Markov fluid models for energy and performance analysis

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Outline

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- Introduction to Fluid Models
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Motivations

Traditional performance analysis:
system behavior: discrete state model

Energy and performance model:
energy level: continuous variable

System models with energy level
$\Rightarrow$ hybrid (continuous and discrete) state space.

Motivations

Real systems: complex dependencies

$\Rightarrow$ difficult to describe

$\Rightarrow$ only analysis method is simulation

Simulation:

[+] general complex models

[−] computational complexity (e.g., rare events)

Simplified/restricted system model: memoryless behavior

[+] numerical solution (up to a given limit)

[−] often far from real system behavior

Battery usage

Two main kinds of devices
- rechargeable battery
- non-rechargeable battery

Rechargeable battery
- There is a minimum (0) and a maximum (B) energy level.
- Energy consumption and recharging can happen at the same time with different intensities.

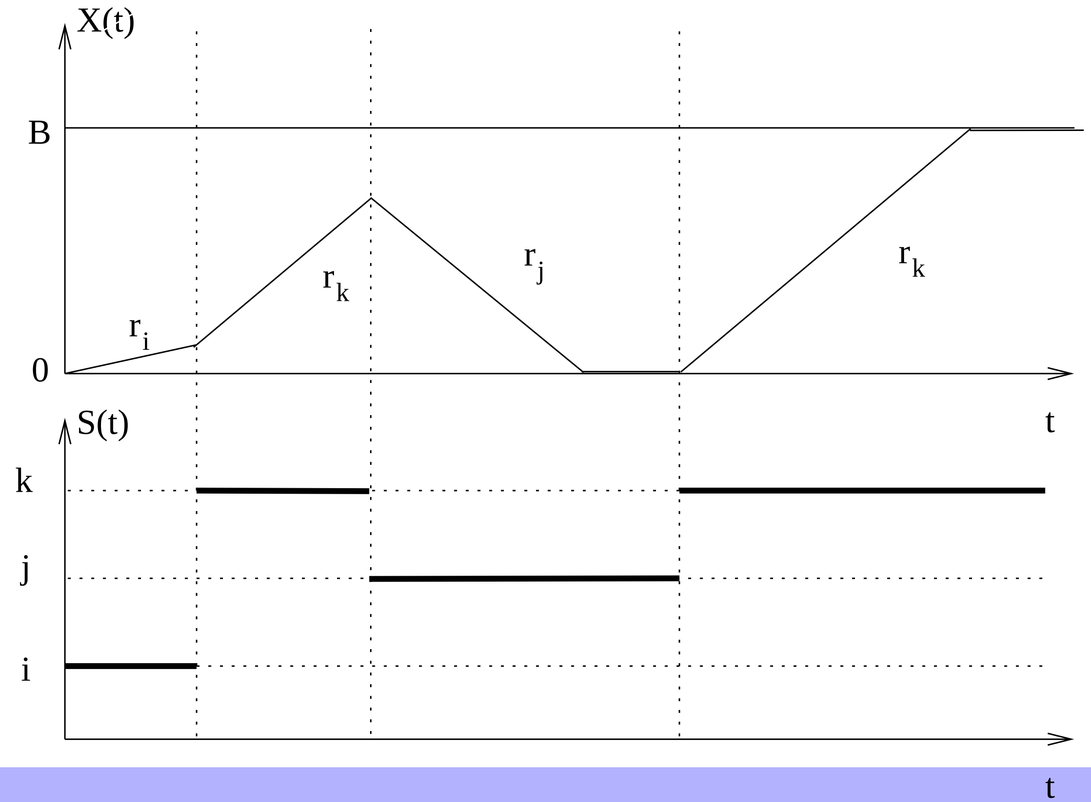
Non-rechargeable battery
(or analyzes of strictly consuming period of rechargeable battery)
- The energy level starts from B and monotone decreases to 0.

Markov fluid model

Memoryless behavior + rechargeable battery =

= Markov fluid model

- bounded evolution,
- different roles at the border.

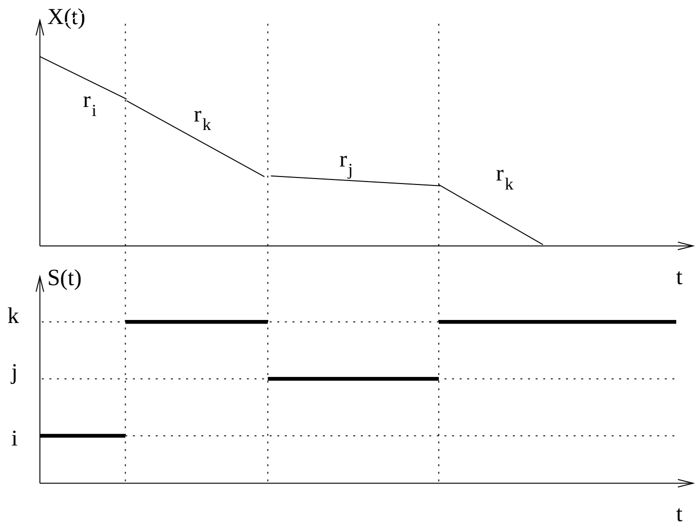


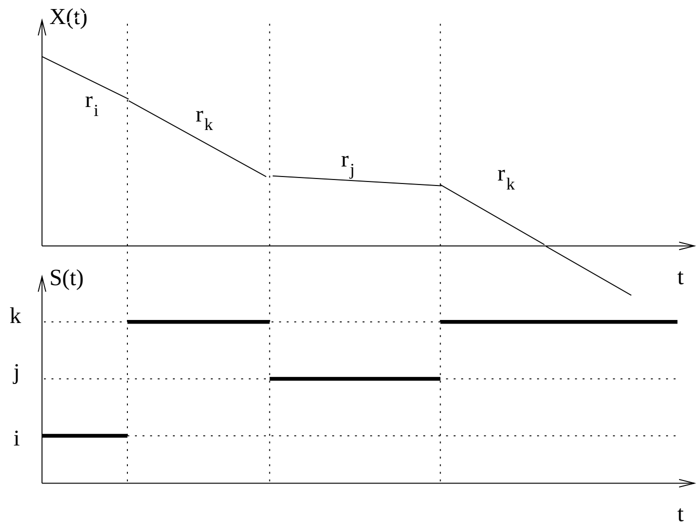
Special Markov fluid model

Memoryless behavior $+$ non-rechargeable battery $=$

$=$ special Markov fluid model

- monotone decreasing evolution,
- equivalent model without border:
  where negative energy means 0 energy





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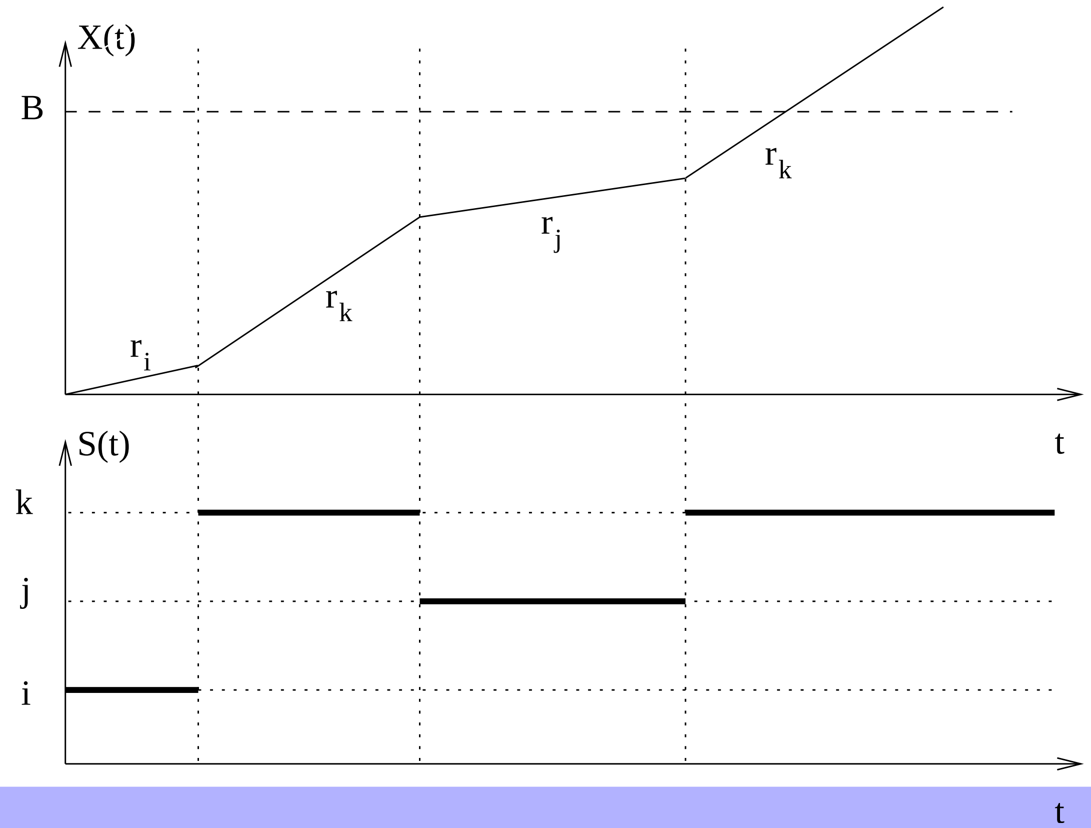
Markov reward model

Replacing the energy level with the consumed energy
$\Rightarrow$ Markov reward model

- starts from level 0,
- monotone increasing evolution,
- energy level larger than B means energy level B



Relation of the models

Fluid models are more general then special fluid models and reward models.

Solution methods of Markov fluid models are applicable for the other two.

Additionally special solution methods are available for Markov reward models.

Introduction to fluid models

Continuous time stochastic processes with

- discrete value (state),
  CTMC,
- continuous value,
  energy level,
- hybrid (continuous and discrete) value,
  discrete system state and energy level.

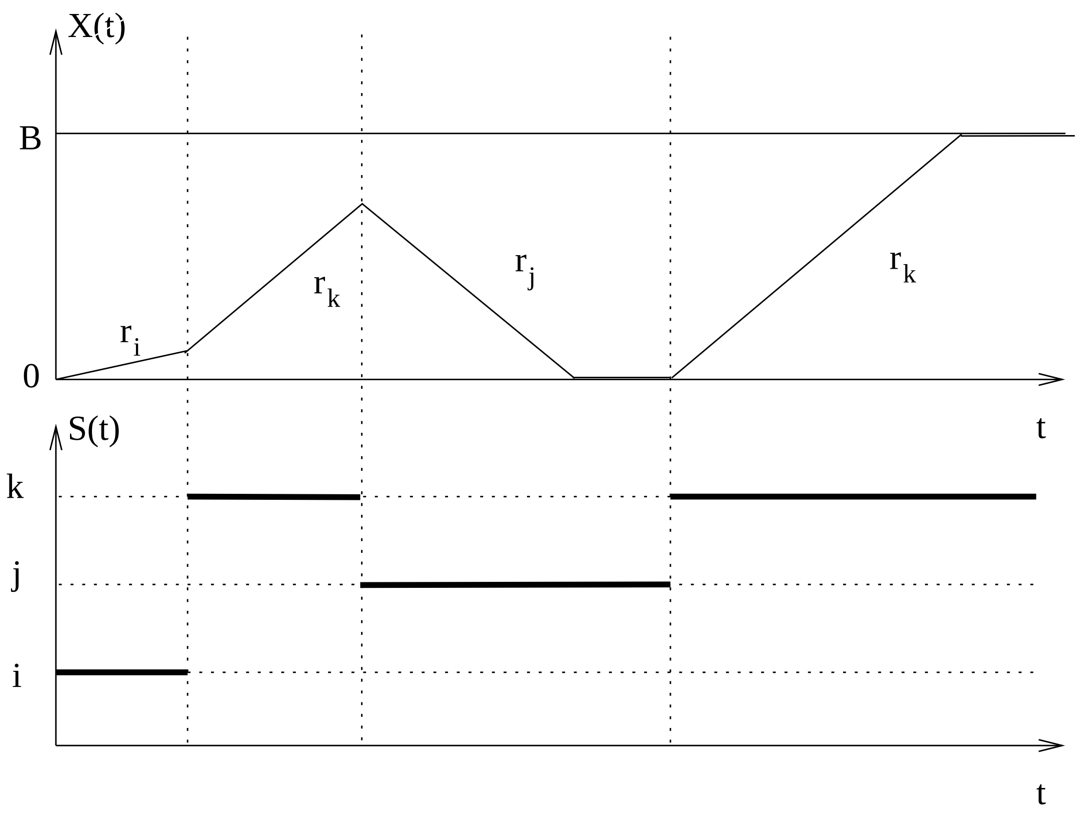
General hybrid valued stochastic processes are hard to analyze.

We focus on the case when a simple function of a discrete state stochastic process governs the evolution of the continuous variable in a memoryless (Markovian) way.

Introduction to fluid models

Fluid models:

- bounded evolution,
- different roles at the border.



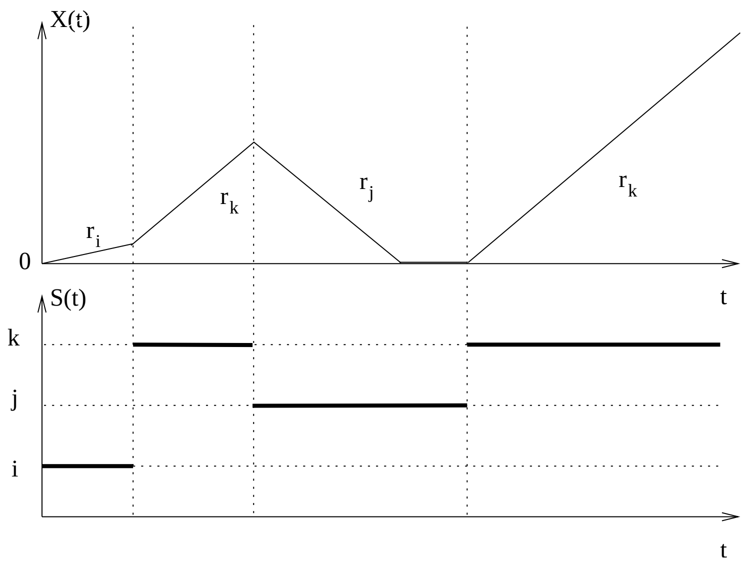
Introduction to fluid models

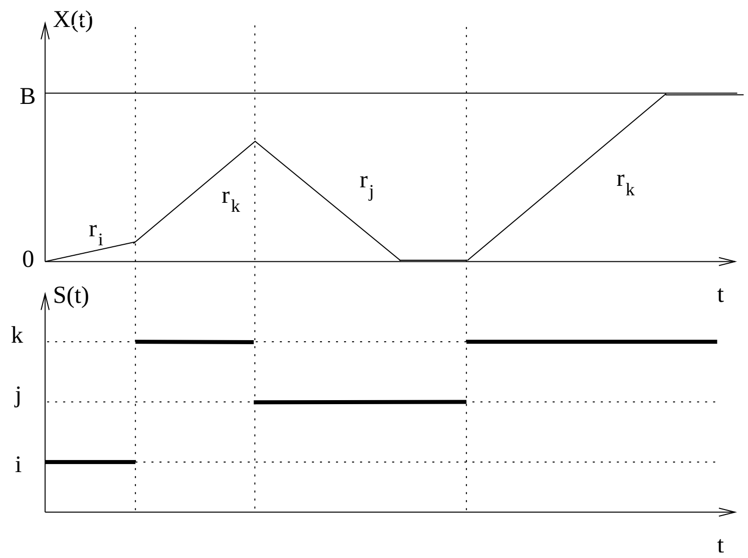
Classes of fluid models:

- finite buffer – infinite buffer,
- first order – second order,
- homogeneous – fluid level dependent,
- barrier behavior in second order case
  - reflecting – absorbing.

Introduction to fluid models

Infinite buffer: the continuous quantity is only lower bounded at zero.
Finite buffer: the continuous quantity is lower bounded at zero and upper bounded at B.

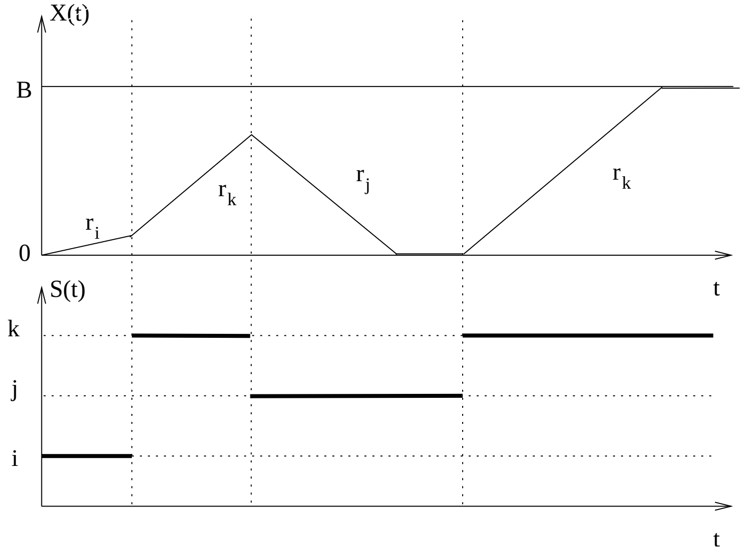


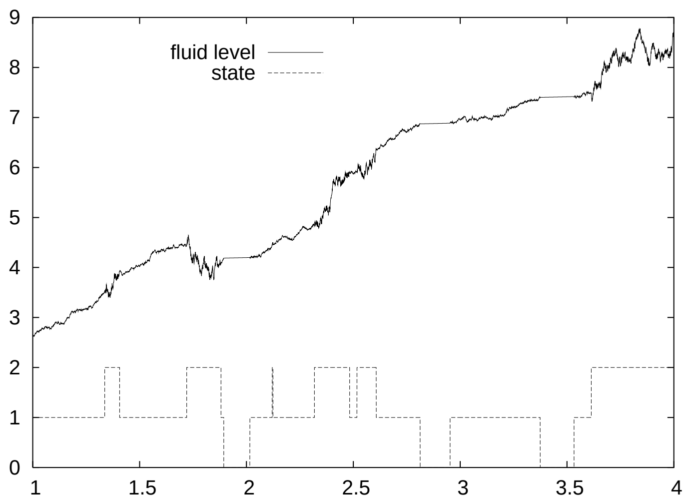


Introduction to fluid models

First order: the continuous quantity is a deterministic function of a CTMC.

Second order: the continuous quantity is a stochastic function of a CTMC.

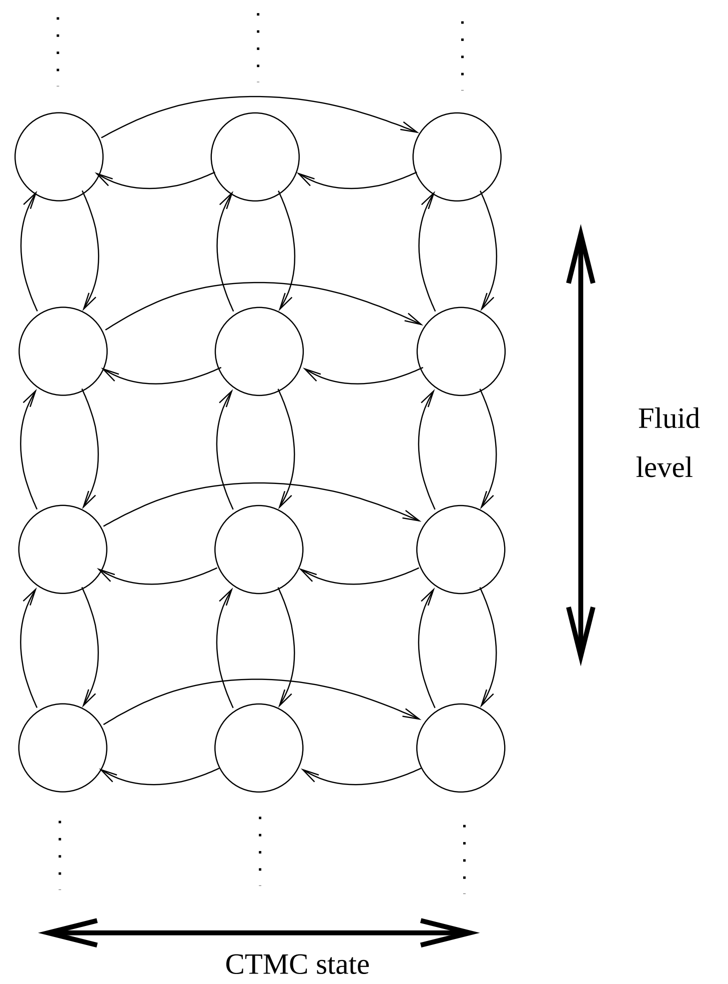




Introduction to fluid models

Interpretation of second order fluid models.

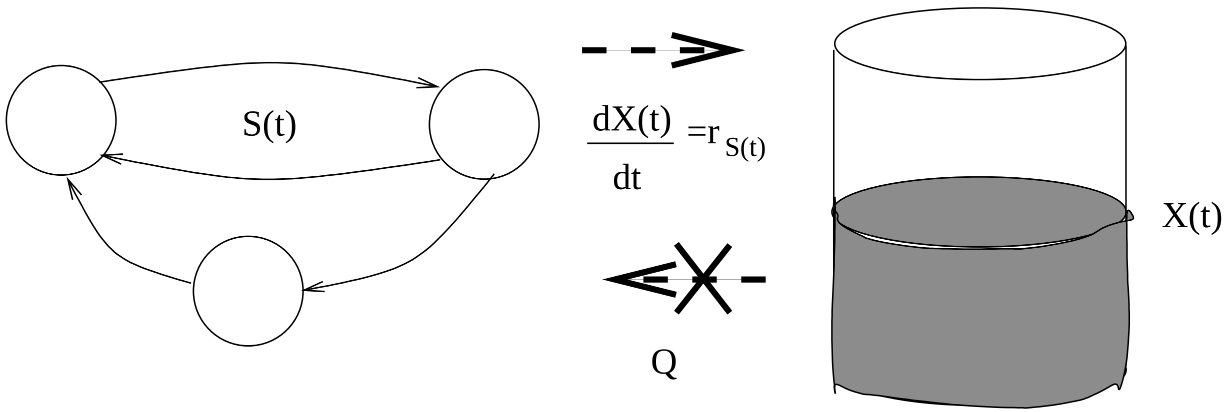
Random walk with decreasing time and fluid granularity.

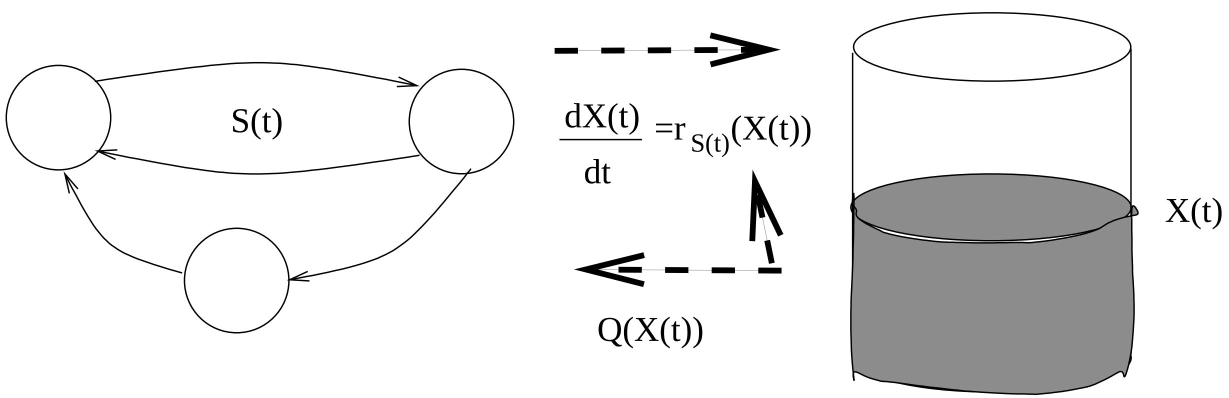


Introduction to fluid models

Homogeneous: the evolution of the CTMC is independent of the fluid level.

Fluid level dependent: the generator of the CTMC is a function of the fluid level.



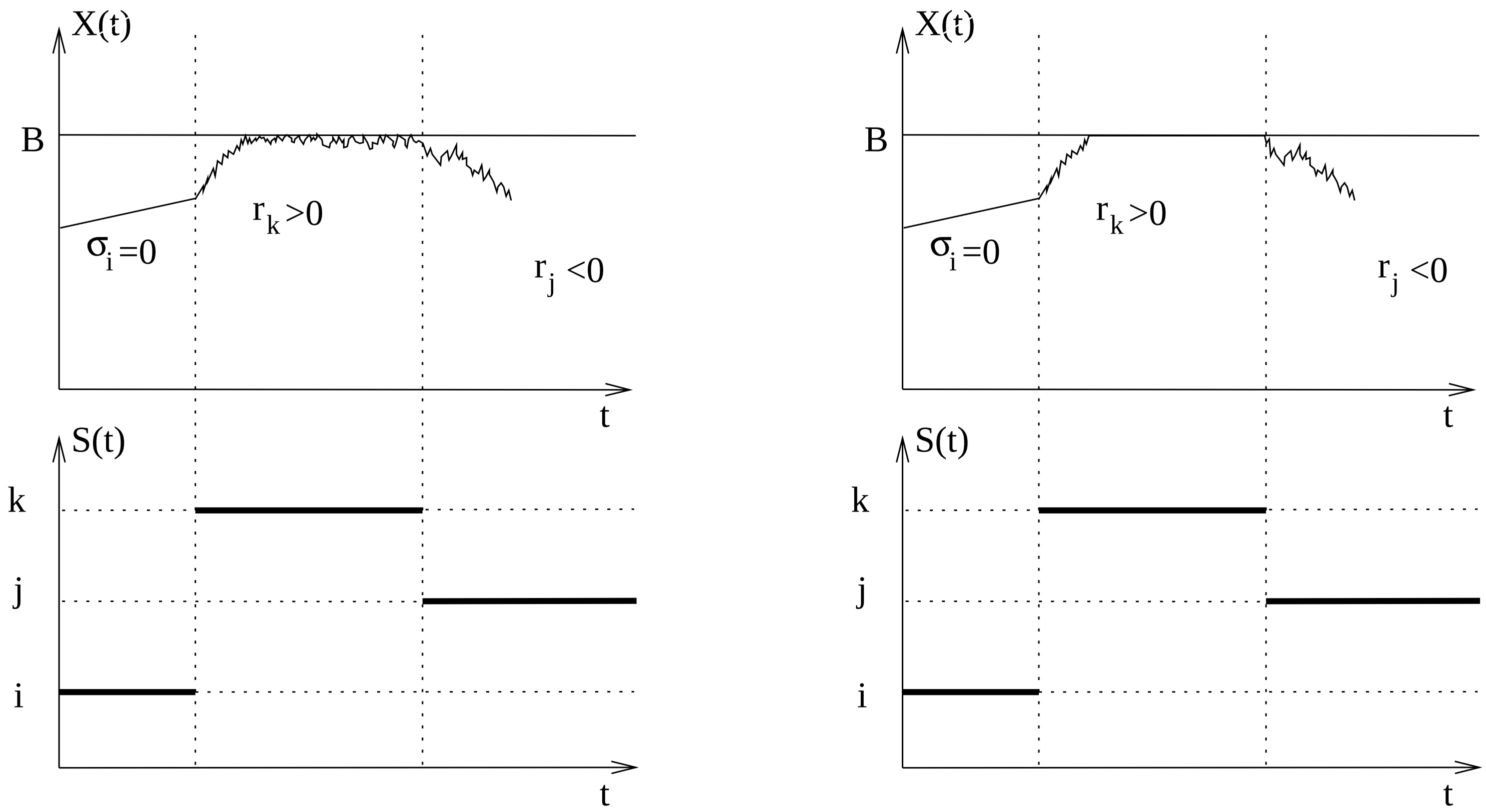


Introduction to fluid models

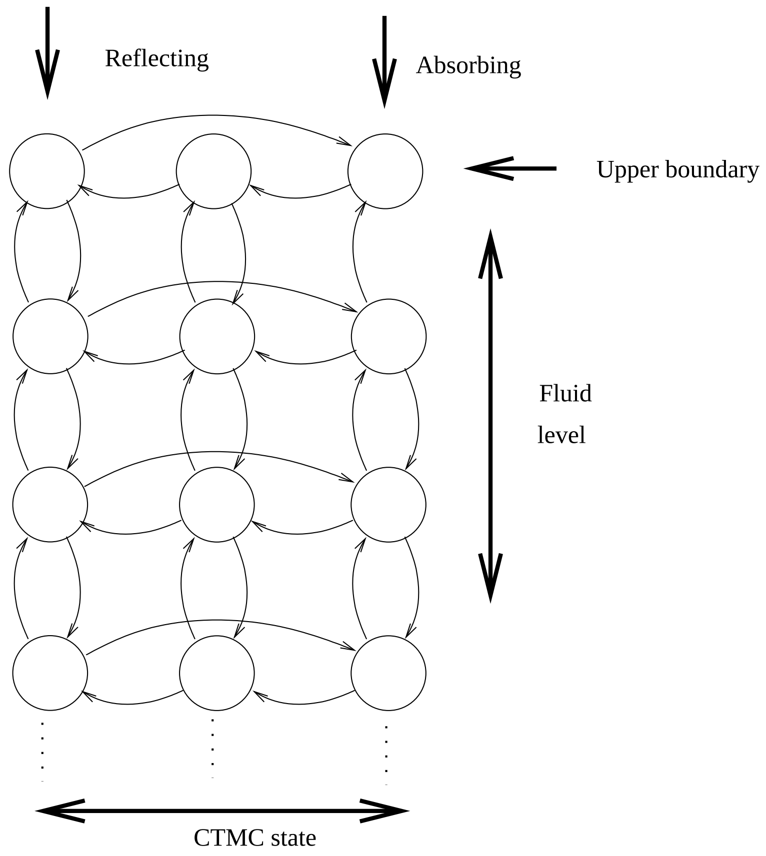
Boundary behavior of second order fluid models.

Reflecting: the fluid level is immediately reflected at the boundary.

Absorbing: the fluid level remains at the boundary up to a state transition of the Markov chain.



Interpretation of the boundary behaviors:



Transient behavior of fluid models

Transient behavior of *first order infinite buffer homogeneous* Markov fluid models,

Extensions:

- finite buffer,
- second order,
- fluid level dependency.

Transient behavior of fluid models

First order, infinite buffer, homogeneous Markov fluid models

During a sojourn of the CTMC in state i ($S(t) = i$) the fluid level ($X(t)$) increases at rate r_i when $X(t) > 0$:

$$X(t+\Delta) - X(t) = r_i \Delta \quad \rightarrow \quad \frac{d}{dt} X(t) = r_i \quad \text{if } S(t) = i, X(t) > 0.$$

When $X(t) = 0$ the fluid level cannot decrease:

$$\frac{d}{dt} X(t) = \max(r_i, 0) \quad \text{if } S(t) = i, X(t) = 0.$$

That is

$$\frac{d}{dt} X(t) = \begin{cases} r_{S(t)} & \text{if } X(t) > 0, \\ \max(r_{S(t)}, 0) & \text{if } X(t) = 0. \end{cases}$$

Transient behavior of fluid models

First order, finite buffer, homogeneous Markov fluid models

When $X(t) = B$ the fluid level can not increase:

$$\frac{d}{dt}X(t) = \min(r_i, 0), \quad \text{if } S(t) = i, X(t) = B.$$

That is

$$\frac{d}{dt}X(t) = \begin{cases} r_{S(t)}, & \text{if } X(t) > 0, \\ \max(r_{S(t)}, 0), & \text{if } X(t) = 0, \\ \min(r_{S(t)}, 0), & \text{if } X(t) = B. \end{cases}$$

Transient behavior of fluid models

Second order, infinite buffer, homogeneous Markov fluid models with reflecting barrier

During a sojourn of the CTMC in state i ($S(t) = i$) in the sufficiently small $(t, t + \Delta)$ interval the distribution of the fluid increment ($X(t + \Delta) - X(t)$) is normal distributed with mean $r_i\Delta$ and variance $\sigma_i^2\Delta$:

$$X(t + \Delta) - X(t) = \mathcal{N}(r_i\Delta, \sigma_i^2\Delta),$$

$$\text{if } S(u) = i, u \in (t, t + \Delta), X(t) > 0.$$

At $X(t) = 0$ the fluid process is reflected immediately,
$\longrightarrow Pr(X(t) = 0) = 0$.

Transient behavior of fluid models

Second order, infinite buffer, homogeneous Markov fluid models with absorbing barrier

Between the boundaries the evolution of the process is the same as before.

First time when the fluid level decreases to zero the fluid process stops,
$\longrightarrow Pr(X(t) = 0) > 0$.

Due to the absorbing property of the boundary the probability that the fluid level is close to it is very low,
$\longrightarrow \lim_{\Delta \to 0} \frac{Pr(0 < X(t) < \Delta)}{\Delta} = 0$.

Transient behavior of fluid models

Inhomogeneous (fluid level dependent), first order, infinite buffer Markov fluid models

The evolution of the fluid level is the same:

$$\frac{d}{dt}X(t) = \begin{cases} r_{S(t)}(X(t)), & \text{if } X(t) > 0, \\ \max(r_{S(t)}(X(t)), 0), & \text{if } X(t) = 0. \end{cases}$$

But the evolution of the CTMC depends on the fluid level:

$$\lim_{\Delta \to 0} \frac{Pr(S(t+\Delta) = j | S(t) = i)}{\Delta} = q_{ij}(X(t)) \ .$$

The generator of the CTMC is $Q(X(t))$ and the rate matrix is $R(X(t))$.

Transient description of fluid models

Notations:

$\pi_i(t) = Pr(S(t) = i)$ – state probability,

$u_i(t) = Pr(X(t) = B, S(t) = i)$ – buffer full probability,

$\ell_i(t) = Pr(X(t) = 0, S(t) = i)$ – buffer empty probability,

$$p_i(t,x) = \lim_{\Delta\to 0} \frac{1}{\Delta} Pr(x \le X(t) < x + \Delta, S(t) = i)$$

– fluid density.

$$\Longrightarrow \quad \pi_i(t) = \ell_i(t) + u_i(t) + \int_x p_i(t,x)dx.$$

Transient description of fluid models

First order, infinite buffer, homogeneous behavior.

Forward argument:

If $S(t+\Delta) = i$, then between t and $t+\Delta$ the CTMC

- stays in i with probability $1 + q_{ii}\Delta$,
- moves from k to i with probability $q_{ki}\Delta$,
- has more than 1 state transition with probability $\sigma(\Delta)$.

Transient description of fluid models

Fluid density:

$$
\begin{aligned}
p_i(t+\Delta, x) \; = \; & (1+q_{ii}\Delta)\, p_i(t, x-r_i\Delta) + \\
& \sum_{k\in\mathcal{S}, k\neq i} q_{ki}\Delta\, p_k(t, x-\mathcal{O}(\Delta)) + \\
& \sigma(\Delta)\ ,
\end{aligned}
$$

where $\lim_{\Delta\to 0}\sigma(\Delta)/\Delta = 0$ and $\lim_{\Delta\to 0}\mathcal{O}(\Delta) = 0$.

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Transient description of fluid models

$$p_i(t+\Delta,x) - p_i(t,x-r_i\Delta) = \sum_{k\in\mathcal{S}} q_{ki}\Delta\, p_k(t,x-\mathcal{O}(\Delta)) + \sigma(\Delta)\ ,$$

$$\frac{p_i(t+\Delta,x) - p_i(t,x)}{\Delta} + r_i \frac{p_i(t,x) - p_i(t,x-r_i\Delta)}{r_i\Delta} = \sum_{k\in\mathcal{S}} q_{ki}\, p_k(t,x-\mathcal{O}(\Delta)) + \frac{\sigma(\Delta)}{\Delta}\ ,$$

$$\boxed{\frac{\partial}{\partial t} p_i(t,x) + r_i \frac{\partial}{\partial x} p_i(t,x) = \sum_{k\in\mathcal{S}} q_{ki}\, p_k(t,x)\ .}$$

Transient description of fluid models

Empty buffer probability:

If $r_i > 0$,

$\longrightarrow$ the fluid level increases in state i,

$\longrightarrow \ell_i(t) = Pr(X(t) = 0, S(t) = i) = 0$.

If $r_i < 0$, then in the same way as before

$$\boxed{\frac{d}{dt}\ell_i(t) \; = \; -r_i \, p_i(t,0) + \sum_{k\in\mathcal{S}} q_{ki} \, \ell_k(t) \; .}$$

Transient description of fluid models

By the definition of fluid density and empty buffer probability we have:

$$\int_0^\infty p_i(t,x)dx + \ell_i(t) = \pi_i(t) \ .$$

In the homogeneous case:

$$\frac{d}{dt}\pi_i(t) \ = \sum_{k\in\mathcal{S}} q_{ki} \ \pi_k(t), \quad \longrightarrow \quad \boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)e^{Qt}.$$

Transient description of fluid models

Second order, infinite buffer, homogeneous behavior.

Fluid density:

$$p_i(t+\Delta,x) \; = \; (1+q_{ii}\Delta) \underbrace{\int_{-\infty}^{\infty} p_i(t,x-u) f_{\mathcal{N}(\Delta r_i, \Delta\sigma_i^2)}(u) du}_{**} + \sum_{k\in\mathcal{S},k\neq i} q_{ki}\Delta \, p_k(t,x-\mathcal{O}(\Delta)) + \sigma(\Delta)$$

Transient description of fluid models

Using

$$p_i(t, x-u) = p_i(t,x) - u p_i'(t,x) + \frac{u^2}{2} p_i''(t,x) + \mathcal{O}(u)^3$$

we have:

$$** =$$

$$p_i(t,x) \underbrace{\int_{-\infty}^{\infty} f_{\mathcal{N}(\Delta r_i, \Delta\sigma_i^2)}(u)du}_{1} - p_i'(t,x) \underbrace{\int_{-\infty}^{\infty} u f_{\mathcal{N}(\Delta r_i, \Delta\sigma_i^2)}(u)du}_{\Delta r_i} +$$

$$p_i''(t,x) \underbrace{\int_{-\infty}^{\infty} \frac{u^2}{2} f_{\mathcal{N}(\Delta r_i, \Delta\sigma_i^2)}(u)du}_{\Delta^2 r_i^2 + \Delta\sigma_i^2/2 = \Delta\sigma_i^2/2 + \sigma(\Delta)} + \underbrace{\int_{-\infty}^{\infty} \mathcal{O}(u)^3 f_{\mathcal{N}(\Delta r_i, \Delta\sigma_i^2)}(u)du}_{\mathcal{O}(\Delta)^2 = \sigma(\Delta)} .$$

Transient description of fluid models

From which:

$$p_i(t+\Delta,x) \;=\; (1+q_{ii}\Delta)\,\big(p_i(t,x)-p_i'(t,x)\Delta r_i+p_i''(t,x)\Delta\sigma_i^2/2\big)+\sum_{k\in\mathcal{S},k\neq i} q_{ki}\Delta\, p_k(t,x-\mathcal{O}(\Delta))+\ \sigma(\Delta)\ ,$$

$$p_i(t+\Delta,x)-p_i(t,x) \;=\; q_{ii}\Delta p_i(t,x)-p_i'(t,x)\Delta r_i+p_i''(t,x)\Delta\sigma_i^2/2+\sum_{k\in\mathcal{S},k\neq i} q_{ki}\Delta\, p_k(t,x-\mathcal{O}(\Delta))+\ \sigma(\Delta)\ ,$$

$$\boxed{\frac{\partial}{\partial t}p_i(t,x)+\frac{\partial}{\partial x}p_i(t,x)r_i-\frac{\partial^2}{\partial x^2}p_i(t,x)\frac{\sigma_i^2}{2} \;=\sum_{k\in\mathcal{S}} q_{ki}\, p_k(t,x).}$$

Transient description of fluid models

General case:
Second order, *finite buffer*, *inhomogeneous behavior*.

Differential equations:

$$\frac{\partial p(t,x)}{\partial t} + \frac{\partial p(t,x)}{\partial x} \boldsymbol{R}(x) - \frac{\partial^2 p(t,x)}{\partial x^2} \boldsymbol{S}(x) = p(t,x) \boldsymbol{Q}(x) ,$$

$$p(t,0) \boldsymbol{R}(0) - p'(t,0) \boldsymbol{S}(0) = \ell(t) \boldsymbol{Q}(0) ,$$

$$-p(t,B) \boldsymbol{R}(B) + p'(t,B) \boldsymbol{S}(B) = u(t) \boldsymbol{Q}(B) ,$$

where $\boldsymbol{R}(x) = \text{Diag}\langle r_i(x) \rangle$ and $\boldsymbol{S}(x) = \text{Diag}\langle \frac{\sigma_i^2(x)}{2} \rangle$.

Transient description of fluid models

General case:

Second order, *finite buffer*, *inhomogeneous behavior*.

Bounding behavior:

$\sigma_i = 0$ and positive/negative drift: $\ell_i(t) = 0$/$u_i(t) = 0$.

$\sigma_i > 0$, reflecting lower/upper barrier: $\ell_i(t) = 0$/$u_i(t) = 0$.

$\sigma_i > 0$, absor. lower/upper barrier: $p_i(t,0) = 0$/$p_i(t,B) = 0$.

Normalizing condition:

$$\int_0^B p(t,x)\,dx\mathbb{I} + \ell(t)\mathbb{I} + u(t)\mathbb{I} = 1.$$

Stationary description of fluid models

Condition of ergodicity:

For $\forall x, y \in \mathbb{R}^+, \forall i, j \in \mathcal{S}$ the transition time

$$T = \min_{t>0}(X(t) = y, S(t) = j | X(0) = x, S(0) = i)$$

has a finite mean (i.e., $E(T) < \infty$).

... AND TIME DEPENDENCE IS ELIMINATED FROM THE RELATED TRANSIENT EQUATION.

Solution methods

Transient analysis:

- (normalized) initial condition,
- set of differential equations,
- bounding behavior.

Stationary analysis:

- set of differential equations,
- bounding behavior,
- normalizing condition.

Transient solution methods

- Numerical solution of differential equations,
- Randomization,
- (Markov regenerative approach,)
- (Transform domain.)

Transient solution methods

Numerical solution of differential equations (Chen et al.)

All cases.

The approach

- starts from the initial condition, and
- follows the evolution of the fluid distribution in the $(t, t+\Delta)$ interval at some fluid levels based on the differential equations and the boundary condition.

This is the only approach for inhomogeneous models.

Transient solution methods

Randomization (Sericola)

First order, infinite buffer, homogeneous behavior.

$$F_i^c(t,x) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{k=0}^{n}\binom{n}{k} x_j^k (1-x_j)^{n-k} b_i^{(j)}(n,k),$$

where $F_i^c(t,x) = Pr(X(t) > x, S(t) = i)$,

$x_j = \frac{x - r_{j-1}^+ t}{r_j t - r_{j-1}^+ t}$ if $x \in [r_{j-1}^+ t, r_j t)$, and

$b_i^{(j)}(n,k)$ is defined by initial value and a simple recursion.

Transient solution methods

Properties of the randomization based solution method:

- the expression with the given recursive formulas is a solution of the differential equation,
  the initial value of $b_i^{(j)}(n,k)$ is set to fulfill the boundary condition,
- $0 \leq x_j \leq 1$
  $\longrightarrow$ convex combination of non-negative numbers
  $\longrightarrow$ numerical stability,
- the initial fluid level is $X(0) = 0$.
  (extension to $X(0) > 0$ and to finite buffer is not available.)

Stationary solution methods

Condition of stability of infinite buffer first/second order homogeneous fluid models.

Suppose $S(t)$ is a finite state irreducible CTMC with stationary distribution π.

The fluid model is stable if the overall drift is negative:

$$d = \sum_{i \in \mathcal{S}} \pi_i r_i < 0.$$

$\longrightarrow$ the variance does not play role.

General stability condition is not available for inhomogeneous models.

Stationary solution methods

Stationary solution methods

- Spectral decomposition,
- Matrix exponent,
- Numerical solution of differential equations,
- Spectral partitioning.

State space partitioning:

- $\mathcal{S}^+$: $i \in \mathcal{S}^+$ iff $\sigma_i > 0$,
  second order states,
- $\mathcal{S}^0$: $i \in \mathcal{S}^0$ iff $r_i = 0$ and $\sigma_i = 0$,
  zero states,
- $\mathcal{S}^{0+}$: $i \in \mathcal{S}^{0+}$ iff $r_i > 0$ and $\sigma_i = 0$,
  positive first order states,
- $\mathcal{S}^{0-}$: $i \in \mathcal{S}^{0-}$ iff $r_i < 0$ and $\sigma_i = 0$,
  negative first order states,
- $\mathcal{S}^* = \mathcal{S}^{0-} \bigcup \mathcal{S}^{0+}$,
  first order states.

Stationary solution methods

First order, infinite/finite buffer, homogeneous case.

Spectral decomposition (Kulkarni)

Differential equation: $p'(x)\ \boldsymbol{R}\ = p(x)\ \boldsymbol{Q}\ ,$

Form of the solution vector: $p(x) = e^{\lambda x}\phi,$

Substituting this solution we get the characteristic equation:

$$\phi(\lambda \boldsymbol{R} - \boldsymbol{Q}) = 0,$$

whose solutions are obtained at $\det(\lambda \boldsymbol{R} - \boldsymbol{Q}) = 0$.

Stationary solution methods

Spectral decomposition

The characteristic equation has $|\mathcal{S}^{0+}| + |\mathcal{S}^{0-}|$ solutions, with

$$\begin{cases} |\mathcal{S}^{0+}| & \text{negative eigenvalue,} \\ 1 & \text{zero eigenvalue,} \\ |\mathcal{S}^{0-}| - 1 & \text{positive eigenvalue.} \end{cases}$$

From which the solution is: $$p(x) = \sum_{j=1}^{|\mathcal{S}^{0+}|+|\mathcal{S}^{0-}|} a_j e^{\lambda_j x} \phi_j,$$

and the a_j coefficients are set to fulfill the boundary and normalizing conditions.

Stationary solution methods

Spectral decomposition

In the *infinite buffer* case these conditions are:

- $p(0)\ \boldsymbol{R}\ = \ell\ \boldsymbol{Q}\ ,$
- $\ell_i = 0$ if $r_i > 0$, and
- $\int_0^\infty p_i(x) + \ell_i = \pi_i .$

From which $a_j = 0$ for $\lambda_j > 0$
and the rest of the coefficients are obtained from a linear system of equations.

In the *finite buffer* case these conditions are:

- $p(0)\ \boldsymbol{R}\ = \ell\ \boldsymbol{Q}\ ,\ \ p(B)\ \boldsymbol{R}\ = u\ \boldsymbol{Q}\ ,$
- $\ell_i = 0$ if $r_i > 0$, $u_i = 0$ if $r_i < 0$, and
- $\int_0^\infty p_i(x) + \ell_i + u_i = \pi_i .$

From which the a_j coefficients are obtained from a linear system of equations.

Stationary solution methods

First order, finite buffer, homogeneous case.

Matrix exponent: (Gribaudo)
Assume that $|\mathcal{S}^0| = 0$ and $\mathcal{S} = \mathcal{S}^*$.
Introduce $v = \ell + u$, $\boldsymbol{Q}^-$, $\boldsymbol{Q}^+$,
where $q^-_{ij} = q_{ij}$ if $i \in \mathcal{S}^-$ and otherwise $q^-_{ij} = 0$.

The set of equations becomes:

$$\frac{\partial p(x)}{\partial x}\boldsymbol{R} = p(x)\boldsymbol{Q} \quad \longrightarrow \quad p(B) = p(0)\, e^{\boldsymbol{Q}\boldsymbol{R}^{-1}B} = p(0)\,\boldsymbol{\Phi},$$

$$p(0)\boldsymbol{R} = v\boldsymbol{Q}^- \quad \longrightarrow \quad p(0) = v\boldsymbol{Q}^-\boldsymbol{R}^{-1},$$

$$-p(B)\boldsymbol{R} = v\boldsymbol{Q}^+ \quad \longrightarrow \quad \boxed{v(\boldsymbol{Q}^-\boldsymbol{R}^{-1}\boldsymbol{\Phi}\boldsymbol{R} + \boldsymbol{Q}^+) = 0},$$

And the normalizing condition is

$$\ell\mathbb{1} + u\mathbb{1} + p(0)\underbrace{\int_0^B e^{\boldsymbol{Q}\boldsymbol{R}^{-1}x} dx}_{\boldsymbol{\Psi}}\,\mathbb{1} = \boxed{v(\boldsymbol{I} + \boldsymbol{Q}^-\boldsymbol{R}^{-1}\boldsymbol{\Psi})\mathbb{1} = 1}.$$

Stationary solution methods

Second order, infinite/finite buffer, homogeneous case.

Spectral decomposition (Karandikar-Kulkarni)

Differential equation: $p'(x)\,\boldsymbol{R} - p''(x)\,\boldsymbol{S} = p(x)\,\boldsymbol{Q}$,

Form of the solution vector: $p(x) = e^{\lambda x}\phi$,

Substituting this solution we get the characteristic equation:

$$\phi(\lambda\boldsymbol{R} - \lambda^2\boldsymbol{S} - \boldsymbol{Q}) = 0,$$

whose solutions are obtained at $\det(\lambda\boldsymbol{R} - \lambda^2\boldsymbol{S} - \boldsymbol{Q}) = 0$.

Stationary solution methods

Spectral decomposition

The characteristic equation has $2|\mathcal{S}^+| + |\mathcal{S}^*|$ solutions, with

$$\begin{cases} |\mathcal{S}^+| + |\mathcal{S}^{0+}| & \text{negative eigenvalue,} \\ 1 & \text{zero eigenvalue,} \\ |\mathcal{S}^+| + |\mathcal{S}^{0-}| - 1 & \text{positive eigenvalue.} \end{cases}$$

From which the solution is: $p(x) = \sum_{j=1}^{2|\mathcal{S}^+|+|\mathcal{S}^*|} a_j e^{\lambda_j x} \phi_j,$

and the a_j coefficients are set to fulfill the boundary and normalizing conditions.

Stationary solution methods

Second order, infinite/infinite buffer, homogeneous case.

A transformation of the quadratic equation to a linear one

Assume that $|\mathcal{S}^0| = |\mathcal{S}^*| = 0$ and $\mathcal{S} = \mathcal{S}^+$.

$$\begin{aligned} \frac{d}{dx} p(x)\, \boldsymbol{R} - \frac{d}{dx} p'(x)\, \boldsymbol{S} &= p(x)\, \boldsymbol{Q}\,, \\ \frac{d}{dx} p(x)\, \boldsymbol{I} &= p'(x)\, \boldsymbol{I}\,, \end{aligned}$$

$$\frac{d}{dx} \begin{bmatrix} p(x) & p'(x) \end{bmatrix} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{I} \\ -\boldsymbol{S} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} p(x) & p'(x) \end{bmatrix} \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}$$

$$\Longrightarrow \frac{d}{dx} \hat{p}(x)\, \hat{\boldsymbol{R}} = \hat{p}(x)\, \hat{\boldsymbol{Q}} \longrightarrow \hat{p}(B) = \hat{p}(0)\, e^{\hat{\boldsymbol{Q}} \hat{\boldsymbol{R}}^{-1} B}.$$

Stationary solution methods

Numerical solution of differential equations (Gribaudo et al.)

For cases with finite buffer.

Numerically solve the matrix function $\boldsymbol{M}(x)$ with initial condition $\boldsymbol{M}(0) = \boldsymbol{I}$ based on

$$\boldsymbol{M}'(x)\,\boldsymbol{R}(x) - \boldsymbol{M}''(x)\,\boldsymbol{S}(x) = \boldsymbol{M}(x)\,\boldsymbol{Q}(x)$$

and calculate the unknown boundary conditions based on

$$p(B) = p(0)\,\boldsymbol{M}(B)$$

This is the only approach for inhomogeneous models.

Weakness of the previous stationary solution

Major step:

- Spectral decomposition (Kulkarni),
- Computation based on matrix exponent

$$\boldsymbol{M} = e^{\boldsymbol{Q}\boldsymbol{R}^{-1}B} \text{ and } p(B) = p(0)\boldsymbol{M}$$

Problems:

- Spectral decomposition:
  expensive and numerically sensitive
- Computation based on matrix exponent:
  matrix $\boldsymbol{Q}\boldsymbol{R}^{-1}$ has positive and negative eigenvalues
  $\rightarrow$ numerical rank degradation of $\boldsymbol{M}$ when B is large
  $\rightarrow$ there is no solution for the linear system.

Possible solution:
Iterative computation methods which separates the positive and negative eigenvalues.

Spectral partitioning

(proposed by A. Nail and his coauthors)

Let $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}^{-1}$ and π the stationary distribution of the Markov chain.

One eigenvalue of $\boldsymbol{A}$ is 0 because

$$\pi\boldsymbol{A} = \underbrace{\pi\boldsymbol{Q}}_{0}\,\boldsymbol{R}^{-1} = 0$$

Find $\boldsymbol{T}$, e.g., using generalized Schur decomposition such that

$$\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{A}_{22} \end{pmatrix}$$

where the eigenvalues of $\boldsymbol{A}_{11}$ ($\boldsymbol{A}_{22}$) have negative (positive) real part.

Spectral partitioning

Let $p(x)\boldsymbol{T} = [t(x), u(x), v(x)]$, then

$$t'(x) = 0, u'(x) = u(x)\boldsymbol{A_{11}}, v'(x) = v(x)\boldsymbol{A_{22}}.$$

and

$$t(x) = c, u(x) = u(0)e^{\boldsymbol{A_{11}}x}, v(x) = v(0)e^{\boldsymbol{A_{22}}x}.$$

The key idea is

$$v(x) = v(0)e^{\boldsymbol{A_{22}}x} = v(B)e^{-\boldsymbol{A_{22}}(B-x)}.$$

where the eigenvalues of $-\boldsymbol{A_{22}}$ have negative real part.

Finally, the fluid density is

$$p(x) = c\boldsymbol{L_1} + u(0)e^{\boldsymbol{A_{11}}x}\boldsymbol{L_2} + v(B)e^{-\boldsymbol{A_{22}}(B-x)}\boldsymbol{L_3},$$

where $\boldsymbol{L_i}$ is the related block of $\boldsymbol{T}^{-1}$.

... and the set of linear equations remains stable.

Spectral partitioning

(proposed by the matrix analytic community)

The standard solution method of QBD processes computes the "minimal non-negative solution" of the quadratic matrix equation

$$\mathbf{B} + \mathbf{L}\mathbf{G} + \mathbf{F}\mathbf{G}^2 = \mathbf{0}$$

where $\mathbf{G}$ is an unknown matrix.

The quadratic equation has 2^n solutions and the "minimal non-negative solution" is such that all eigenvalues of $\mathbf{G}$ are inside the unit disk.

Spectral partitioning

Transformation to unit fluid rates:
Let $\hat{\boldsymbol{R}} = \text{Diag}\langle 1/|r_i|\rangle$.

Multiplying

$$\frac{d}{dx}\boldsymbol{\pi}(x)\boldsymbol{R} = \boldsymbol{\pi}(x)\boldsymbol{Q} \ ,$$

with $\hat{\boldsymbol{R}}$ from the right results in a modified Markov fluid model with identical fluid density

$$\bar{\boldsymbol{R}} = \boldsymbol{R}\hat{\boldsymbol{R}} = \text{Diag}\langle \pm 1\rangle, \quad \bar{\boldsymbol{Q}} = \boldsymbol{Q}\hat{\boldsymbol{R}}$$

We partition the states according to the sign of the fluid rate

$$\bar{\boldsymbol{Q}} = \begin{array}{|c|c|} \hline \mathbf{Q}_{++} & \mathbf{Q}_{+-} \\ \hline \mathbf{Q}_{-+} & \mathbf{Q}_{--} \\ \hline \end{array}$$

Spectral partitioning

Main observation:
$\boldsymbol{\pi}(x)$ has a matrix exponential solution when $B = \infty$ and the queue is stable

$$\boldsymbol{\pi}(x) = \boldsymbol{v}_+ e^{\boldsymbol{K}_{++}x}[\boldsymbol{I}_{++}, \boldsymbol{\Psi}]$$

consequently, the eigenvalues of $\boldsymbol{K}_{++}$ have negative real part.

The matrix $[\boldsymbol{I}_{++}, \boldsymbol{\Psi}]$ represents the fact that an upward level crossing at level x has a pair, a downward level crossing at level x.

The stochastic behavior between consecutive level crossings at level x is identical with the one at level 0.

Matrix $\boldsymbol{\Psi}$ describes the state transition between the beginning and the end of a busy period

$$\boldsymbol{\Psi}_{ij} = P(S(T) = j | S(0) = i, X(0) = 0)$$

where T is the time of the busy period.

Spectral partitioning

Computation of $\mathbf{\Psi}$:
Let $c = \max_{i \in S} |\mathbf{Q}_{ii}|$ and define matrix $\mathbf{P} = \mathbf{I} + \mathbf{Q}/c$ which is identically partitioned as $\mathbf{Q}$,

$$\mathbf{F} = \begin{array}{|c|c|} \hline \frac{1}{2}\mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \\ \hline \end{array}, \quad \mathbf{L} = \begin{array}{|c|c|} \hline \frac{1}{2}\mathbf{P}_{++} - \mathbf{I} & \mathbf{0} \\ \hline \mathbf{P}_{-+} & -\mathbf{I} \\ \hline \end{array}, \quad \mathbf{B} = \begin{array}{|c|c|} \hline \mathbf{0} & \frac{1}{2}\mathbf{P}_{+-} \\ \hline \mathbf{0} & \mathbf{P}_{--} \\ \hline \end{array}.$$

Find the minimal non-negative solution of the quadratic equation

$$\mathbf{B} + \mathbf{L}\mathbf{G} + \mathbf{F}\mathbf{G}^2 = \mathbf{0}$$

$\mathbf{\Psi} = \mathbf{G}_{+-}$ obtained from the minimal non-negative solution.

Finally,

$$\mathbf{K}_{++} = \mathbf{Q}_{++} + \mathbf{\Psi}_{+-}\mathbf{Q}_{-+},$$

Analysis of Markov reward models

Markov reward models are similar to Markov fluid models except their boundary behavior.

That is, we have

$$\frac{\partial}{\partial t} p_i(t,x) + \ r_i(x) \ \frac{\partial}{\partial x} p_i(t,x) = \sum_{k \in \mathcal{S}} \ q_{ki}(x) \ \ p_k(t,x)$$

but instead of

$$\frac{d}{dt} X(t) = \begin{cases} r_{S(t)}, & \text{if } X(t) > 0, \\ \max(r_{S(t)}, 0), & \text{if } X(t) = 0, \\ \min(r_{S(t)}, 0), & \text{if } X(t) = B. \end{cases}$$

we have

$$\frac{d}{dt} X(t) = r_{S(t)}$$

Markov reward models

The interesting performance measures

- fluid level distribution at time t,
- distribution of time to reach fluid level w,

are available from the solution of the PDE.

But there is a computationally efficient way to avoid the solution of the PDE.

Computation of moments

The nth moment ($n \geq 1$) of the fluid level distribution at time t $V_{ij}^{(n)}(t) = \int_{w=0}^{\infty} w^n dY_{ij}(t,w)$ satisfies the PDE

$$\frac{d}{dt} \boldsymbol{V}^{(n)}(t) = n\, \boldsymbol{V}^{(n-1)}(t)\, \boldsymbol{R} + \boldsymbol{V}^{(n)}(t)\, \boldsymbol{Q},$$

with initial condition $\boldsymbol{V}^{(n)}(0) = \boldsymbol{0}, \forall n \geq 1$ and $\boldsymbol{V}^{(0)}(0) = \boldsymbol{I}$.

Computation of moments

With strictly positive reward rates the nth moment ($n \geq 1$) of the time to reach fluid level w satisfies the PDE

$$\frac{d}{dw}\,\boldsymbol{D}^{(n)}(w) = n\,\boldsymbol{D}^{(n-1)}(w)\,\boldsymbol{R}^{-1}(w) + \boldsymbol{D}^{(n)}(w)\,\boldsymbol{R}^{-1}(w)\,\boldsymbol{Q}(w),$$

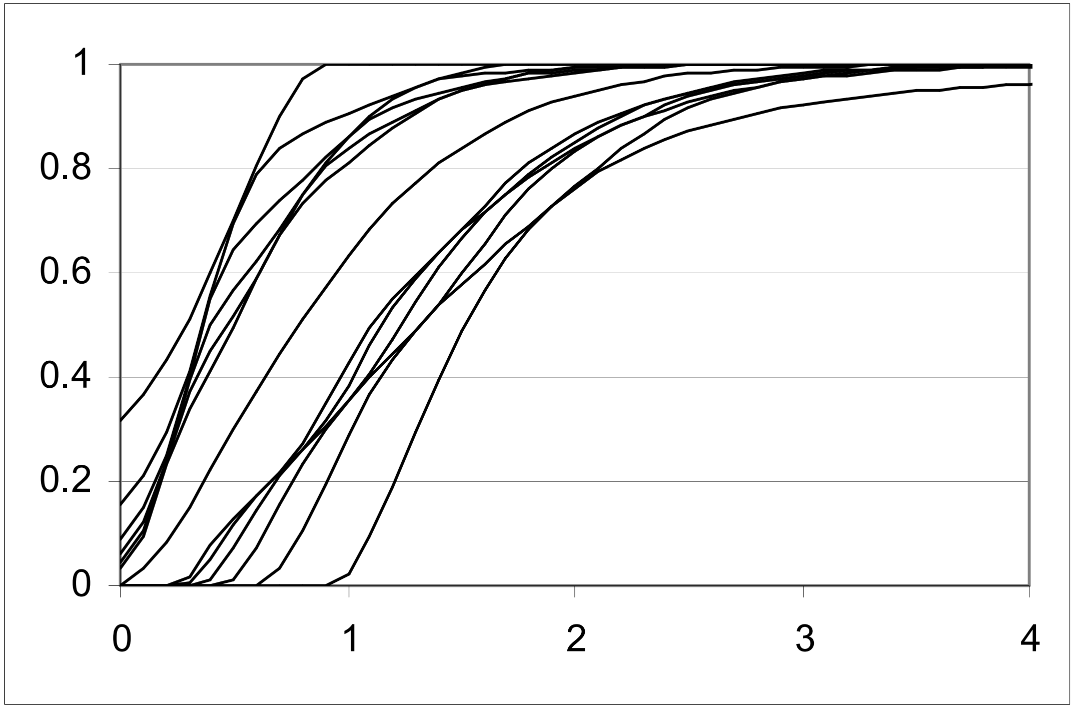
with initial conditions $\boldsymbol{D}^{(n)}(0) = \boldsymbol{0}, n \geq 1$ and $\boldsymbol{D}^{(0)}(0) = \boldsymbol{I}$.

$\Rightarrow$ cheap computation of the moments.

Moments based distribution bounding

A procedure is available for moments based distribution bounding based on an arbitrary number of moments.

The procedure evaluate the extreme values of all distributions with the given moments.



(Weibull with shape parameters 1.5)

Moments based distribution bounding

The procedure is based on a result of Stieltjes (1899):

The moments of a random variable ($\mu_i = E(X^i)$) are such that

$$\text{Det}\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix} \geq 0$$

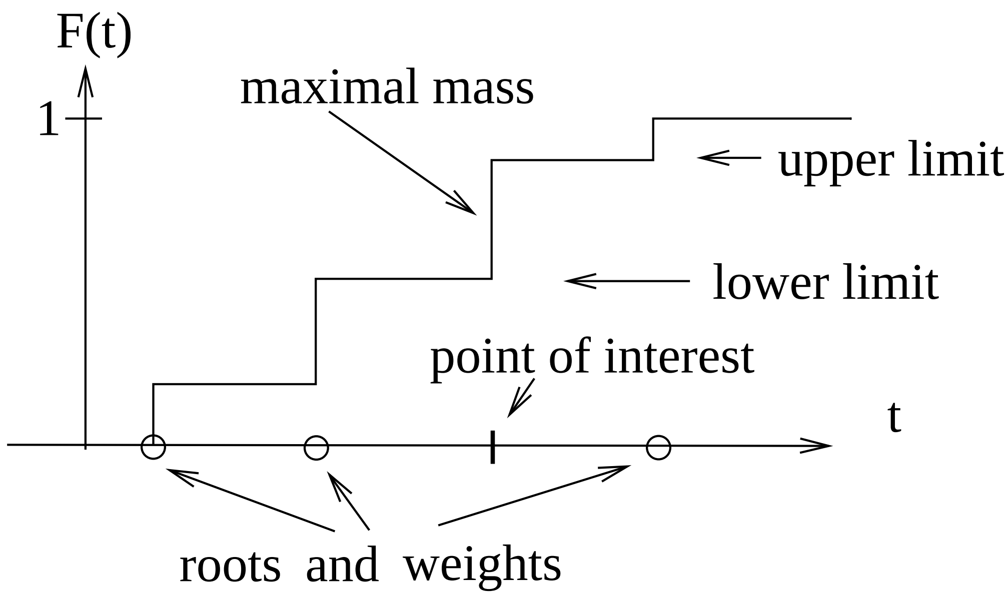
for all $n \geq 1$

and if the determinant is positive for $n < N$ and zero for $n = N$, then X is a discrete distribution with N points.

Moments based distribution bounding

Summary of the procedure:

- calculates maximal mass at the point of interest,
- computes the roots and weights of the other points of an extreme discrete distribution.



An implementation is available as part of the MRMSolve package.

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Conclusions

- Stochastic models with continuous variables (Fluid models, Hybrid models, FSPNs) often allows proper modeling of real systems.
- Their analysis is more complex than the one of only discrete variables, but feasible for a wide class of models.
- The analytical description of these models and a set of solution techniques have been introduced.