Statistics of Atomic Frequency Standards

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Abstract—A theoretical development is presented which results in a relationship between the expectation value of the standard deviation of the frequency fluctuations for any finite number of data samples and the infinite time average value of the standard deviation, which provides an invariant measure of an important quality factor of a frequency standard. A practical and straightforward method of determining the power spectral density of the frequency fluctuations from the variance of the frequency fluctuations, the sampling time, the number of samples taken, and the dependence on system bandwidth is also developed. Additional insight is also given into some of the problems that arise from the presence of "flicker noise" (spectrum proportional to $|\omega|^{-1}$) modulation of the frequency of an oscillator.

The theory is applied in classifying the types of noise on the signals of frequency standards made available at NBS, Boulder Laboratories, such as: masers (both H and $N^{15}H_3$), the cesium beam frequency standard employed as the U. S. Frequency Standard, and rubidium gas cells.

"Flicker noise" frequency modulation was not observed on the signals of masers for sampling times ranging from 0.1 second to 4 hours. In a comparison between the NBS hydrogen maser and the NBS III cesium beam, uncorrelated random noise was observed on the frequency fluctuations for sampling times extending to 4 hours; the fractional standard deviations of the frequency fluctuations were as low as 5 parts in 10^{14}.

I. INTRODUCTION

AS ATOMIC TIMEKEEPING has come of age, it has become increasingly important to identify quality in an atomic frequency standard. Some of the most important quality factors are directly related to the inherent noise of a quantum device and its associated electronics. For example, a proper measurement and statistical classification [1] of this inherent noise makes it possible to determine the probable rate of time divergence of two independent atomic time systems, as well as giving insight concerning the precision and accuracy obtainable from an atomic frequency standard.

In the realm of precise frequency measurements, the properties of noise again play an important role. The relative precision obtainable with atomic frequency standards is unsurpassed in any field, and the precision limitations in this field are largely due to inherent noise in the atomic device and the associated electronic equipment. The standard deviation of the frequency fluctuations can be shown to be directly dependent on the type of noise in the system, the number of samples taken, and the dead-time between samples.

A very common and convenient way of making measurements of the noise components on a signal from

a frequency standard is to compare two such standards by measuring the period of the beat frequency between the two standards. It is again the intent of the author to show a practical and easy way of classifying the statistics, i.e., of determining the power spectral density of the frequency fluctuations using this type of measuring system.

An analysis has already been made of the noise present in passive atomic frequency standards [1], such as cesium beams, but a classification of the types of noise exhibited by the maser type of quantum-mechanical oscillator has not been made in the long term area, i.e., for low frequency fluctuations. Though this paper is far from exhaustive, the intent is to give additional information on the noise characteristics of masers. Because a maser's output frequency is more critically parameter dependent than a passive atomic device, it has been suggested [2] that the output frequency might appear to be "flicker noise" modulated, where "flicker noise" is defined as a type of power spectral density which is inversely proportional to the spectral frequency $\omega/2\pi$. It has been shown that if "flicker noise" frequency modulation is present on a signal from a standard, some significant problems arise, such as the logarithmic divergence of the standard deviation of the frequency fluctuations as the number of samples taken increases, and also the inability to define precisely the time average frequency. It thus becomes of special interest to determine whether "flicker noise" is or is not present on the signal from a maser so that one might better evaluate its quality as a frequency standard.

Throughout the paper, the paramount mathematical concern is the functional form of the equations with the hope of maintaining simplicity and of providing better understanding of the material to be covered.

II. METHODS EMPLOYED TO MEASURE NOISE

A. Power Spectrum and Variance Relationship

The average angular frequency $\Omega_\tau(t)$ of an oscillator (to distinguish it from spectral frequency ω) over a time interval τ can be written

$$\Omega_\tau(t) = \frac{1}{\tau}\left[\phi(t+\tau) - \phi(t)\right], \tag{1}$$

where ϕ is the phase angle in radians. Now the variance of the frequency deviations is the square of the standard deviation σ. Define the time average of a function as

$$\langle f(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, dt. \tag{2}$$

One may, therefore, write the square of the standard deviation as follows:

$$\sigma^2 = \langle \Omega_r(t)^2 \rangle - \langle \Omega_r(t) \rangle^2. \qquad (3)$$

One may assume with no loss of generality that the second term in (3) can be set equal to zero by a proper translation, since it is the square of the time average frequency. Therefore, $\Omega_r(t)$ is now the frequency deviation from the average value, and $\phi(t)$ the integrally related phase deviation.

Substituting (1) into (3) gives

$$\sigma^2 = \frac{1}{\tau^2} \left[\langle \phi(t+\tau)^2 \rangle - 2\langle \phi(t+\tau) \cdot \phi(t) \rangle + \langle \phi^2(t) \rangle \right]. \qquad (4)$$

The time average of $\phi(t+\tau) \cdot \phi(t)$ is the autocovariance function of the phase—denoted $R_\phi(\tau)$. One is justified in assuming that a time translation has no effect on the autocovariance function [1], therefore [4],

$$\sigma^2 = \frac{2}{\tau^2} \left[R_\phi(0) - R_\phi(\tau) \right]. \qquad (5)$$

It is now possible to relate the variance of the squared frequency deviations to the power spectral density by use of the Wiener-Khinchin Theorem, which states that the autocovariance function of the phase is equal to the Fourier transform (F.T.) of the power spectral density of the phase $S_\phi(\omega)$. The power spectral density of the frequency is related to this by the useful equation

$$S_{\dot\phi}(\omega) = \omega^2 S_\phi(\omega).$$

Most of the discussion that follows is based on the restriction

$$S_\phi(\omega) = h \, |\omega|^\alpha. \qquad (6)$$

That a singular type of power spectrum predominates over a reasonable range of ω has been verified experimentally. The region of interest for α is $-3 \leq \alpha \leq -1$, and $\alpha = 0$. This covers white noise phase modulation $(S_\phi(\omega) = h)$, "flicker noise" frequency modulation $(S_{\dot\phi}(\omega) = h\,|\omega|^{-1})$, and includes, of course, white noise frequency modulation $(S_{\dot\phi}(\omega) = h)$. Fortunately, the Fourier transforms of functions of the above form [3] have been tabulated, and the following transforms can be established:

F.T. $|\omega|^\alpha = a'(\alpha) \cdot |\tau|^{-\alpha-1}$ for $\alpha \neq 0$ or not an integer

F.T. $|\omega|^{-0} = \delta(\tau)$

F.T. $|\omega|^{-2} = a'(-2) \cdot |\tau| \qquad (7)$

where a' is an α dependent coefficient. A useful substitution is the following [1]:

$$U(\tau) = 2[R_\phi(0) - R_\phi(\tau)]. \qquad (8)$$

Because of finite system bandwidths, ω_B, a better representation of $U(\tau)$ is obtained by replacing $R_\phi(0)$ with $R_\phi(1/\omega_B)$ (see Appendix). If the sample time τ is large compared to the reciprocal system bandwidth $1/\omega_B$, then $R_\phi(1/\omega_B)$ is negligible compared to $R_\phi(\tau)$ in the

region where $-2 < \mu < 0 (-1 > \alpha > -3)$. $R_\phi(1/\omega_B)$ becomes the larger term, however, in the region of $-1 < \alpha < 0$, and if one assumes $(\omega_B\tau)^{\alpha+1} \gg 1$, then $R_\phi(\tau)$ is neglectable (see Appendix). The following equations for $U(\tau)$ may, therefore, be written:

$$U(\tau) = \begin{cases} |a(\alpha)| \, |\tau|^{-\alpha-1}; & -3 < \alpha < -1 \\ |a(\alpha)| \, \omega_B|^{\alpha+1}; & -1 < \alpha \leq 0. \end{cases} \qquad (9)$$

The standard deviation squared may, therefore, be written:

$$\sigma^2 = \begin{cases} a(\mu) \cdot |\tau|^\mu; & -3 < \alpha < -1 \\ a(\alpha) \, |\omega_B|^{\alpha+1} \, |\tau|^{-2}; & -1 < \alpha \leq 0 \end{cases} \qquad (10)$$

where $\mu = -\alpha - 3$.

$A(\mu)$ has a small dependence on ω_B, implicit within the previous assumptions. Considering the results of (10), the Appendix, and that to be discussed in Section II-D on "flicker noise" modulation $(\alpha = -1, -3)$, an informative graph of μ into α may be established as illustrated below.
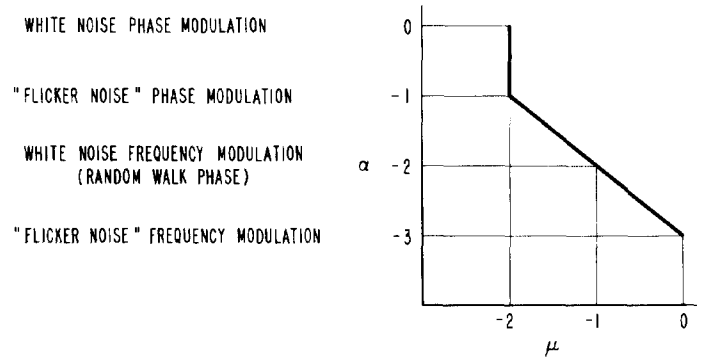
Using (6) and (10) along with the above graph, one may, therefore, deduce the power spectral density from the dependence of the standard deviation of the frequency fluctuations on the sampling time, with restrictions on the experimental parameters as will be shown in the following.

B. Adjacent Sampling of Data

In actual practice, of course, the number of frequency or phase samples must be finite. The case to be considered now is one for which the phase or the frequency is monitored on a continuous basis. Two of the techniques used by the author to accomplish this were as follows: A device, described elsewhere [1], was used to monitor the phase of the beat frequency between two oscillators at prescribed time intervals; the second technique was to measure the period of the beat frequency between two oscillators with two counters so that the dead-time of one counter corresponded to the counting time of the other.

A very powerful and meaningful method of analysis of data taken by a phase monitoring technique has been developed by J. A. Barnes [1] at the National Bureau of Standards, Boulder, Colo. The method employs the use of finite differences and is especialy useful in analyzing

"flicker noise" and long-term frequency fluctuations in general. On the other hand, if one uses the period counting technique for data acquisition, the following form of analysis is useful. It also may be cast in a form where one may use the finite difference technique.

Data are often obtained with one counter measuring the frequency or the period of the beat note between two oscillators with a dead-time between counts.[1] The concern of Section II is with the dead-time being zero, but it is convenient to develop the general case for which the dead-time is nonzero for use in Section III, and specialize this to the continuous sampling case for which the dead-time is zero, which is the case of interest in this section.

Let T be the period of sampling, τ the sample time, and N the number of samples. The standard deviation, $\sigma(N, T, \tau)$, of the frequency fluctuations[2] may, therefore, be written as:

$$\sigma^2(N, T, \tau) = \frac{1}{N-1} \left\{ \sum_{n=0}^{N-1} \left[\frac{\phi(nT+\tau) - \phi(nT)}{\tau} \right]^2 \right.$$
$$\left. - \frac{1}{N} \left[\sum_{n=0}^{N-1} \frac{\phi(nT+\tau) - \phi(nT)}{\tau} \right]^2 \right\}. \quad (11)$$

Taking the expectation value of $\sigma^2(N, T, \tau)$ and making the substitution given in (8) yields

$$\langle \sigma^2(N, T, \tau) \rangle = \frac{1}{\tau^2} \left\{ U(\tau) + \frac{1}{N(N-1)} \right.$$
$$\left. \sum_{n=0}^{N-1} (N-n)[2U(nT) - U(nT+\tau) - U(nT-\tau)] \right\}. \quad (12)$$

If the dead-time were zero, then $T = \tau$, and (12) becomes [1]

$$\langle \sigma^2(N, \tau) \rangle = \frac{1}{(N-1)\tau^2} \left[NU(\tau) - \frac{1}{N} U(N\tau) \right]. \quad (13)$$

Remembering that $\mu = -\alpha - 3$ and substituting (9) into (13) gives

$$\langle \sigma^2(N, \tau) \rangle = \frac{a(\mu)N |\tau|^\mu}{N-1} [1 - N^\mu]; \quad -2 < \mu < 0 \quad (14)$$

which establishes the interesting result of the dependence of the expectation value of the standard deviation of the frequency fluctuations on the number of samples, the sample time, and the power spectral density. It will be noted that

$$\langle \sigma^2(\infty, \tau) \rangle = a(\mu) |\tau|^\mu; \quad -2 < \mu < 0 \quad (15)$$

in agreement with (10).

[1] See L. S. Cutler and C. L. Searle, "Some aspects of the theory and measurement of frequency fluctuations in frequency standards," this issue, page 136. This paper shows that the fluctuations $\Delta\tau$ in the period τ are a good approximation to the frequency fluctuations if $\Delta\tau \gg \tau$.

[2] Note that this $\sigma(N, T, \tau)$ is not the same as the σ in (3)-(5), and (10). $\sigma(N, T, \tau)$ is over a finite number of data samples N and, to avoid confusion, the variable N will always be used with σ in the finite sampling case as in (11).

By keeping N constant and assuming μ to be constant over several different values of τ, it may be seen that the value of μ is the slope on a log-log plot of $\langle \sigma^2(N, \tau) \rangle$ vs. τ. This provides a means of determining the power spectral density simply by varying the sample time over the region of interest [4].

It is informative to look at the family of curves obtained from a plot of the dependence of $\langle \sigma^2(N, \tau) \rangle$ as a function of N for various pertinent values of μ to see how it approaches $\langle \sigma^2(\infty, \tau) \rangle$. The family of curves is shown in Fig. 1. One may notice that the convergence is much faster in the region between white noise frequency modulation and white noise phase modulation than between white noise frequency modulation and "flicker noise" frequency modulation.[3] In fact, as $\mu \to 0$, the ratio approaches zero, and one would conjecture that the

$$\lim_{N \to \infty} \langle \sigma^2(N, \tau) \rangle$$

is infinite in the presence of "flicker noise" frequency modulation—a result proven by J. A. Barnes et al. [1].
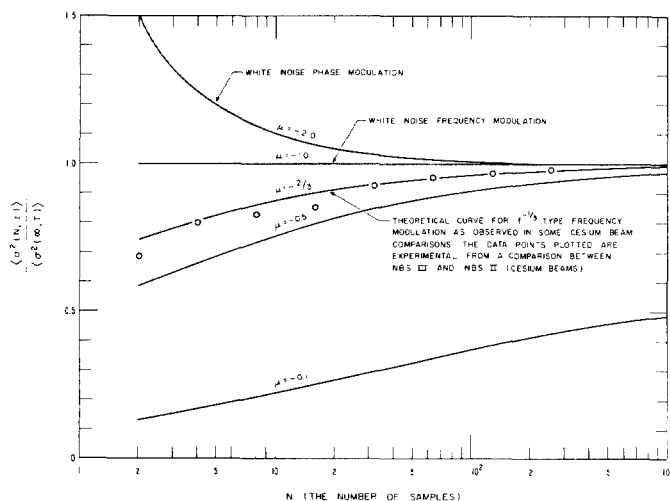

Fig. 1. A plot showing the dependence of the standard deviation of the frequency fluctuation on the number of samples and the type of noise present.

The data points plotted in Fig. 1 were extracted from a cesium beam-cesium beam comparison analyzed elsewhere [1], and exhibit in this new formulation a type of frequency modulation proportional to $|\omega|^{-1/3}$, giving confirmation to this strange type of power spectral density.

It is possible to utilize the dependence of the standard deviation on the number of samples to determine a value of μ by considering a function which takes into account the extreme values of N obtainable from a finite set of data, namely,[4]

[3] To see that $\mu = 0$ corresponds to "flicker noise" see Section II-D.

[4] It will be noted that the τ dependence cancels in the expression for χ and hence it is N and μ dependent only. In the table and graphs, the τ dependence is not shown and is, therefore, suppressed since the μ dependence is the thing emphasized.

$$\chi(N, \mu) = \frac{\langle \sigma^2(N, \tau) \rangle}{\langle \sigma^2(2, \tau) \rangle}. \tag{16}$$

Tabulated values of this function are given in Table I, and a plot of $\chi(N, \mu)$ as a function of μ for various values of N is given in Fig. 2. It may be noted that the function is most sensitive in the region between "flicker noise" frequency modulation $\mu = 0$ and white noise frequency modulation $\mu = -1$—one of basic interest. In practice, the table and graph have proven very useful. $\chi(\infty, \mu)$ is plotted for comparison and computational purposes.

Note that $\chi(\infty, 0) = \infty$. In the development thus far,

TABLE I

THE DEPENDENCE OF $\langle \sigma(N, \tau) \rangle$ ON THE NUMBER OF SAMPLES FOR A FIXED SAMPLE TIME τ, THUS DETERMINING THE VALUE OF μ AND THE STATISTICS. THE VALUES LISTED ARE OF

$$\chi(N, \mu) = \frac{\langle \sigma^2(N, \tau) \rangle}{\langle \sigma^2(2, \tau) \rangle}.$$

| μ	N (Number of Samples)					
	4	16	64	256	1024	∞
0.0	1.337	2.133	3.048	4.016	5.004	∞
−0.1	1.288	1.928	2.580	3.190	3.736	7.464
−0.2	1.247	1.753	2.215	2.598	2.899	3.855
−0.3	1.208	1.604	1.928	2.167	2.332	2.660
−0.4	1.171	1.475	1.700	1.847	1.937	2.062
−0.5	1.138	1.365	1.517	1.606	1.655	1.705
−0.6	1.106	1.270	1.369	1.422	1.447	1.467
−0.7	1.077	1.188	1.249	1.278	1.291	1.299
−0.8	1.049	1.116	1.150	1.165	1.171	1.174
−0.9	1.023	1.054	1.068	1.074	1.076	1.076
−1.0	1.000	1.000	1.000	1.000	1.000	1.000
−1.1	0.977	0.952	0.942	0.938	0.937	0.937
−1.2	0.956	0.910	0.893	0.887	0.886	0.886
−1.3	0.937	0.873	0.851	0.844	0.842	0.842
−1.4	0.919	0.841	0.815	0.807	0.805	0.804
−1.5	0.902	0.812	0.784	0.776	0.774	0.773
−1.6	0.886	0.786	0.756	0.748	0.746	0.745
−1.7	0.871	0.763	0.733	0.725	0.723	0.722
−1.8	0.858	0.743	0.712	0.704	0.702	0.701
−1.9	0.845	0.724	0.693	0.685	0.683	0.682
−2.0	0.833	0.708	0.677	0.669	0.667	0.667
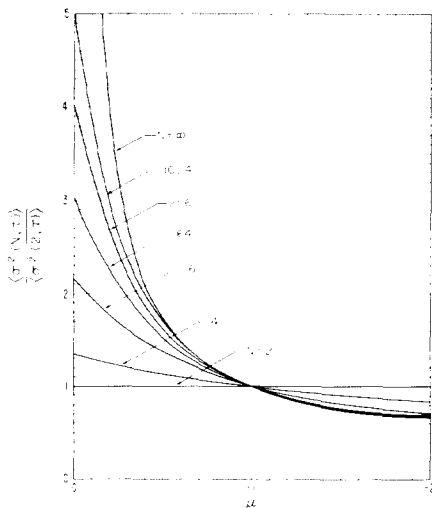
Fig. 2. A plot enabling one to experimentally extract the statistics of an oscillator by knowing the standard deviation for N samples and for two samples using $S_\phi(\omega) = h |\omega|^{-\mu-1}$ except at $\mu = -2$.

no consideration has been given to the experimental fact that there must exist a lower cutoff frequency that keeps the functions considered from going to infinity, corresponding to certain types of noise, such as "flicker noise." The value of the cutoff frequency is not important other than to say that the functions considered are valid for times up to the order of $1/(\omega \text{ cutoff})$. This time is apparently more than a year for quartz crystal oscillators [5]. If "flicker noise" is present in some atomic frequency standards, the value of $1/(\omega \text{ cutoff})$ is probably less than for quartz crystal oscillators for reasons discussed later, and if "flicker noise" is not present, infinities do not occur in the functions considered for most other types of pertinent noise and hence there is no concern.

C. Non-Adjacent Sampling of Data

The next consideration is to determine the effect of counter dead-time on one's ability to deduce the statistics of an oscillator using the techniques developed in Section II-B. This form of data acquisition is one of the most common, and hence merits attention.

It was shown earlier that (12) is applicable to the present case, and if (9) is substituted into (12), with the assignment that $r = T/\tau$ (the ratio of the period of sampling to the sample time), then

$$\langle \sigma^2(N, T, \tau) \rangle$$

$$= a(\mu) |\tau|^\mu \left\{ 1 + \sum_{n=1}^{N-1} \frac{(N-n)(nr)^{\mu+2}}{N(N-1)} \left[2 - \left(1 + \frac{1}{nr}\right)^{\mu+2} \right. \right.$$

$$\left. \left. - \left(1 - \frac{1}{nr}\right)^{\mu+2} \right] \right\}; \qquad -2 < \mu < 0. \tag{17}$$

An important result from (17) is that if N and r are held constant, it is still possible to determine the value of μ by varying τ. Therefore, the relationship between the power spectral density and the standard deviation has the same form as for the continuous sampling case. Additional insight may be obtained by considering some special cases. If $\mu = -1$, $(S_\phi(\omega) = h)$ [6], the series in (17) goes to zero for all possible values of r and hence

$$\langle \sigma^2(N, T, \tau) \rangle = \frac{a(-1)}{|\tau|} \tag{18}$$

for white noise frequency modulation; this is the same result obtained in the continuous data sampling case. One notices that (18) is independent of N as would be espected since the frequency fluctuations are uncorrelated.

If $\mu = -2$, the series in (17) again goes to zero for all values of $r > 1$. The value of α is degenerate except that one may say $-1 < \alpha$. In this domain, the frequency fluctuations appear to be uncorrelated as long as the measurement dead-time is nonzero. Using the proper form of $U(\tau)$ from (9) and substituting into (13) for the case where $r = 1$ (zero dead-time), gives:

$$\langle \sigma^2(N, \tau) \rangle = [a(\alpha) |\omega_B|^{\alpha+1}(N+1)/N\tau^2].$$

One, therefore, has the unusual result that

$$\langle \sigma^2(N, T, \tau) \rangle = \frac{N + \delta(r - 1)}{N} \cdot \frac{a(\alpha) \mid \omega_B \mid^{\alpha+1}}{\tau^2} \quad (19)$$

where

$$\delta(r - 1) = \begin{cases} 1, & r = 1 \\ 0, & r \neq 1, \end{cases} \quad \text{for } -1 < \alpha \text{ and } (\omega_B\tau)^{\alpha+1} \gg 1.$$

A slight N dependence then appears only in the continuous sampling case. Experimentally, one can show that the Kronecker δ-function is replaced by $R_\phi(r-1)/R_\phi(0)$ because of the finite bandwidths involved.

It will be recalled that the curves in Figs. 1 and 2 are for $r = 1$ (the dead-time equal zero). The results of (19) show a character change in the curves for $r > 1$ and $-2 < \mu < -1$, for now the curve for $\mu = -2$ is coincident with the curve for $\mu = -1$ in Fig. 1 and $\chi(N, -2) = \chi(N, -1) = 1.0$ in Fig. 2. No profound character change occurs for $-1 < \mu < 0$.

It is possible to show that the series in (17) approaches zero as N approaches infinity for all values of $r > 1$ and $\mu < 0$, hence $\langle \sigma^2(N, T, \tau) \rangle$ has the same asymptotic value as for the continuous sampling technique, independent of the counter dead-time.

If a binomial expansion is made of the second two terms in the series expression of (17), and fourth-order terms and higher in $1/nr$ are neglected, the following simplification occurs:

$$\langle \sigma^2(N, T, \tau) \rangle$$

$$= a(\mu) \mid \tau \mid^\mu \left[1 - \frac{(\mu + 2)(\mu + 1)}{N(N - 1)} r^\mu \sum_{n=1}^{N} (N - n)n^\mu \right]. \quad (20)$$

On the first observation of (20), one may notice that as r becomes large, the standard deviation approaches its asymptotic value. This occurs when one is taking samples much shorter than the capable reset time of the counter. The dependence on N is, therefore, reduced as r^μ. In fact, it has been determined by a computer analysis of (17) that the net effect of increasing the dead-time is to collapse the curves in Figs. 1 and 2 towards the unit axis.

D. The "Flicker Noise" Problem

The existence of "flicker noise" frequency modulation on the signal of quartz crystal oxcillators has caused difficulty in handling such quantities as the autocovariance function of the phase and the standard deviation of the frequency fluctuations. As mentioned previously, the development by J. A. Barnes [1] makes it possible to classify "flicker noise" frequency modulation without any divergence difficulties or dependence on the value of the low-frequency cutoff.

Some of the other difficulties associated with the presence of "flicker noise" frequency modulation, as Barnes has shown, are illustrated in the following equations:

$$\langle \sigma^2(N, \tau) \rangle = \frac{2hN \ln N}{N - 1}, \quad (21)$$

and

$$\lim_{\tau \to 0} + \left\langle \left[\frac{\phi(t + \tau) - \phi(t)}{\tau} \right] \right\rangle = \langle \Omega(t) \rangle. \quad (22)$$

The fact that the standard deviation diverges with N, as shown in (21), is an annoyance, aside from the fact that it becomes more difficult to write specifications on the frequency fluctuations of an oscillator. The limit expressed in (22) does not exist or is dependent on the low-frequency cutoff; this, of course, affords difficulty in defining a frequency, and would be an unfortunate property to be present in a frequency standard. As has been stated previously for all values of μ considered thus far, $-2 < \mu < 0$, the limit as N approaches infinity of $\langle \sigma^2(N, \tau) \rangle$ exists; and as the sampling time increases, $\langle \sigma^2(N, \tau) \rangle$ converges toward zero or perfect precision.

Data have been analyzed that indicate the presence of "flicker noise" frequency modulation on the signals from rubidium gas cell frequency standards. It is of concern to determine if this type of noise is present on the signals of other atomic frequency standards, and specifically masers. It is not the intent to determine the source or sources of "flicker noise," as this is a ponderous problem in and of itself, but perhaps only infer where such noise might arise.

Consider, now, ways to establish the presence of "flicker noise." If the data were on a continuous basis, the method of finite differences [1] developed by Barnes would be very useful. One may also notice from (21) that if N is held constant, the value of μ is zero for values of τ in the "flicker noise" frequency modulation region. If data were taken by the common technique of non-adjacent samples, the following considerations are of value.

For "flicker noise" frequency modulation, $U(\tau)$ takes on a different form as a result of the divergence of the autocovariance function of the phase [1],

$$U(\tau) = \lim_{\mu \to 0^-} \frac{k \mid \tau \mid^{\mu+2}}{4 - 2^{\mu+2}}. \quad (23)$$

The constant k is dependent on the quality of the oscillator. If (23) is substituted into (12), letting $\mu \to 0$ ($\mu = 0$ corresponds to "flicker noise"), an indeterminate form results for $\langle \sigma^2(N, T, \tau) \rangle$. Applying L'Hospital's Rule, and then passing to the limit, gives the following equation:

$$\langle \sigma^2(N, T, \tau) \rangle = \frac{kr^2}{N(N - 1)} \sum_{n=1}^{N-1} (N - n) \left[-2n^2 \ln (nr) \right.$$

$$\left. + \left(n - \frac{1}{r} \right)^2 \ln (nr+1) + \left(n - \frac{1}{r} \right)^2 \ln (nr-1) \right]. \quad (24)$$

Hence, for a fixed ratio $r = T/\tau$, and a determined number of samples N, the standard deviation is constant,

independent of τ. So for both adjacent and nonadjacent data sampling, the value of μ is zero. One may, therefore, determine if "flicker noise" frequency modulation is present on a signal sampled in a nonadjacent fashion, subject to the above constraints.

A consideration of interest at this point is "flicker noise" phase modulation $(S_\phi(\omega) = h\,|\omega|^{-1})$. One can show from the work of J. A. Barnes[5] that the value of the function $U(\tau)$ for $\alpha = -1$ is

$$U(\tau) = 4h(2 + \ln \tau\omega_B),$$

$$\tau\omega_B \gg 1, \qquad (25)$$

where ω_B is the system bandwidth. Substituting (25) into (13) yields the following:

$$\langle\sigma^2(N, \tau)\rangle = \frac{(N+1)4h}{N\tau^2}\left[2 + \ln \tau\omega_B - \frac{\ln N}{N^2 - 1}\right]$$

$$\alpha = -1, \quad \tau\omega_B \gg 1. \qquad (26)$$

One thus obtains the somewhat unfortunate result that experimentally it would be unlikely that one could distinguish between "flicker noise" phase modulation and any other noise with $-1 < \alpha$ up to and including white noise frequency modulation using the dependence of $\langle\sigma^2(N, T, \tau)\rangle$ on τ, since $\mu \simeq -2$ in this range. However, one might be able to infer from the experimental setup which of the two types of noise was being observed. If this were not possible, one would be forced to determine the type of noise present by some other technique, such as the one employed by Vessot in which he varies ω_B.[6]

If the number of samples N, and the ratio of the period of sampling to the sample time r, are held constant, the following general equation can be written for both adjacent and nonadjacent sampling of data:

$$\langle\sigma^2(N, T, \tau)\rangle = K(N, r)\,|\tau|^\mu. \qquad (27)$$

Using (27) and (6), $(S_\phi(\omega) = h\,|\omega|^\alpha)$, coupled with the results established thus far, one is now able to extract the power spectral density from the dependence of the standard deviation on sample time for values of μ ranging from $-2 < \mu \leq 0$. All of the results have therefore been obtained to make the previous mapping of μ into α for finite data sampling.

III. ATOMIC FREQUENCY STANDARDS

A. Passive Atomic Standards

Devices such as atomic beam machines and rubidium gas cells have been made to generate impressively stable frequencies [7], [8]. In October, 1964, the appropriately authorized International Committee of Weights and Measures adopted as a provisional definition for the

⁵ See Barnes [1], equation (74), page 219.
⁶ R. Vessot, L. Mueller, and J. Vanier, "The specification of oscillator characteristics from measurements made in the frequency domain," this issue, page 199. (Also see Appendix.)

measurement of time, the transition between the $F = 4$, $m_F = 0$ and $F = 3$, $m_F = 0$ hyperfine levels of the ground state $^2S_{1/2}$ of the atom of cesium 133, unperturbed by external fields, with the assigned frequency for the transition of 9 192 631 770 Hz. The cesium beam at NBS has also been established as the United States Frequency Standard. The analysis of the theory of operation of these quantum devices has been covered in many publications [7], [8], along with the methods of slave-locking an oscillator to a given transition [8]. An analysis of the noise that should be present in an oscillator servoed to an atomic transition has been made by Kartaschoff [9], Cutler [6], and others. The author desires to reiterate at this point some of the results of these analyses: the frequency should appear white noise modulated; therefore, the phase fluctuations will go as the random walk phenomena, and hence the mean square time error in a clock running from one of these frequency standards would be proportional to the running time. Barnes [1] has also shown that

$$\langle(\Delta^3\phi)^2\rangle \sim \langle(\delta t)^2\rangle, \qquad (28)$$

where Δ^3 denotes the 3rd finite difference and δt the clock error time for a running time t, and also that

$$\langle(\Delta^3\phi)^2\rangle \sim |\tau|^{\mu+2} \qquad -2 < \mu \leq 0. \qquad (29)$$

The results from combining (28) and (29) are obvious, but still very important, i.e., if either the power spectrum or the dependence of $\langle\sigma^2\rangle$ on the sample time is known, one may then determine the rate of time divergence and conversely [see (7) and 10)],

$$\langle(\delta t)^2\rangle \sim |\tau|^{\mu+2}; \qquad -2 < \mu \leq 0. \qquad (30)$$

An example of the above is illustrated by the following: if in fact "flicker noise" frequency modulation is present on the signal of rubidium gas cells, and if one assumes the rms time errors were equal on clocks driven by a rubidium gas cell and a cesium beam of theoretical form at $\frac{1}{3}$ of a day—say, for example, 0.1 microsecond—then the accumulated rms time error after 1 year would be of the order of 100 microseconds for the rubidium cell and 3 microseconds for the cesium beam.

B. Masers

Masers, in contrast to the passive atomic devices discussed in Section III-A, may be used as quantum mechanical oscillators [10]. Since the sources of "flicker noise" frequency modulation could arise from many different mechanisms, the suggestion exists that perhaps the active character of masers could be influenced by one of these mechanisms. Most of the basic experimental research associated with this paper and performed by the author has been to determine if the above suggestion is valid or not, and specifically to determine the type or types of noise present on the signals from masers.

One $N^{15}H_3$ maser has been in operation at NBS for

several years. It is desirable to compare similar atomic devices in looking at noise, so another $N^{15}H_3$ maser was put into operation for direct comparison purposes. Worth considering at this time is the basic operation of a maser so that arguments made later on will be understandable.

Three basic elements of a maser are the source, the energy-state selector, and the resonant cavity. A needle valve and a collimating nozzle are coupled to the source to provide a highly directed beam of $N^{15}H_3$ down the axis of the four-pole focusers. The four-pole focusers are the energy-state selectors and consist of four electrodes with alternate high voltages so that on-axis (in line with the resonant cavity) the electric field is zero, but slightly off-axis the field is large and increases as the distance off axis. Because ammonia has an induced electric dipole moment, an interaction occurs as it enters this electric field region, and as it has been shown [10] the low energy states of the inversion levels of ammonia are defocused while the high energy states are focused on axis and into the resonant cavity. Therefore, if a noise component of the proper frequency is present and the ammonia beam flux is sufficient, a regenerative process will take place in which a high energy-state molecule will decay and radiate a quantum of energy "$h\nu$" causing the field to increase in the cavity and hence inducing other molecules to undergo the transition, etc. The radiation may be coupled off with a waveguide and then, with an appropriate detection scheme, one may observe the maser oscillations at a frequency ν where, neglecting any frequency pulling effects, ν is approximately 22 789 421 700 Hz for $N^{15}H_3$.

There are many parameters that affect the output frequency of an ammonia maser. It is from these parameters that one may expect correlation of frequency fluctuations over long times such as exist for "flicker noise" frequency modulation. To see the mechanism, consider the fundamental pulling equation [10]

$$\nu_0 - \nu \doteq \frac{\Delta\nu_1}{\Delta\nu_c}(\nu_0 - \nu_c) + B, \tag{31}$$

where ν_0 is the unperturbed transition frequency, ν the maser output frequency, $\Delta\nu_1$ the transition line width, $\Delta\nu_c$ the cavity bandwidth, ν_c the cavity's frequency, and B is a term involving the basic parameters such as ammonia beam flux, the voltage on the state selectors, and the magnetic and electric field intensities inside the cavity. Most of the quantities in (31) are dependent on temperature either directly or through the coupling electronics. Some of the quantities in (31) are dependent on the maser's alignment affording another mechanism for correlation of the long-term frequency fluctuations.

As described in detail elsewhere [11], it is possible to construct an electronic servo that will tune the resonant frequency of the cavity to that of the ammonia transition frequency ($\nu_0 = \nu_c$). The tuning is not perfect because of the noise present. The effect this has, as can be

seen from (31), is profound, greatly reducing the effect of the basic parameters and almost entirely eliminating cavity dimensional instability. The correlation now existing in the long term frequency fluctuations may well be expected to be masked out by other noise in the system.

Since the servomechanism is of the frequency lock type, and if the noise in the servo is white over some finite bandwidth, then the fluctuations on the output frequency would be white inasmuch as they were caused by the servo. This is the same as for passive atomic devices—a result not altogether unexpected.

IV. EXPERIMENTS PERFORMED

A. $N^{15}H_3$ Maser Comparison

Two $N^{15}H_3$ masers were compared by the following technique. Each maser has coupled onto its output waveguide a balanced crystal detector and onto each detector a 30 MHz IF amplifier. Since the frequency of the inversion transition in $N^{15}H_3$ is about 22 790 MHz, a local oscillator signal at 22 760 MHz is inserted into each remaining leg of the balanced crystal detectors. The 30 MHz beat frequencies resulting from the output of the IF amplifiers are then compared in a balanced mixer and its audio output is analyzed; the bandwidth of this mixer is about 20 kHz. The frequency of the audio signal can be adjusted to a reasonable value by offset tuning the cavity of one of the masers. The period of this audio signal is determined with a counter; the data are punched on paper tape for computer analysis. The computer determines the average value of the fractional standard deviation σ, along with its confidence limit for pertinent discrete values of the sampling time τ.

In all of the maser data taken it was necessary to subtract out a systematic but well-understood linear drift— due to a change of the ammonia beam flux in the $N^{15}H_3$ maser and due to cavity dimensional drift from changing temperature in the case of the H maser. This was accomplished by the method of least squares [12]. The ability to change the number of samples N was also built into the computer program.

A plot of the computer output for some of the data analyzed in the $N^{15}H_3$ maser comparison is shown in Fig. 4. The output frequencies of the masers were not servo-controlled for these data. The slope is very nearly that of white noise frequency modulation. It will be noted that for a time of one second, the fractional standard deviation of the frequency fluctuations is less than 2×10^{-12}.

B. Cesium Beam, Maser, Quartz Oscillator Intercomparisons

Many comparisons were made between all the different types of atomic devices made available in the Atomic Frequency and Time Standards Section of NBS. Most of the comparisons were made at 5 MHz by measuring the period of the beat frequency with a counter, and the resulting data were processed by a computer.
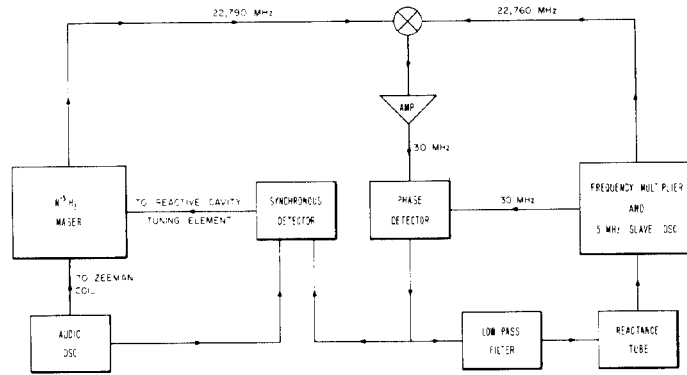
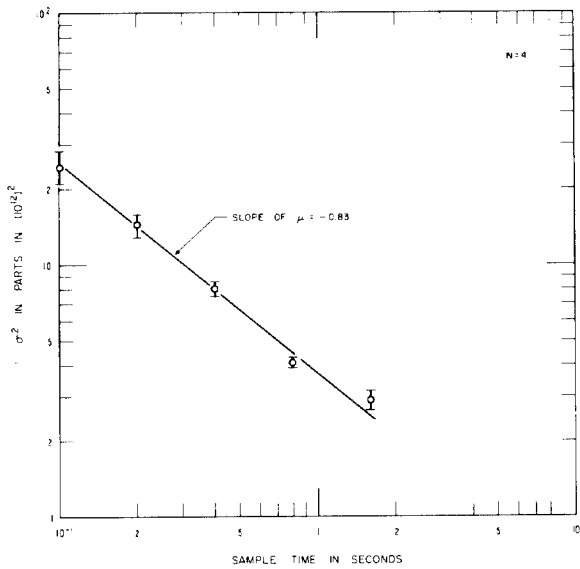Fig. 3.   Block diagram of maser cavity servo system.

Fig. 4.   Comparison of two $N^{15}H_3$ masers; standard deviation of the frequency fluctuations as a function of sampling time.
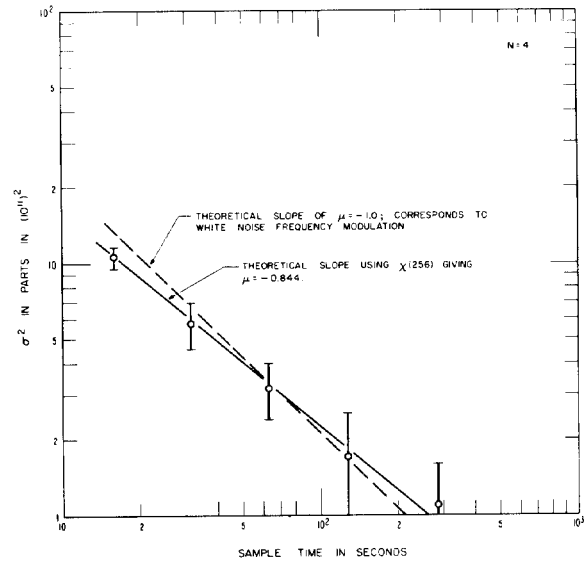

Fig. 5.   Cesium beam (NBS III), $N^{15}H_3$ maser (2) comparison; standard deviation of the frequency fluctuations as a function of sampling time.
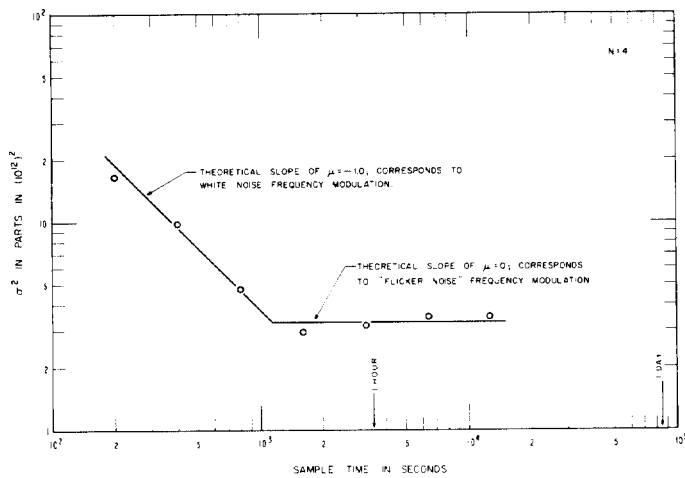

Fig. 6.   Cesium beam (NBS III), quartz crystal oscillator comparison; standard deviation of the frequency fluctuations as a function of sampling time.
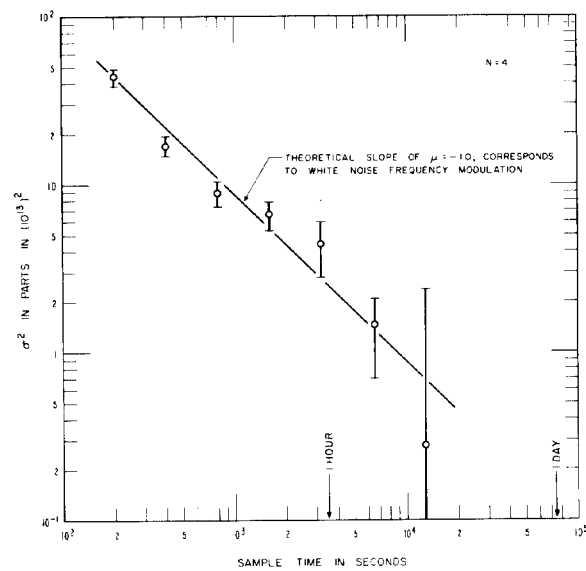

Fig. 7.   Cesium beam (NBS III), hydrogen maser comparison; standard deviation of the frequency fluctuations as a function of sampling time.

The bandwidth of the 5 MHz mixer that produces the beat frequency was about 33 Hz.

A comparison was made between the double beam $N^{15}H_3$ maser with the cavity servo in operation as illustrated in Fig. 3, and the NBS III cesium beam. A 5 MHz quartz crystal oscillator of good spectral purity was phase locked by a double heterodyne technique to the output frequency of the maser, and this oscillator was compared with a high quality quartz crystal oscillator that was frequency locked to NBS III. A plot of the data is shown in Fig. 5. The noise is undoubtedly that of the maser cavity servo system, and though it exhibits very nearly its theoretical value of $\mu = -1$ for white noise frequency modulation, the noise level will be seen to be well over an order of magnitude higher than for the free-running maser.

The free-running, single beam, $N^{15}H_3$ maser was compared with NBS III on a longer time basis and indicated the presence of "flicker noise" frequency modulation. There was indicated a fair amount of uncertainty to the data, however.

In Fig. 6 one sees a very interesting plot in the comparison of the NBS III cesium beam with a very high quality quartz crystal oscillator. The comparison was again made at 5 MHz as indicated above. The first part of the curve shows the white noise frequency modulation in the cesium beam servo system, and then the curve changes slope indicating the presence of "flicker noise" frequency modulation as is typically exhibited by quartz crystal oscillators. This shows experimentally the theoretical result discussed previously, that one may have quite a high level of white noise in a system and eventually the system will be better than one with "flicker noise" frequency modulation.

An extended comparison was made (58 hours) between the H maser, NBS III cesium beam, a rubidium gas cell, and a quartz crystal oscillator. Some of the data are still to be analyzed, but some interesting and impressive results have been obtained thus far. Figure 7 shows the analysis of some of the data obtained from the H maser, cesium beam (NBS III) comparison. White noise frequency modulation is exhibited for sampling times extending to 4 hours with the very impressive standard deviation of $\sim 6 \times 10^{-14}$ for $\tau = 4$ hours. "Flicker noise" frequency modulation was observed on both the rubidium gas cell and on the quartz crystal oscillator in this comparison.

V. CONCLUSIONS

An invariant quality factor of a frequency standard having definite esthetic value would be the infinite time average, standard deviation of the frequency fluctuations. From the theoretical development a least biased estimate of this factor may be established in terms of the expectation value of the standard deviation for any finite number of samples:

$$\sigma^2 = \frac{(N-1)\langle \sigma^2(N, \tau)\rangle}{N(1-N^\mu)},$$

$$-2 \leq \mu \leq 0 \text{ and } |\tau\omega_B|^{|\alpha+1|} \gg 1. \quad (32)$$

σ^2 also has the desirable feature of being dependent on only one variable, τ, except for $\mu = -2$, where it is also necessary to specify ω_B, the system bandwidth. Equation (32) is valid for all types of noise between and including "flicker noise" frequency modulation and white noise phase modulation $(-3 \leq \alpha \leq 0)$. However, if $\alpha = -3$, some other means of specification need be employed, such as Barnes' [1] finite difference technique or $\langle \sigma^2(2, \tau)\rangle$ which, interestingly, is equal to $\langle(\Delta^2\phi)^2\rangle/2\tau^2$ (Δ^2 denotes the second finite difference).

"Flicker noise" frequency modulation was not observed on the frequency fluctuations of the maser type of atomic frequency standard for any of the reliable data obtained—the time coverage here being from about 0.1 second to 4 hours. The presence of "flicker noise" was indicated for free-running masers (both H and $N^{15}H_3$) in the long term range, though the data were somewhat unreliable. This type of noise was eliminated in the case of the $N^{15}H_3$ maser by use of a cavity servo mechanism, and one might infer from the similarities in the masers that the same would hold true for the H maser.

It is acknowledged that if "flicker noise" is present on the frequency fluctuations of a free-running maser for longer times, the lower cutoff frequency might be greater than for quartz crystal oscillators—for indeed the maser operator is himself a part of a cavity servo when he retunes the cavity or perhaps recoats the quartz bulb.

The existence of "flicker noise" frequency modulation on the frequency fluctuations of rubidium gas cells is just another indication that they would not at present make a reliable primary frequency standard. One cannot conclude from the data analyzed that "flicker noise" is not present on masers—even if a cavity servo is employed. It is possible, however, to determine an upper limit for the level of "flicker noise." If $G_\phi(\omega) = h|\omega|^{-3}$, then using (22), the upper limit of h can be determined from the minimum value of $\sigma^2(N, \tau)$. The value of h for $N^{15}H_3$ (determined from Fig. 4) is $\sim 7 \times 10^{-25}$ seconds^{-2}. It is of interest to compare this with the value of h determined for the quartz crystal oscillator shown in Fig. 6—$\sim 8 \times 10^{-25}$ seconds^{-2}. Though comparable, it should be stated that this oscillator is one of the best that has been analyzed at NBS. The upper limit on h for the H maser, cesium beam (NBS III) comparison shown in Fig. 7 is $\sim 2 \times 10^{-27}$ seconds^{-2}—being a factor of 350 and 400 times better than the $N^{15}H_3$ maser and quartz crystal oscillator, respectively.

There are many areas where additional data analysis would be informative. It would be very interesting to look at the behavior of the cesium beam, hydrogen

maser comparison, shown in Fig. 7, for extended times. Another analysis of interest would be that of a cavity-servoed H maser. Such a servo system is under development for the NBS H maser. To then compare the cavity-servoed H maser with a cesium beam and its associated servo system would afford great insight into determining which of these competitive atomic standards should be primary.

APPENDIX

The general expression for the autocovariance function for infinite bandwidth is:

$$R_\varphi(\tau) = a'(\alpha) \, | \, \tau \, |^{-\alpha-1}, \tag{33}$$

$\alpha \neq 0$, and not an integer, where

$$a'(\alpha) = 2 \left[\cos\left(\frac{\pi(\alpha + 1)}{2}\right) \right] \cdot \Gamma(\alpha + 1).$$

Physically, $R_\varphi(0)$ is kept from diverging in the region $-1 < \alpha$ because of finite system bandwidths, ω_B; i.e., for any $\epsilon < 1/\omega_B$, $R_\varphi(\epsilon) \approx R_\varphi(1/\omega_B)$. A good approximation to $R_\varphi(0)$ in this region is, therefore, $R_\varphi(1/\omega_B)$. One may notice that the variance of the frequency fluctuations diverges [see (5)] with increasing ω_B. The variance of the frequency fluctuations may, therefore, be written:

$$\sigma^2 = \frac{a(\alpha)}{\tau^2} \left[\, | \, \omega_B \, |^{\alpha+1} - \, | \, \tau \, |^{-\alpha-1} \right]. \tag{34}$$

In the region $-3 < \alpha < -1$, $R_\varphi(0)$ is approximately zero; and if $| \tau \omega_B |^{|\alpha+1|} \gg 1$, then the equations for $U(\tau)$ are as written in (9).

Consider now a finite number of data samples N in the domain where $-1 < \alpha < 0$, and $| \tau \omega_B |^{\alpha+1} \gg 1$. Using the appropriate form of $U(\tau)$ from (9) and substituting into (13) gives:

$$\langle \sigma^2(N, \tau) \rangle = \frac{N + 1}{N} a(\alpha) \, | \, \tau \, |^\mu \, | \, \tau \omega_B \, |^{\alpha+1}, \tag{35}$$

where

$$\mu = -\alpha - 3.$$

Hence, by keeping N and $\tau \omega_B$ constant, one can determine the value of μ (of α) on a log $\langle \sigma^2(N, \tau) \rangle$ vs. log τ plot, thus eliminating the degeneracy in this region. Equation (35) may also be written:

$$\langle \sigma^2(N, \tau) \rangle = \frac{(N + 1) \cdot a(\alpha) \cdot \, | \, \omega_B \, |^{\alpha-1}}{N \tau^2}. \tag{36}$$

Hence, if N and τ are held constant and ω_B is varied, the slope on a log-log plot of $\langle \sigma^2(N, \tau) \rangle$ vs. ω_B would be $\alpha + 1$—affording another technique of determining the spectral density in this domain.

It is of interest to note that the most sensitive parameters to use in arriving at the value of α are: the bandwidth, ω_B, if $-1 < \alpha < 0$; the sample time τ, if $-2 < \alpha < -1$; and the number of samples N, if $-3 < \alpha < -2$.

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