

Principles of  
**Constraint  
Programming**

Krzysztof R. Apt



CAMBRIDGE

# Contents

<i>Acknowledgements</i>	page xi
<b>1</b> <i>Introduction</i>	1
1.1 Basic characteristics of constraint programming	1
1.2 Applications of constraint programming	3
1.3 A very short history of the subject	5
1.4 Our approach	6
1.5 Organisation of the book	6
<b>2</b> <i>Constraint satisfaction problems: examples</i>	8
2.1 Basic concepts	9
2.2 Constraint satisfaction problems on integers	11
2.3 Constraint satisfaction problems on reals	16
2.4 Boolean constraint satisfaction problems	19
2.5 Symbolic constraint satisfaction problems	21
2.6 Constrained optimization problems	43
2.7 Summary	47
2.8 Exercises	48
2.9 Bibliographic remarks	51
2.10 References	52
<b>3</b> <i>Constraint programming in a nutshell</i>	54
3.1 Equivalence of CSPs	55
3.2 Basic framework for constraint programming	58
3.2.1 PREPROCESS	59
3.2.2 HAPPY	60
3.2.3 ATOMIC	61
3.2.4 SPLIT	61
3.2.5 PROCEED BY CASES	64
3.2.6 CONSTRAINT PROPAGATION	66

3.2.7	Constraint propagation algorithms	70
3.3	Example: Boolean constraints	71
3.4	Example: polynomial constraints on integer intervals	74
3.5	Summary	80
3.6	Bibliographic remarks	81
<b>4</b>	<b><i>Some complete constraint solvers</i></b>	<b>82</b>
4.1	A proof theoretical framework	83
4.1.1	Proof rules	84
4.1.2	Derivations	87
4.2	Term equations	92
4.2.1	Terms	93
4.2.2	Substitutions	94
4.2.3	Unifiers and mgus	95
4.2.4	Unification problem and solving of CSPs	98
4.2.5	The <i>UNIF</i> proof system	99
4.2.6	The MARTELLI–MONTANARI algorithm	103
4.3	Linear equations over reals	107
4.3.1	Linear expressions and linear equations	107
4.3.2	Substitutions, unifiers and mgus	110
4.3.3	Linear equations and CSPs	111
4.3.4	The <i>LIN</i> proof system	112
4.3.5	The GAUSS–JORDAN ELIMINATION algorithm	115
4.3.6	The GAUSSIAN ELIMINATION algorithm	118
4.4	Linear inequalities over reals	121
4.4.1	Syntax	121
4.4.2	Linear inequalities and CSPs	122
4.4.3	The <i>INEQ</i> proof system	123
4.4.4	The FOURIER–MOTZKIN ELIMINATION algorithm	124
4.5	Summary	131
4.6	Exercises	131
4.7	Bibliographic remarks	132
4.8	References	133
<b>5</b>	<b><i>Local consistency notions</i></b>	<b>135</b>
5.1	Node consistency	136
5.2	Arc consistency	138
5.3	Hyper-arc consistency	143
5.4	Directional arc consistency	144
5.5	Path consistency	147
5.6	Directional path consistency	155

5.7	$k$ -consistency	157
5.8	Strong $k$ -consistency	164
5.9	Relational consistency	166
5.10	Graphs and CSPs	170
5.11	Summary	175
5.12	Exercises	175
5.13	Bibliographic remarks	176
5.14	References	176
<b>6</b>	<b><i>Some incomplete constraint solvers</i></b>	<b>178</b>
6.1	A useful lemma	180
6.2	Equality and disequality constraints	181
6.3	Boolean constraints	184
6.3.1	Transformation rules	185
6.3.2	Domain reduction rules	186
6.3.3	Example: full adder circuit	188
6.3.4	A characterisation of the system <i>BOOL</i>	191
6.4	Linear constraints on integer intervals	192
6.4.1	Domain reduction rules for inequality constraints	194
6.4.2	Domain reduction rules for equality constraints	196
6.4.3	Rules for disequality constraints	199
6.4.4	Rules for strict inequality constraints	200
6.4.5	Shifting from intervals to finite domains	200
6.4.6	Example: the <i>SEND + MORE = MONEY</i> puzzle	201
6.4.7	Bounds consistency	202
6.4.8	A characterisation of the <i>LINEAR EQUALITY</i> rule	206
6.5	Arithmetic constraints on integer intervals	211
6.5.1	Domain reduction rules: first approach	211
6.5.2	Domain reduction rules: second approach	213
6.5.3	Domain reduction rules: third approach	217
6.5.4	Implementation of the third approach	221
6.5.5	Shifting from intervals to finite domains	223
6.6	Arithmetic constraints on reals	224
6.6.1	Interval arithmetic	226
6.6.2	Domain reduction rules	227
6.6.3	Implementation issues	233
6.6.4	Using floating-point intervals	236
6.6.5	Correctness and efficiency issues	238
6.7	Arithmetic equations over reals	242
6.8	Summary	245

6.9	Exercises	245
6.10	Bibliographic remarks	248
6.11	References	251
<b>7</b>	<b><i>Constraint propagation algorithms</i></b>	<b>254</b>
7.1	Generic iteration algorithms	256
7.1.1	Iterations	256
7.1.2	Algorithms for arbitrary partial orderings	261
7.1.3	Algorithms for cartesian products of partial orderings	264
7.2	From partial orderings to CSPs	268
7.3	A node consistency algorithm	269
7.4	An arc consistency algorithm	271
7.5	A hyper-arc consistency algorithm	273
7.6	A directional arc consistency algorithm	275
7.7	A path consistency algorithm	277
7.8	A directional path consistency algorithm	281
7.9	A $k$ -consistency algorithm	283
7.10	A relational consistency algorithm	286
7.11	Implementations of incomplete constraint solvers	287
7.12	Summary	290
7.13	Exercises	291
7.14	Bibliographic remarks	295
7.15	References	297
<b>8</b>	<b><i>Search</i></b>	<b>299</b>
8.1	Search trees	301
8.2	Labeling trees	303
8.2.1	Complete labeling trees	304
8.2.2	Reduced labeling trees	308
8.2.3	<i>prop</i> labeling trees	310
8.3	An example: <i>SEND + MORE = MONEY</i>	313
8.4	Instances of <i>prop</i> labeling trees	315
8.4.1	Forward checking	315
8.4.2	Partial look ahead	319
8.4.3	Maintaining arc consistency (MAC)	321
8.5	Search algorithms for the labeling trees	324
8.5.1	Backtrack-free search	325
8.5.2	Backtrack-free search with constraint propagation	327
8.5.3	Backtracking	329
8.5.4	Backtracking with constraint propagation	330
8.6	Instances of backtracking with constraint propagation	332

8.6.1 Forward checking	332
8.6.2 Partial look ahead	333
8.6.3 Maintaining arc consistency (MAC)	334
8.6.4 Searching for all solutions	335
8.7 Search algorithms for finite constrained optimization problems	335
8.7.1 Branch and bound	337
8.7.2 Branch and bound with constraint propagation	339
8.7.3 Branch and bound with constraint propagation and cost constraint	339
8.8 Heuristics for search algorithms	341
8.8.1 Variable selection	341
8.8.2 Value selection	343
8.9 An abstract branch and bound algorithm	344
8.10 Summary	347
8.11 Exercises	347
8.12 Bibliographic remarks	348
8.13 References	349
<b>9 <i>Issues in constraint programming</i></b>	<b>351</b>
9.1 Modeling	352
9.1.1 Choosing the right variables	352
9.1.2 Choosing the right constraints	353
9.1.3 Choosing the right representation	356
9.1.4 Global constraints	358
9.2 Constraint programming languages	359
9.2.1 Constraint logic programming	360
9.2.2 ILOG solver	362
9.2.3 Generation of constraints	363
9.3 Constraint propagation	364
9.4 Constraint solvers	366
9.4.1 Building constraint solvers	366
9.4.2 Incrementality	367
9.4.3 Simplification of constraints	368
9.5 Search	369
9.5.1 Search in modeling languages	369
9.5.2 Depth-first search: backtracking and branch and bound	370
9.5.3 Breadth-first search and limited discrepancy search	371
9.5.4 Local search	372
9.5.5 Search in constraint programming languages	375
9.5.6 Biology-inspired approaches	378

9.6	Over-constrained problems	379
9.6.1	Partial, weighted and fuzzy CSPs	380
9.6.2	Constraint hierarchies	381
9.6.3	Generalisations	383
9.6.4	Reified constraints	383
9.7	Summary	384
9.8	Bibliographic remarks	384
	<i>Bibliography</i>	387
	<i>Author index</i>	401
	<i>Subject index</i>	404