On properties of geometric random problems in the plane

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In this paper, we present results dealing with properties of well-known geometric random problems in the plane, together with their motivations. The paper specifically concentrates on the traveling salesman and minimum spanning tree problems, even though most of the results apply to other problems such as the Steiner tree problem and the minimum weight matching problem.

Keywords: Traveling salesman, minimum spanning tree, Euclidean and probabilistic analyses, rates of convergence, geometric constants.

1. Overview and motivations

In Beardwood et al. [3], the authors prove that for any bounded uniform i.i.d. random variables $\{X_i : 1 \le i < \infty\}$ with values in $\mathbb{R}^2$, the length of the shortest tour through $\{X_1,\ldots,X_n\}$ is asymptotic to a constant times $\sqrt{n}$ with probability one (the same being true in expectation). In fact, this result is valid for any uniform i.i.d. random variables with compact support of measure one in $\mathbb{R}^d$, $d \ge 2$, provided $\sqrt{n}$ is replaced by $n^{(d-1)/d}$, the constant depending only on the dimension of the space and not on the shape of the compact support.

This theoretical result has become widely recognized to be at the heart of the probabilistic evaluation of the performance of heuristic algorithms for vehicle routing problems. It is used as the main argument in the probabilistic analysis of partitioning algorithms for the traveling salesman problem (TSP) in Karp [18]. It also plays a crucial role in Haimovich and Rinnooy Kan [9] in which a probabilistic analysis of a class of heuristics is performed for the capacitated vehicle routing problems. For an overview of these algorithms and related ones, the reader is referred to Karp and Steele [19] and Haimovich et al. [10], respectively. Another analysis of partitioning algorithms for the Euclidean traveling salesman problem is contained in Halton and Terada [11]. More recent probabilistic analyses of heuristics for routing problems include the works of Bramel et al. [5] and Bramel and Simchi-Levi [4]. In the non-routing context, one of the earliest and nicest contributions is contained in Papadimitriou [20] who, through a careful analysis of Beardwood et al.'s paper, is able to extract the main ideas and then provide a rigorous probabilistic analysis of matching heuristics.

Because of these algorithmic applications, results of the Beardwood et al. type have gained considerable practical interest, and have inspired a growing interest in the area of probabilistic analysis of combinatorial optimization problems. In turn, this encouraged the development of other results of the same nature. An important contribution is contained in Steele [31] in which the author uses the theory of independent subadditive processes to obtain strong limit laws for a class of problems in geometrical probability which exhibit nonlinear growth. Examples include the traveling salesman problem, the Steiner tree problem, and the minimum weight matching problem. Among other problems, not in this class, but with a similar asymptotical behavior, is the minimum spanning tree problem and some weighted versions of it (see Steele [33]). Other problems of interest in transportation include probabilistic version of the previous problems as analyzed in Jaillet [14]. In a somewhat different flavor, Steele [32] generalized Beardwood et al.'s result in order to obtain complete convergence for the traveling salesman functional. Such an extension was motivated by a remark in Weide [37] showing that one needs complete convergence instead of almost sure convergence for the rigorous justification of partitioning algorithms as proposed in [18]. Finally, general techniques have recently been proposed for getting and/or extending all previous results. For example, the use of martingale inequalities as introduced by Rhee and Talagrand [28] is now an essential tool. In an effort to generalize and streamline the work of Steele [31] even further, Rhee [27] and Redmond and Yukich [24] have recently proposed less restrictive and more natural conditions for the derivation of limit theorems in geometric probability.

For most of these analyses, the main results concern the almost sure or complete convergence of a sequence of normalized random variables, say $L_P(n)/\sqrt{n}$, to a constant β_P, as well as the convergence of the normalized means (here "P" is a generic symbol representing any of the problems pre-cited). Questions about rates of convergence for these limit laws have been raised many times in the literature (see for example [3, 18, 31, 32]). There are in fact two issues concerning information on rates of convergence:

1. What is the asymptotic size of $L_P(n) - \mathbf{E}L_P(n)$?
2. What can be said about the rate of convergence of the normalized means $\mathbf{E}L_P(n)/\sqrt{n}$ to β_P?

With respect to the first question, Rhee and Talagrand [29] prove that, for the TSP, there is a constant k such that $\|L_{TSP}(n) - \mathbf{E}L_{TSP}(n)\|_p \le k\sqrt{p}$ for each p for all n. This interesting result indicates that $L_{TSP}(n)$ is quite concentrated around its mean. With respect to the second question, partial results are obtained in Jaillet [13], who proves that $|\mathbf{E}L_P(n)/\sqrt{n} - \beta_P| = O(1/\sqrt{n})$. For geometric problems in higher dimension ($n \geq 3$), Alexander [1] and Redmond and Yukich [24] independently proved stronger results than those contained in [13]. However, the important question remained open: Is $1/\sqrt{n}$ the *exact* rate of convergence, or does $\mathbf{E}L_P(n)/\sqrt{n}$ go faster to β_P? In Jaillet [16] and Rhee [26], the authors independently prove that $1/\sqrt{n}$ is indeed the exact rate of convergence for the minimum spanning tree and traveling salesman problem, respectively. In [13], an attempt is also made to find the best constants for these rates of convergence.

Such considerations are extremely important in practice. Indeed, following Karp's paper, many results have appeared in the literature about the asymptotical optimality (with probability one, in probability, and/or in expectation) of heuristics for various problems in the area of routing and location theory. For the practitioner, it is important to know whether an asymptotically optimal heuristic is really applicable for realistic problem sizes, or whether its asymptotical behavior is only of theoretical importance. Obtaining the exact rate of convergence for these limit laws is a mandatory first step toward the general program of evaluating the error one makes by using an asymptotically valid formula when dealing with a finite size problem. Note that this second issue has also practical applications of its own, such as in strategic planning. In the context of routing for example, one can approximate the cost of serving n customers in an area of measure A by using formulas of the type $\beta\sqrt{An}$, where β is the appropriate limiting constant mentioned above (see Larson and Odoni [21] for a detailed discussion of these applications). One of the persistently important open problems in this area is the determination of the exact value of the constant for any interesting functional. Progress has been made by Avram and Bertsimas [2], who have recently obtained an exact expression (as a series expansion) for the MST constant when the points are drawn uniformly from the torus. The authors use the torus in order to avoid boundary effects and obtain tractable derivations. They also conjecture that their resulting constant is in fact the same as for the traditional cube model. A proof of this conjecture is contained in Jaillet [15], who shows that the length of the optimal solutions for all previous problems in the torus and cube models are almost surely asymptotically equivalent.

In this paper, we present some of these results as well as the main ideas and techniques behind the proofs. Our goal is not to be encyclopedic, but to give an overview of the most popular techniques employed in this field. In sections 2 and 3, we provide the necessary notation and discuss classical results. Then we present results on rates of convergence in section 4, and finally on the estimation of constants in section 5. In the last section (section 6), we briefly discuss other topics and list some open problems.

2. Notations

The (geometric) traveling salesman problem and (geometric) minimum spanning tree problem consist of finding the shortest tour and shortest spanning tree through a given finite set of points of the two-dimensional real space $\mathbb{R}^2$. The distance between two points x_i and x_j is taken to be either the ordinary Euclidean (l_2) metric $\|x_i - x_j\|_2$ or the right-angle (l_1) metric $\|x_i - x_j\|_1$. We are concerned here with four stochastic versions of these problems.

The first stochastic model assumes that the positions of the points, X_i, $1 \le i < \infty$, are uniformly and independently distributed in $[0, 1]^2$ and is referred hereafter as the "uniform square" model. This is the most studied model. However, other models can be useful for the purpose of analysis.

The second stochastic model eliminates the boundary effects of the previous one. As we will see later in this paper, this is useful in order to obtain tractable analytical derivation for the limiting constant β_{MST}, or to speed up the estimation of the constant β_P by numerical simulation. The sequence of points is considered modulo 1 in each component. Alternatively, one can think of a sequence on the torus $T^2 = ([0, 1] \bmod 1)^2$ (intuitively, the torus is obtained by identifying opposite faces of the square). The Euclidean distance between two points x_i and x_j is now taken to be $\| \{x_i - x_j\} (\bmod 1)^2 \|_2$ (where, for $y \in [-1, 1]$, $y (\bmod 1)$ is the minimum of $|y|$ and $1 - |y|$). This is the "uniform torus" model.

Finally, the third and fourth models correspond to a "poissonization" of the first and second model, respectively. More precisely, points correspond to a Poisson point process π_n of intensity n times the Lebesgue measure over $[0, 1]^2$, or over the torus. Let N_n be a Poisson random variable with parameter n representing the number of points of this process in $[0, 1]^2$.

$L_{TSP}(n)$, $L_{MST}(n)$, $L_{TSP}^{(t)}(n)$, and $L_{MST}^{(t)}(n)$ will denote the length of the shortest tour and shortest spanning tree through $\{X_1, X_2, \dots, X_n\}$ in the first and second model, respectively. Finally, $L_{TSP}(N_n)$, $L_{MST}(N_n)$, $L_{TSP}^{(t)}(N_n)$, and $L_{MST}^{(t)}(N_n)$ will denote the length of the shortest tour and shortest spanning tree through N_n in the third and fourth model, respectively.

3. Classical results

3.1. COMBINATORIAL PROPERTIES

Let us first present two results, common to the above, that are fundamental in the development of asymptotic analyses and in the design of heuristics. We use the minimum spanning tree problem as a specific example.

LEMMA 1

Consider $X = \{x_i : 1 \le i < \infty\}$ to be an arbitrary infinite sequence of points in $[0, 1]^2$, and let $x^{(n)} = \{x_1, x_2, \dots, x_n\}$ be its first n points. Let $\{Q_i : 1 \le i \le m^2\}$ be a partition of $[0, 1]^2$ into m^2 squares with edges parallel to the axes and of side length $1/m$. Then there exists a constant k_1 such that

$$L_{MST}(x^{(n)}) \leq \sum_{i=1}^{m^2} L_{MST}(x^{(n)} \cap Q_i) + k_1 m. \tag{1}$$

Proof

The classical argument has its origin in [3, lemma 1] and has been used subsequently in many papers. Consider the following tree construction connecting $x^{(n)}$ (see figure 1 for an illustration): first construct optimal trees connecting $x^{(n)} \cap Q_i$ for $1 \leq i \leq m^2$. Then, in each square Q_i where $x^{(n)} \cap Q_i$ is not empty, choose one point as a representative and finally construct an optimal tree connecting the set S of all representatives (at most m^2 points). The combination of the small trees together with the large tree gives a spanning tree connecting $x^{(n)}$ of length $\sum_{i=1}^{m^2} L_{MST}(x^{(n)} \cap Q_i) + L_{MST}(S)$. Now it is easy to show (see [7]) that there exists a constant k_1 such that $L_{MST}(S) \leq k_1 \sqrt{|S|}$, and this establishes (1). □

LEMMA 2

Consider $x = \{x_i : 1 \leq i < \infty\}$ to be an arbitrary infinite sequence of points in $[0, 1]^2$, and let $x^{(n)} = \{x_1, x_2, \ldots, x_n\}$ be its first n points. Let $\{Q_i : 1 \leq i \leq m^2\}$ be a partition of $[0, 1]^2$ into m^2 squares with edges parallel to the axes and of side length $1/m$. Then there exists a constant k_2 such that

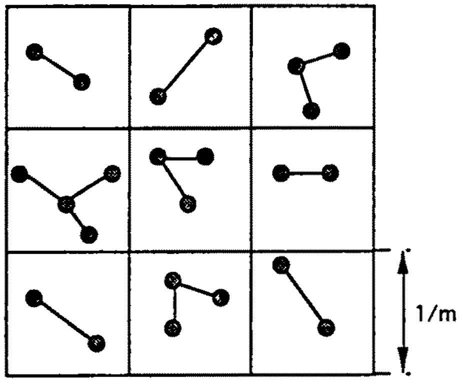
$$L_{MST}(x^{(n)}) \geq \sum_{i=1}^{m^2} L_{MST}(x^{(n)} \cap Q_i) - k_2 m. \tag{2}$$

Proof

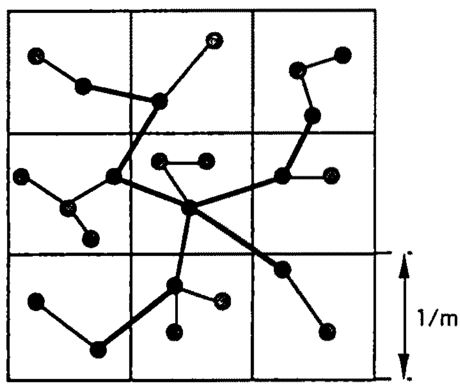
The argument for proving (2) is also classical and is adapted from [3, lemma 2] (see figure 2 for an illustration). Let T^* be an optimal tree through $x^{(n)}$ and let us suppose that $x^{(n)} \cap Q_i$ is not empty. Let $T_i^* = T^* \cap Q_i$ and let T_{ij} for $1 \leq j \leq \mu_i$ ($\mu_i \leq |x^{(n)} \cap Q_i|$) be the connected pieces of T_i^* which contain at least one element of $x^{(n)}$. Let l_i be the total length of all these connected pieces. By using some portion of the perimeter of Q_i, one can connect endpoints of these pieces (lying on the perimeter of Q_i) in order to form a tree spanning $x^{(n)} \cap Q_i$. Now, the additional points, used for this connection, lie outside of the convex hull of $x^{(n)} \cap Q_i$. Hence, we have $L_{MST}(x^{(n)} \cap Q_i) \leq l_i + per(Q_i) = l_i + 4/m$. By summing both sides for all i, we get the validity of (2), with $k_2 = 4$. □

3.2. ASYMPTOTIC ANALYSIS

The results of an asymptotic analysis for the geometric random problems discussed in this paper usually concern the derivation of *strong* limit laws in the following sense. Again we use "P" as a generic symbol representing any of the previous problems.



(i) Find MST trees in the subsquares



(ii) Patch the MST trees together by connecting their representatives

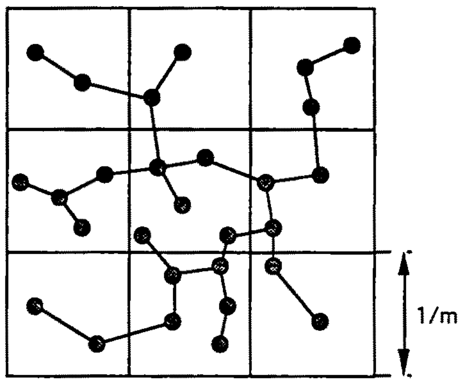
Figure 1. Tree construction on $x^{(n)}$ in $[0, 1]^2$.

THEOREM 1

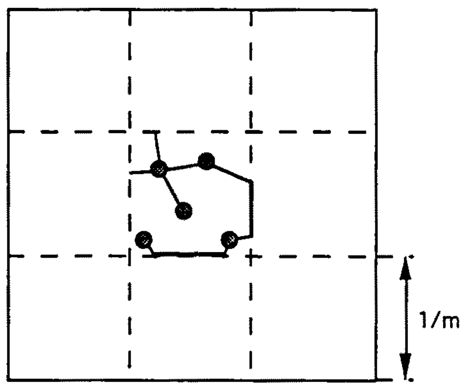
Let $(X_i)_i$ be an infinite sequence of points independently and uniformly distributed over $[0, 1]^2$. Then there exists a positive constant β_P such that:

$$\lim_{n \to \infty} \frac{L_P(n)}{\sqrt{n}} = \beta_P \text{ (a.s.).} \tag{3}$$

It is interesting to note that this result remains valid for any general distribution μ with compact support in $\mathbb{R}^2$. If $d\mu_a = f(x)dx$ is the absolute continuous part of the distribution, $\sqrt{n}$ would be replaced by $\int f(x)^{1/2} dx$. We will not, however, discuss this result here and we refer the interested reader to [3,31,33] for some of the



(i) Find a MST tree spanning all points



(ii) Connect the components in a subsquare, using part of the perimeter

Figure 2. Tree construction on $x^{(n)} \cap Q_i$ in $[0, 1]^2$.

additional techniques necessary to obtain such a result. One way to prove theorem 1 is to use, if possible, the general framework defined by Steele [31] about subadditive functionals. Before stating this result, let us give some definitions. By a functional Φ, we mean a real-valued function of the finite subsets of $\mathbb{R}^2$. We say that (a) Φ is *Euclidean* if it is linear and invariant under translation; (b) Φ is *monotone* if $\Phi(y \cup A) \geq \Phi(A)$ for any y in $\mathbb{R}^2$ and finite subsets A of $\mathbb{R}^2$; (c) Φ is *bounded* if $var[\Phi(X^{(n)})] < \infty$ whenever the points of $X^{(n)}$ are independent and uniformly distributed in $[0, 1]^2$; (d) Φ is *subadditive* if whenever $(Q_i)_{1 \leq i \leq m^2}$ is a partition of the square $[0, t]^2$ into squares with edges parallel to the axes and of length t/m, and whenever (x_i) is an arbitrary sequence of points in $\mathbb{R}^2$, then there exists a positive constant B such that $\Phi(x^{(n)} \cap [0, t]^2) \leq \sum_{i=1}^{m^2} \Phi(x^{(n)} \cap Q_i) + Btm$. In [31], the author proves the following result:

THEOREM (STEELE)

Let Φ be a subadditive, Euclidean, monotone and bounded functional. If $(X_i)_i$ is a sequence of points independently and uniformly distributed over $[0, 1]^2$, then there exists a constant φ such that $\Phi(X^{(n)}/\sqrt{n}$ goes to φ almost surely when n goes to infinity.

It is not difficult to verify that the traveling salesman functional is Euclidean, monotone and bounded. The most demanding is to show that it is subadditive and this is a consequence of the type of results described in lemma 1. Indeed, using similar arguments, lemma 1 can be shown to hold when MST is replaced by TSP. Other functionals such as the Steiner tree and the minimum weight matching can be treated as well using this general framework.

For functionals like the minimum spanning tree however, this general technique does not work anymore due to the lack of monotononicity. Let us give an overview of some of the main techniques involved in proving such theorems when the functional departs from the main conditions stated above. Let us again use the minimum spanning tree as an example.

The first step is to obtain the behavior of the expected value $\mathbf{E}L_{MST}(n)$. And this is usually done (see [3,31,14]) by a technique of Poisson smoothing followed by a Tauberian argument.

POISSON SMOOTHING

From (1) of lemma 1, we have, starting with a Poisson point process π_{m^2n} on $[0, 1]^2$ and using an obvious scaling property, the following subadditive inequality:

$$\mathbf{E}L_{MST}(N_{m^2n}) \leq m\mathbf{E}L_{MST}(N_n) + k_1 m. \tag{4}$$

Dividing both sides of (4) by $\sqrt{m^2n}$, and using the continuity of the functional L_{MST}, one can then prove (see, for example, [31,14]) that there exists a positive constant β_{MST} such that:

$$\lim_{n\to\infty} \frac{\mathbf{E}L_{MST}(N_n)}{\sqrt{n}} = \beta_{MST}. \tag{5}$$

TAUBERIAN ARGUMENT

By definition of $\mathbf{E}L_{MST}(N_n)$ and the fact that N_n is a Poisson random variable with parameter n, we have

$$\frac{\mathbf{E}L_{MST}(N_n)}{\sqrt{n}} = n^{-1/2} \sum_{k=0}^{\infty} L_{MST}(k) \frac{e^{-n}n^k}{k!}, \tag{6}$$

and thus we have

$$\lim_{n \to \infty} \frac{\mathbf{E} L_{MST}(N_n)}{\sqrt{n}} = \lim_{n \to \infty} n^{-1/2} \sum_{k=0}^{\infty} L_{MST}(k) \frac{e^{-n} n^k}{k!} = \beta_{MST} . \tag{7}$$

We can now use a Tauberian theorem such as the one due to Schmidt [30] (see [8]):

THEOREM (SCHMIDT)

If we have

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_k \frac{e^{-n} n^k}{k!} = s, \tag{8}$$

then

$$\lim_{n \to \infty} a_n = s$$

if and only if

$$\lim_{\varepsilon \to 0^+} \liminf_{n \to \infty} \min_{n \le m \le n + \varepsilon \sqrt{n}} \{a_m - a_n\} \ge 0. \tag{9}$$

In order to use this theorem, one first has to show that from (7) we have

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} \frac{L_{MST}(k)}{\sqrt{k}} \frac{e^{-n} n^k}{k!} = \beta_{MST}, \tag{10}$$

and this is done using the well known fact that $L_{MST}(n) \le c\sqrt{n}$, where c is a constant (see, for example, [7]). Finally, it remains to prove that $L_{MST}(k)/\sqrt{k}$ verifies condition (9), and this can be worked out from combinatorial properties of the functional.

The second step is to go from the asymptotics of the expected value to the asymptotics of the random variable itself. For most functionals, one of the easiest ways to do that is to use the technique of martingale inequalities as described in Rhee and Talagrand [28]. The nice feature of this technique is to usually provide a complete convergence result, stronger than the classical almost sure convergence. We refer the reader to reference [28] for a detailed and clear presentation of the technique.

4. Rates of convergence

In this section, we present techniques contained in [13, 16] for dealing with the following question: What can be said about the rate of convergence of the normalized means $\mathbf{E} L_P(n)/\sqrt{n}$ to β_P?

Let us again use the minimum spanning tree as an illustration. In [13], the results were presented for the uniform square model, but they are also valid for the Poisson square model. On the other hand, the exact rate of convergence of [16] has been proved for the Poisson model only. Hence, for reasons of homogeneity, we will consider the Poisson square model throughout this section.

4.1. UPPER BOUND

If one follows the usual subadditivity argument as presented in the previous section, one can go one step further and show that $\mathbf{E}L_{MST}(N_n) \geq \beta_{MST}\sqrt{n} - k_1$ for a positive constant k_1. Also, adapting a classical argument given in [3] for the TSP, one can show that $\mathbf{E}L_{MST}(N_n) \leq \beta_{MST}\sqrt{n} + k_2$ for a positive constant k_2. More precisely, the arguments go as follows.

LEMMA 3

Let N_n be the number of points of a Poisson point process π_n of intensity n times the Lebesgue measure over $[0, 1]^2$, and let $L_{MST}(N_n)$ be the length of the minimum spanning tree connecting these N_n points. Then we have

$$|\mathbf{E}L_{MST}(N_n)/\sqrt{n} - \beta_{MST}| = 0(1/\sqrt{n}). \tag{11}$$

Proof

Let us first prove that there exists a constant k_1 such that

$$\mathbf{E}L_{MST}(N_n) \geq \beta_{MST}\sqrt{n} - k_1. \tag{12}$$

We have seen in section (3.2) that the following inequality holds:

$$\mathbf{E}L_{MST}(N_{m^2n}) \leq m\mathbf{E}L_{MST}(N_n) + k_1 m. \tag{13}$$

Dividing both sides of (13) by $\sqrt{m^2n}$, letting m go to infinity, and using the fact that

$$\lim_{m\to\infty} \mathbf{E}L_{MST}(N_{m^2n})/\sqrt{m^2n} = \beta_{MST},$$

we obtain (12).

Now from lemma 2, it is easy to prove, using the same kind of arguments, that there exists a constant k_2 such that

$$\mathbf{E}L_{MST}(N_n) \leq \beta_{MST}\sqrt{n} + k_2. \tag{14}$$

□

In an attempt to find the best upper bound, we have obtained in [13] the following numerical results for the traveling salesman and minimum spanning tree problems in the uniform square model:

$$\begin{aligned} |\mathbf{E}L_{TSP}(n)/\sqrt{n} - \beta_{TSP}| &\leq 7/\sqrt{n} + 2/\sqrt{n-1}, \\ |\mathbf{E}L_{MST}(n)/\sqrt{n} - \beta_{MST}| &\leq 6.4/\sqrt{n} + 4.8/\sqrt{n-1}. \end{aligned} \tag{15}$$

4.2. EXACT RATE

The main result of this section is the following theorem:

THEOREM 2

Let N_n be the number of points of a Poisson point process π_n of intensity n times the Lebesgue measure over $[0, 1]^2$, and let $L_{MST}(N_n)$ be the length of the optimal spanning tree connecting these N_n points. Then

$$|\mathbf{E}L_{MST}(N_n)/\sqrt{n} - \beta_{MST}| = \Theta(1/\sqrt{n}). \tag{16}$$

From lemma 3, it suffices to prove that there exists a positive constant c such that

$$\mathbf{E}L_{MST}(N_n)/\sqrt{n} \geq \beta_{MST} + c/\sqrt{n}. \tag{17}$$

Let us first replace the traditional {partitioning and patching} way of getting the subadditivity inequality (as in figure 1) by a recursive way. We divide $[0, 1]^2$ into four squares with edges parallel to the axes and of side length 1/2 and we solve the MSTP in each of them. Then we select in each (not empty) quadrant the point closest to the center of $[0, 1]^2$, and finally we construct a tree connecting these points (see figure 3, parts (i) and (ii)). Starting with the Poisson point process π_{4n} in $[0, 1]^2$, we obtain

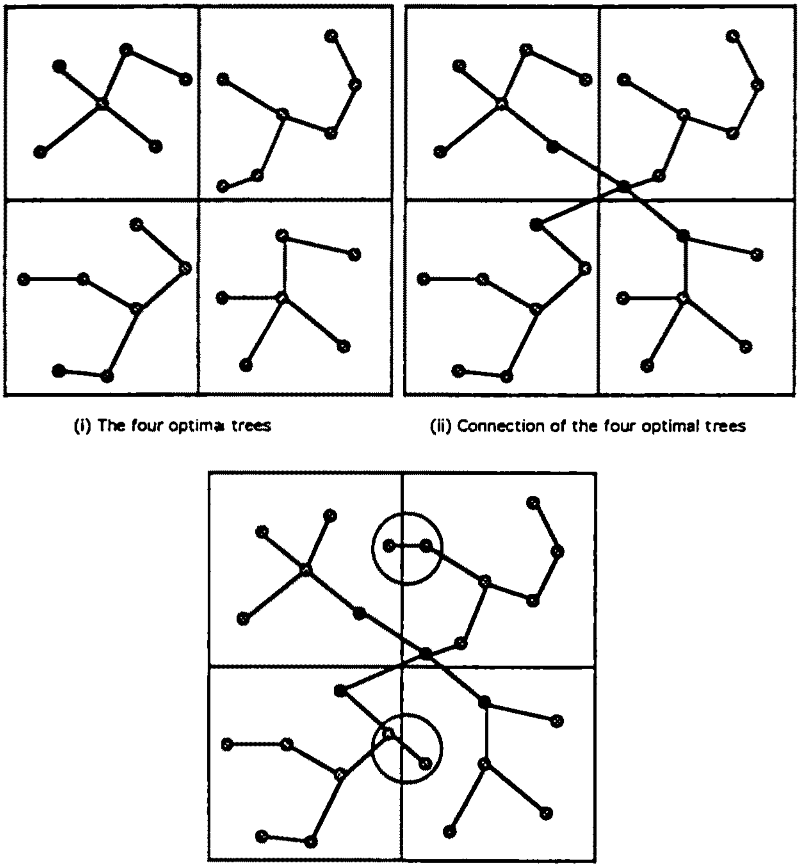
$$\mathbf{E}L_{MST}(N_{4n}) \leq 2\mathbf{E}L_{MST}(N_n) + k/\sqrt{n}, \tag{18}$$

where k is a positive constant.

However, one needs to go one step further in order to get the desired result. The main idea is to improve the feasible solution, obtained from the connection of the four trees, by considering potential savings along the borderline of two given adjacent subsquares. Figure 3, part (iii), illustrates such savings. Note that there will be savings each time there exists a point in one of the subsquares which has its closest point (among π_{4n}) that is located in another subsquare. In order to evaluate the size and likelihood of these savings, let us refer to figure 4, where we show two concentrated balls centered on the borderline of two subsquares, of radius r and $4r$, respectively. Now consider the following event $\mathcal{H}$:

> There is exactly one point in region A, no point in region B, and at least one point in region C.

If such an event is true, then, by connecting the point of A to one of the points of C, one gets savings of at least $3r - 2r = r$ (see figure 4(ii)). For $r = \alpha/\sqrt{n}$, where α is any positive constant, one can always find n large enough, so that the probability of this event, when one considers a Poisson point process π_{4n} in $[0, 1]^2$, is greater



(i) The four optimal trees

(ii) Connection of the four optimal trees

(iii) The circles indicate areas where savings have been possible.

Figure 3. Construction of a feasible solution and "post-savings".

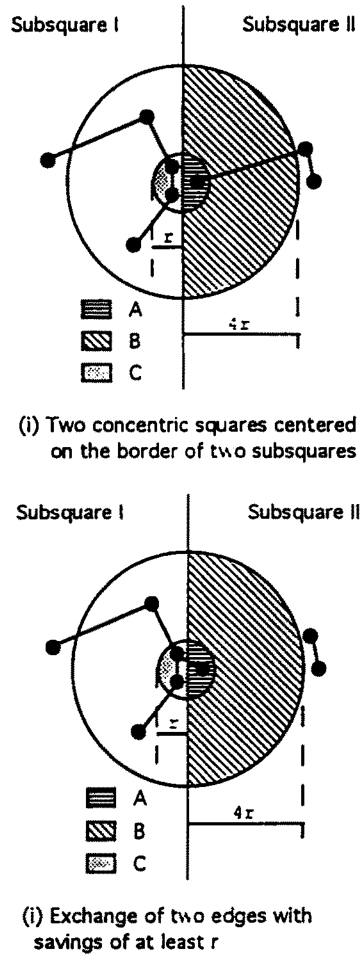
than or equal to a positive constant, say a. Along the side of two adjacent squares (of length 1/2), one can pack at least $\lfloor \sqrt{n}/(8a) \rfloor$ non-overlapping, and thus independent, such combinations of two concentric balls. The expected total savings will then be bounded from below by

$$a(\alpha/\sqrt{n})(\lfloor \sqrt{n}/(8\alpha) \rfloor) \geq c_1, \tag{19}$$

where c_1 is a positive constant. Instead of [18], we now have

$$\mathbf{E}L_{MST}(N_{4n}) \leq 2\mathbf{E}L_{MST}(N_n) + k/\sqrt{n} - c_1, \tag{20}$$

which implies that, for any positive constant $c < c_1$, there exists $n(c)$ large enough such that for all $n \geq n(c)$,



(i) Two concentric squares centered on the border of two subsquares

(i) Exchange of two edges with savings of at least r

Figure 4. Conditions for savings.

$$\mathbf{E}L_{MST}(N_{4n}) \leq 2\mathbf{E}L_{MST}(N_n) - c. \tag{21}$$

By using this inequality recursively, we get

$$\mathbf{E}L_{MST}(N_{4^m n}) \leq 2^m \mathbf{E}L_{MST}(N_n) - c(2^m - 1). \tag{22}$$

Dividing each side by $\sqrt{4^m n}$, and letting m go to infinity, we finally get the desired result. □

5. Estimation of constants

As discussed in the introduction, an important open problem is the determination of the exact value of the limiting constant for any interesting functional.

For example, the TSP constant is not known, and has only been estimated numerically by simulation. The value for β_{TSP} is currently believed to be around 0.72 (Johnson [17]).

In this section, we first present some results for the Euclidean metric and then give bounds between constants under Euclidean and right-angle metrics.

5.1. THE MST CONSTANT

From geometrical arguments, Gilbert [7] has proved that $0.5 \leq \beta_{MST} \leq \sqrt{2}/2$. The numerical estimation of β_{MST} has given successively 0.68, 0.656, 0.62,... . As mentioned in the introduction, Avram and Bertsimas [2] have recently obtained an exact expression (as a series expansion) for the MST constant in the torus model. The result goes as follows:

THEOREM 3

Let $\{X_i : 1 \leq i < \infty\}$ be a sequence of points independently and uniformly distributed over the torus. Then the limiting constant for the minimum spanning tree problem is given by:

$$\beta_{MST}^{torus} = \frac{1}{2\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{1}{k} \int_{0}^{\infty} \frac{f_k(y)}{\sqrt{y}} dy,$$

where $f_1(y) = e^{-y}$, and for $k \geq 2$,

$$f_k(y) = \frac{y^{k-1}}{\pi^{k-1}(k-1)!} \int_{\Omega_k} e^{(-y/\pi)/g_k(u_0,\ldots,u_{k-1})} du_1 \ldots du_{k-1}, \tag{23}$$

where the integration is performed on the set Ω_k of all points $\{ {}_0,\ldots,u_{k-1}\}$ of the torus (u_0 being the "center" of the torus) such that the spheres $S(u_j, 1/2)$, $0 \leq j \leq k-1$, form a connected set and $g_k(u_0,\ldots,u_{k-1})$ is the volume of $\cup_j S(u_j, 1)$.

In order to prove this theorem, Avram and Bertsimas directly analyze the greedy algorithm, which solves the MST exactly. Their approach is general and is based on a set of conditions to be satisfied. We however refer the reader to [2] for details. The authors conjectured that their resulting constant is the same as for the traditional square model.

In [15], we prove this conjecture by showing that the lengths of the optimal solutions in the torus and square models are almost surely asymptotically equivalent. More precisely, the following theorem is obtained:

THEOREM 4

Let $\{X_i : 1 \leq i < \infty\}$ be a sequence of points independently and uniformly distributed over $[0, 1]^2$. Then for the MST we have

$$\lim_{n\to\infty} \frac{L_{MST}^{(t)}(n)}{\sqrt{n}} = \lim_{n\to\infty} \frac{L_{MST}(n)}{\sqrt{n}} = \beta_{MST} \text{ (a.s.).} \tag{24}$$

The idea behind the proof is first to show that in an optimal spanning tree (in the square and/or torus), the length of the largest edge cannot be too large. This can then be used to show that the solutions in the square and in the torus can differ only in the vicinity of the boundary of the square, and in a quantity small enough compared to $\sqrt{n}$ for large n. Let us present some of these steps in more detail, and in a general format.

Let us consider a problem (generically labeled "P"), defined on an undirected graph $G = (V, E)$ with positive weighted edges, which requires finding, among all feasible subsets of edges, a subset of minimum length (the length of a subset of edges being the sum of the length of the edges belonging to this subset). Consider the problem defined in the plane, and let $L_P(n)$ be the length of an optimal solution through n random points in the square. For an arbitrary sequence (x_i) in the torus, let $\mathcal{E}_P^{(t)}(x^{(n)})$ be the set of edges belonging to the optimal solution. Suppose we are interested in comparing $L_P(n)/n^c$ and $L_P^{(t)}(n)/n^c$ for a given positive constant c.

The following properties are sufficient for showing that, for a sequence of points independently and uniformly distributed over $[0, 1]^2$, these quantities are almost surely asymptotically equivalent.

1. (Bounded degree.) For any $x^{(n)} = \{x_1, x_2, \dots, x_n\}$, the degree of each point in $\mathcal{E}_P^{(t)}(x^{(n)})$ is bounded by a constant D.
2. (Bounded passage from torus to square.) Among $\mathcal{E}_P^{(t)}(x^{(n)})$, let k be the number of edges (x_i, x_j) such that $\|\{x_i - x_j\} (\bmod 1)^d\|_2 < \|x_i - x_j\|_2$. Then there exists a feasible solution to the problem in the square, of length bounded from above by $L_P^{(t)}(x^{(n)}) + O(k^c)$.
3. (Probabilistically small largest edge.) For $\{X_i : 1 \le i < \infty\}$, a sequence of points independently and uniformly distributed in the torus, the largest edge is such that, for all $\varepsilon > 0$, $\sum_{n=1}^{\infty} \mathrm{P}(\ell_P^{(t)}(X^{(n)}) > \varepsilon) < \infty$.

As an application, one can show that the traveling salesman problem and the minimum spanning tree problem verify these conditions with $c = 1/2$. Conditions 1 and 2 are easily checked. Condition 3 is the most demanding. As an illustration, let us take the minimum spanning tree problem. In [15], we have shown that the asymptotic growth of the largest edge in an optimal spanning tree is given by the following result:

LEMMA 4

Let $\{X_i : 1 \le i < \infty\}$ be a sequence of points independently and uniformly distributed over $[0, 1]^2$. Then for the corresponding MST in the torus, the length of the largest edge, $\ell^{(t)}_{MST}(n)$, is, for n sufficiently large,

$$\mathbf{P}\left(\ell^{(t)}_{MST}(n) > \sqrt{65}\left(\frac{\log n}{n}\right)^{1/2}\right) \leq \frac{1}{24n^2 \log n}. \tag{25}$$

In order to prove such a result, we need two intermediate results. First, let m be a positive integer, and $(Q_j)_{1 \leq j \leq m^2}$ be a partition of the square $[0, 1]^2$ into squares with edges parallel to the axes and of length $1/m$. If for a sequence of points $\{x_i : 1 \leq i < \infty\}$, $x^{(n)} \cap Q_j$ is not empty for all j, then the MST in the torus is such that

$$\ell^{(t)}_{MST}(x^{(n)}) \leq \frac{\sqrt{5}}{m}. \tag{26}$$

Second, assume now that $\{X_i : 1 \leq i < \infty\}$ is a sequence of points independently and uniformly distributed over $[0, 1]^2$. If N_j denotes the cardinality of $X^{(n)} \cap Q_j$, then we have, for $h \geq 12$ and $n \geq 3$,

$$\mathbf{P}\left(\forall j,\ N_j > n/m^2 - \sqrt{hn \log n(m^2)}\right) \geq 1 - \frac{m^2}{2n^{h/4}}. \tag{27}$$

The result (25) can then be obtained from (26) and (27). We refer the reader to the paper [15] for a detailed presentation of the proofs.

5.2. THE RIGHT-ANGLE METRIC

All the results developed in this paper would remain valid under the l_1 metric, but the limiting constants would be different. Nevertheless, there is a close relationship between the two metrics for all the problems mentioned in this paper. Let us take the traveling salesman problem as an example. Let β^r_{TSP} be the limiting constant under the right-angle. Let us derive bounds between these two quantities. First, consider a sequence of random points in a ball, and consider an optimal solution under the Euclidean metric. Renumber the points so that the optimal tour is given by $\{X_1, \ldots, X_n\}$. We then have

$$\begin{aligned} L^r_{TSP}(n) &\leq \sum_{j=1}^{n-1} \| X_j - X_{j+1} \|_1 + \| X_n - X_1 \|_1 \\ &= \sum_{j=1}^{n-1} (\cos\phi_j + \sin\phi_j) \| X_j - X_{j+1} \|_2 + \cos\phi_n + \sin\phi_n) \| X_n - X_1 \|_2, \end{aligned} \tag{28}$$

where ϕ_j is the angle between the edge $(X_j - X_{j+1})$ and the horizontal axis.

By symmetry of the domain, the choice of the direction of the axes does not influence the optimal tour under the Euclidean metric. So it is easy to argue that for all j, ϕ_j and $\|X_j - X_{j+1}\|_1$ are independent, and that ϕ_j is uniform on $[0, \pi/2]$. Hence, we have

$$\beta_{TSP}^r \le \beta_{TSP} \int_0^{\pi/2} (\cos\phi + \sin\phi)\, d\phi = 4/\pi \beta_{TSP} \sim 1.27\beta_{TSP}. \tag{29}$$

If one reverses the previous argument, one gets

$$\begin{aligned} L_{TSP}(n) &\le \sum_{j=1}^{n-1} \|X_j - X_{j+1}\|_2 + \|X_n - X_1\|_2 \\ &= \sum_{j=1}^{n-1} \|X_j - X_{j+1}\|_1/(\cos\phi_j + \sin\phi_j) + \|X_n - X_1\|_1/(\cos\phi_n + \sin\phi_n). \end{aligned} \tag{30}$$

The problem now is that there is no reason to think that ϕ_j and $\|X_j - X_{j+1}\|_1$ are independent, nor that ϕ_j is uniform on $[0, \pi/2]$, for the optimal tour under the l_1 metric. Nevertheless, we conjecture that

$$\beta_{TSP} \overset{?}{\le} \beta_{TSP}^r \int_0^{\pi/2} d\phi/(\cos\phi - \sin\phi) = 2\sqrt{2}\log(1+\sqrt{2})/\pi\beta_{TSP}^r. \tag{31}$$

From (29) and assuming that (31) is true, we would finally obtain

$$(1.26\beta_{TSP}\sim)(\pi/2\sqrt{2}\log(1+\sqrt{2}))\beta_{TSP} \le \beta_{TSP}^r \le (4/\pi)\beta_{TSP}(\sim 1.27\beta_{TSP}). \tag{32}$$

Such relationships between constants would also be valid for other geometric random problems discussed in this paper.

6. Concluding remarks

6.1. ON RATES OF CONVERGENCE

The techniques and results surveyed in section 4 remain valid for other functionals of geometric probability. For the Steiner tree problem and the minimum weight matching problem, the arguments are in fact almost identical. Also, Rhee [26] has recently and independently proved this result for the traveling salesman problem. The basic idea of her proof is identical to ours, although it involves solving a number of significant technical problems in order to ensure that the techniques go through.

For all these problems, it is natural to expect that $|\mathbf{E}L_P(n)/\sqrt{n} - \beta_P| = \Theta(1/\sqrt{n})$ remains true under the uniform fixed sample size model. However, as pointed out in [26], this does not seem to be an easy consequence of the corresponding result stated under the Poisson model. The usual way to link the two models (see

[13] for details) is to prove that $|\mathbf{E}L_P(k+1) - \mathbf{E}L_P(k)| = O(1/\sqrt{k})$, which then implies that $|\mathbf{E}L_P(N_n) - \mathbf{E}L_P(n)| = O(1)$. This last relationship is not, however, sufficient here and a deeper understanding of how $|\mathbf{E}L_P(k+m) - \mathbf{E}L_P(k)|$ behaves as a function of m and k seems to be necessary for closing the gap.

Also, if one follows carefully the proofs of section 4, one remarks that the exact rate of convergence is obtained on savings done along boundaries of the small squares. This proof would not work for the torus for obvious reasons. In fact, we conjecture that in the torus, the rate of convergence is faster than $1/\sqrt{n}$. More precisely, we conjecture that $\lim_{n\to\infty} \mathbf{E}L_P(n) - \beta_P\sqrt{n} = 0$. A faster convergence in the torus was observed empirically by Johnson [17], who has been using the torus model for his recent estimations of the traveling salesman problem constant.

Finally, a persistently open question related to the issues of rates of convergence is the possible existence of central limit theorems for the combinatorial optimization problems listed in this paper.

6.2. ON SOME REFINEMENTS

In the course of proving the main theorem of section 5, we obtained in [14] several results of independent interest. For example, for n points i.i.d. uniform on $[0, 1]^2$, the length of the largest edge of the optimal MST solutions (in the square or torus) is almost surely asymptotically bounded from above by $\lambda(\log n/n)^{1/2}$. In fact, it is not difficult to show (see for example [6]) that, for a Poisson point process π_n with intensity n times the Lebesgue measure on $[0, 1]^2$, the growth of the largest edge is $\Theta((\log n/n)^{1/2}$ almost surely. Let us mention an interesting algorithmic application. For constructing an optimal minimum spanning tree through n random points, one does not need to consider edges with length greater than $\mu(\log n/n)^{1/2}$, for a carefully chosen constant μ. Out of the $n(n-1)/2$ edges, one needs to consider only $\Theta(n\log n)$, and thus develop faster versions of well-known algorithms such as Kruskal's. This scheme defines an asymptotically optimal algorithm with excellent theoretical convergence properties coupled with very fast practical convergence (with 100 points, the probability that the heuristic does not give the optimal solution is below 0.001).

Also, in [36], the authors prove that for any independent and uniform random variables $\{X_i : 1 \le i < \infty\}$ in $[0, 1]^2$, the number of vertices of degree k in the MST through $\{X_1, \ldots, X_n\}$, is asymptotic to a constant α_k times n with probability one. In the case $k = 1$ (i.e., for the number of leaves of the MST in the square), the authors have shown that the constant $\alpha = \alpha_{1,2}$ is positive and that Monte Carlo simulation results suggest that $\alpha = 2/9$ is a reasonable approximation. If one attempts to get any more information on this constant, one rapidly finds that the boundary effects of the square are a serious limitation on any analytical approach. From [15], any attempts on characterizing these constants could be made within the torus model, with no boundary problems. For example, it is clear, from the symmetry induced by the torus model, that α_k is equal to $\lim_{n\to\infty}\mathbf{P}(H^{(n)}=k)$, where $H^{(n)}$ is the degree of any point, say X_1, in a minimal spanning tree through $\{X_1,\ldots,X_n\}$ in the torus.

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