interpretation description logic, $\mathcal{ALC}^n_\sim$, proposed by Kaneiwa (Kaneiwa, 2007). The logic $\mathcal{PALC}$ is constructed on the basis of *dual-interpretation semantics* and has both the benefits of $\mathcal{ALC}4$ and quasi-classical description logics (i.e., it has the faithful translation and the classical negation connective). The semantics of $\mathcal{PALC}$ is constructed on the basis of the dual-consequence Kripke semantics for N4. A comparison among some previously developed inconsistency-tolerant description logics was addressed by Kamide in (Kamide, 2013), wherein a simple system $\mathcal{SALC}$, which is logically equivalent to $\mathcal{PALC}$, was developed using a simple single interpretation semantics. Some interpolation theorems were proven by Kamide in (Kamide, 2011) for two extended inconsistency-tolerant and temporal description logics using some theorems for embedding these logics into a standard description logic. An extended inconsistency-tolerant description logic with a sequence modal operator has recently been introduced by Kamide in (Kamide, 2020a), wherein it was intended to suitably handle inconsistency-tolerant ontological reasoning with sequential information that is expressed as sequences. This extended inconsistency-tolerant description logic was shown in (Kamide, 2020a) to be decidable via an embedding theorem.

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