VECTOR FIELD METRICS BASED ON DISTANCE MEASURES OF FIRST ORDER CRITICAL POINTS

H. Theisel [⋆]
T. Weinkauf [⋆⋆]

[⋆] University of Rostock, Computer Science Department, D-18051 Rostock, Germany
theisel@informatik.uni-rostock.de, http://wwwicg.informatik.uni-rostock.de/

[⋆⋆] Zuse Institute Berlin, Scientific Visualization Group, Takustr. 7, D-14343 Berlin, Germany
weinkauf@zib.de, http://www.zib.de/visual/

ABSTRACT

Topological methods have been proven to be useful both for the visualization and the definition of distance measures of vector fields. This paper introduces and assesses a new distance measure for first order critical points of 2D vector fields. This distance measure forms the foundation of the definition of vector field metrics. Based on this we give an advanced and complete classification of all first order critical points.

Keywords: flow visualization, vector field topology, first order critical point, distant measure

1 INTRODUCTION

During the last few years the investigation of metrics of vector fields became an interesting and popular research topic in the visualization community. A variety of applications like reconstruction, compression and simplification of vector fields rely on certain metrics on them: they are used to compare the original and the new vector field and consequently give information about the quality of the algorithms.

The first approaches on metrics (distance measures) of vector fields consider local deviations of direction and magnitude of the flow vectors in a certain number of sample points ([Hecke99], [Telea99]). These distance functions give a fast comparison of the vector field but do not take the complete flow behavior into consideration.

The application of topological methods is nowadays a popular method to visualize vector fields. Originally introduced in [Helma89] for 2D vector fields, topological methods have been proven to describe the global flow behavior in a very effective and condensed way. The original work [Helma89] considers first order critical points and a certain number of separatrices between them.

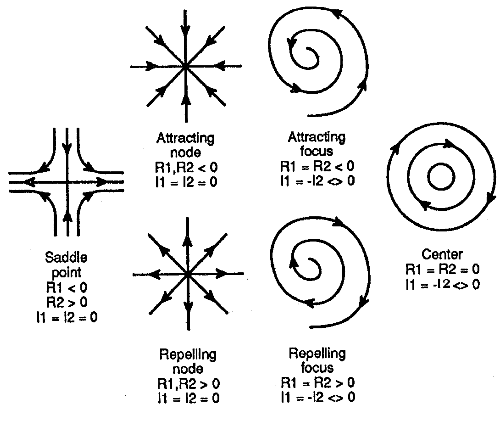

Figure 1: Classification of first order critical points; R_1, R_2 denote the real parts of the eigenvalues of the Jacobian matrix while I_1, I_2 denotes its imaginary parts (from [Helma89]).

This way [Helma89] yields a classification of first order critical points which is shown in figure 1. In the following years also higher order critical points ([Scheu97], [Scheu98]), 3D vector fields ([Batra99a], [Chong90], [Phili97]), and more general separatrices ([Kenwr99], [Trott00]) have been considered. A comprehensive introduction into the topology of 2D vector fields can be found in [Firby82].

Since the topology of vector fields describes the flow behavior in an efficient way, it is an obvious approach to incorporate it into the definition of vector field metrics: vector fields with a similar flow behavior should have a rather small distance to each other. A first approach to find a distance function which is based on the topology of vector fields is introduced in [Lavin98]. There the critical points of the vector fields to be compared are detected and matched: for each critical point in the first vector field a corresponding critical point in the second vector field has to be found, and vice versa. Then the distances between all corresponding critical points are compared: their summation gives the distance of the two vector fields. This way the computation of the distance of two vector fields is reduced to the computation of the distance of critical points. [Batra99b] gives an extension of [Lavin98] by considering not only the critical points but also their connectivity. The problem of finding suitable distance measures between critical points is highly correlated to the problem of finding a suitable classification of all kinds of critical points.

It is the purpose of this paper to assess the quality of the distance measures for first order critical points in [Batra99b] and [Lavin98] and to introduce a new advanced metrics of first order critical points. Based on this metrics we obtain a complete classification scheme of first order critical points. This paper is organized in the following way: section 2 introduces the necessary common concepts of vector fields and critical points. Section 3 surveys the metrics of first order critical points in [Lavin98] which is based on an (α, β) phase plane. The foundation of the new distance measure to be introduced are smooth vector field transformations which are introduced in section 4. The new distance measure which is based on a (γ, r) phase plane is introduced in section 5.

2 VECTOR FIELDS AND CRITICAL POINTS

Given a vector field

$$\begin{array}{rccc} \mathbf{v}: & E_2 & \rightarrow & \mathbb{R}^2 \\ & (x,y) & \rightarrow & \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \end{array} \tag{1}$$

where E_2 is a closed and compact subset of $\mathbb{E}^2$, we assume $\mathbf{v}$ to be continuous and differentiable. The *Jacobian matrix field*

$$\mathbf{J}_{\mathbf{v}}(x,y) = \begin{pmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{pmatrix}. \tag{2}$$

covers all first order derivatives of $\mathbf{v}$. The determinant of $\mathbf{J}_{\mathbf{v}}$ is called *Jacobian* of $\mathbf{v}$. A point $\mathbf{x}_0 \in E_2$ is called a *critical point* iff $\mathbf{v}(\mathbf{x}_0) = (0,0)^T = \mathbf{0}$ and $\mathbf{v}(\mathbf{x}) \neq \mathbf{0}$ for any $\mathbf{x} \neq \mathbf{x}_0$ in a certain neighborhood of $\mathbf{x}_0$. A critical point $\mathbf{x}_0$ in the vector field $\mathbf{v}$ is called a *first order critical point* iff the Jacobian does not vanish in $\mathbf{x}_0$; otherwise the critical point is called *higher order critical point*. The *divergence* of the vector field is given by

$$\mathrm{div}(\mathbf{v}(x,y)) = u_x(x,y) + v_y(x,y). \tag{3}$$

Since this paper considers only first order critical points, we can apply a first order Taylor expansion of the vector field around the critical point. Without loss of generality we can also assume that critical point $\mathbf{x}_0$ lies at $\mathbf{0} = (0,0)$. This yields the following simplified description of the vector field:

$$\mathbf{v}(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4}$$

This linear vector field has one critical point at $\mathbf{0}$ and a constant Jacobian matrix $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. For the rest of the paper we only have to consider the critical point of vector fields described by (4).

3 THE (α, β) PHASE PLANE

The conceptional idea of how to compute the distance of two first order critical points in [Lavin98] is to compute the amount of work which must be performed to transform one critical point into the other. Based on the Jacobian matrix $\mathbf{J}_{\mathbf{v}}$, the critical point $\mathbf{0}$ of the vector field (4) is mapped into an (α, β) phase plane by

$$\begin{array}{c} p = \mathrm{div}(\mathbf{v}) \quad , \quad q = \det(\mathbf{J}_{\mathbf{v}}) \\ \hat{\alpha} = p \quad , \quad \hat{\beta} = \mathrm{sign}(p^2 - 4q) \cdot \sqrt{\|(p^2 - 4q\|} \\ \alpha = \frac{\hat{\alpha}}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2}} \quad , \quad \beta = \frac{\hat{\beta}}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2}}. \end{array} \tag{5}$$

This way the first order critical point $\mathbf{0}$ is mapped onto the unit circle in the (α, β) plane. Figure 2 shows the relation between the classification of first order critical points in [Helma89] (shown in figure 1) and the location in the (α, β) phase plane. Note that the additionally introduced classes of critical points, attracting star and repelling star, correspond to the conditions

attracting star: $R_1 = R_2 < 0\ ,\ I_1 = I_2 = 0$

repelling star : $R_1 = R_2 > 0\ ,\ I_1 = I_2 = 0$

in [Helma89] where R_1, R_2 denote the real parts of the eigenvalues of the Jacobian matrix while I_1, I_2 denotes its imaginary parts. Now the distance of two first order critical points is simply

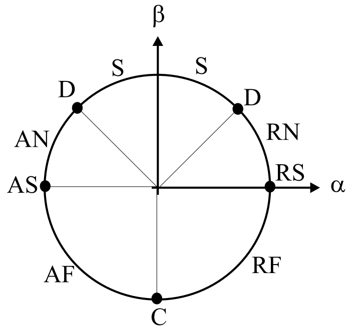

Figure 2: Classification of first order critical points in (α, β) phase plane (following [Lavin98]): RS (repelling star), RN (repelling node), D (degenerate - not a first order critical point), S (saddle), AN (attracting node), AS (attracting star), AF (attracting focus), C (center), RF (repelling star).

the Euclidian distance of their locations in the (α, β) plane. This distance is called EMD (earth mover's distance) in [Lavin98]. There its usefulness has been shown by a number of examples. However, the (α, β) phase plane also has properties which do not correspond to intuition:

- Inconsistent treatment of inverted vector fields. Given a first order critical point $\mathbf{x}_0$ in a vector field $\mathbf{v}$, a certain amount of work is necessary to convert this critical point into the critical point of the vector field $-\mathbf{v}$. Figure 3[1] shows an example of inverting a center and a repelling star. The inverse of the center is a center as well and has therefore the same (α, β) coordinates of $(0, -1)$. The inversion of the repelling star (coordinates $(1, 0)$ in (α, β) space) is an attracting star with the (α, β) coordinates of $(-1, 0)$.

- Collapsing of critical points with different flow behavior (but similar topology concerning [Helma89]) into the same location in (α, β) space. To illustrate this, figure 4 shows the critical point $(0, 0)$ of the linear vector field

$$\mathbf{v}(x, y) = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \cdot \begin{pmatrix} x \\ -\frac{1-2\sqrt{r(1-r)}}{1-2r}\, y \end{pmatrix}$$

for 18 different choices of γ and r. In particular, γ and r have been chosen as

$$\begin{array}{ll} \text{a)}: \gamma = \frac{\pi}{2}, r = 1; & \text{b)}: \gamma = -\frac{\pi}{2}, r = 1; \\ \text{c)}: \gamma = \frac{\pi}{2}, r = 0.8; & \text{d)}: \gamma = -\frac{\pi}{2}, r = 0.8; \\ \text{e)}: \gamma = \frac{\pi}{2}, r = 0.6; & \text{f)}: \gamma = -\frac{\pi}{2}, r = 0.6; \end{array}$$

$$\begin{array}{ll} \text{g)}: \gamma = 0; & \text{h)}: \gamma = -\frac{\pi}{8}; \\ \text{i)}: \gamma = \frac{\pi}{8}; & \text{j)}: \gamma = -\frac{\pi}{4}; \\ \text{k)}: \gamma = \frac{\pi}{4}; & \text{l)}: \gamma = -\frac{3\pi}{8}; \end{array}$$

$$\begin{array}{ll} \text{m)}: \gamma = \frac{\pi}{4}; & \text{n)}: \gamma = -\frac{\pi}{4}; \\ \text{o)}: \gamma = \frac{\pi}{3}; & \text{p)}: \gamma = -\frac{\pi}{3}; \\ \text{q)}: \gamma = \frac{5\pi}{12}; & \text{r)}: \gamma = -\frac{5\pi}{12}; \end{array}$$

$$\text{g)-l)}: r = \frac{1}{1+\sin^2\gamma}; \quad \text{m)-r)}: r = \frac{1}{2\sin^2\gamma}.$$

This way the critical points in figures 4a-f have (α, β) coordinates of $(0, -1)$; the critical points in figures 4g-l have (α, β) coordinates of $(1, 0)$; and the critical points in figures 4m-r have (α, β) coordinates of $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. This contradicts the observation that for instance the figures 4j and 4p are visually more similar than the figures 4m and 4p.

[1]The visualization technique used for this (and the following) illustrations is called *Integrate&Draw* and is described in [Risqu98]. For here it is sufficient to mention that the behavior of the stream lines can be detected quite well in this visualization.

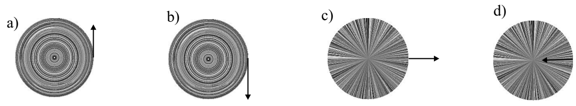

Figure 3: Inverted vector fields in (α, β) plane; a) center with (α, β) coordinates $(0, -1)$; b) inverse vector field of a) has the same (α, β) coordinates; c) repelling star with (α, β) coordinates $(1, 0)$; d) inverse vector field of c) has the (α, β) coordinates $(-1, 0)$.

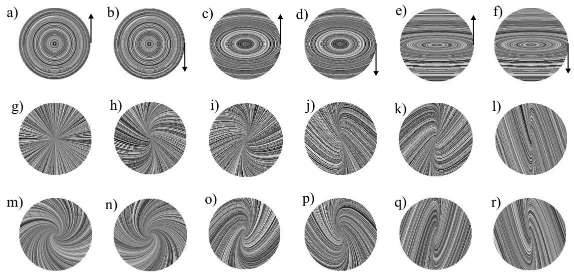

Figure 4: Different critical points with the same (α, β) coordinates; a-f: (α, β) coordinates $(0, -1)$; g-l: (α, β) coordinates $(1, 0)$; m-r: (α, β) coordinates $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

The disadvantages of the (α, β) phase plane mentioned above are mainly the motivation for introducing a new distance measure based on a new (γ, r) phase plane. This new distance measure will be based on the concept of smooth vector field transformations which will be introduced in the following section.

4 SMOOTH VECTOR FIELD TRANSFORMATIONS

The new distance measure to be introduced follows the approach of [Lavin98] that the distance of two critical points is the amount of work which is necessary to transform one into the other. The theoretical foundation of this are smooth transformations of vector fields which yield a parameterized set of new vector fields from an original one. In this section we introduce 4 basic transformations which turn out to form the space of all possible first order critical points.

4.1 Scaled vector fields

Given a vector field $\mathbf{v}$ by (4), we can define a new vector field $\lambda\mathbf{v}$ by choosing a certain $\lambda > 0$. Obviously this transformation does not influence the flow behavior and should therefore not have any influence on the distance measure. However, scaling a vector field changes the Jacobian and the divergence:

$$\begin{aligned} \det(\mathbf{J}_{\lambda\mathbf{v}}) &= \lambda^2 \det(\mathbf{J}_{\mathbf{v}}) \\ \mathrm{div}(\lambda\mathbf{v}) &= \lambda \, \mathrm{div}(\mathbf{v}). \end{aligned}$$

This gives reason to introduce the concepts of *normalized Jacobian* d_{norm} and *normalized divergence* div_{norm} by

$$d_{norm}(\mathbf{v}) = 2\frac{\det(\mathbf{J}_{\mathbf{v}})}{u_x^2+v_x^2+u_y^2+v_y^2} \quad (6)$$

$$\mathrm{div}_{norm}(\mathbf{v}) = \frac{\mathrm{div}(\mathbf{v})}{\sqrt{2\,(u_x^2+v_x^2+u_y^2+v_y^2)}}. \quad (7)$$

The values d_{norm} and div_{norm} can be interpreted as scaling independent versions of Jacobian and divergence. For any first order critical point, d_{norm} and div_{norm} range between -1 and 1.

4.2 Domain rotated vector fields

Given a vector field $\mathbf{v}$ by (4) with the critical point $\mathbf{x}_0 = \mathbf{0}$, the domain rotated vector field $\mathbf{v}^{\langle\delta,\mathbf{0}\rangle}$ which is obtained by a counterclockwise domain rotation around $\mathbf{x}_0 = \mathbf{0}$ by the angle δ can be written as

$$\begin{aligned} \mathbf{v}^{\langle\delta,\mathbf{0}\rangle} = & \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix} \cdot \\ & \cdot\ \mathbf{v}\left((x,y)\cdot\begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}\right). \end{aligned} \quad (8)$$

Figure 5 shows an example of domain rotated vector fields. As we can see there, the behavior of the flow is not influenced by a domain rotation.

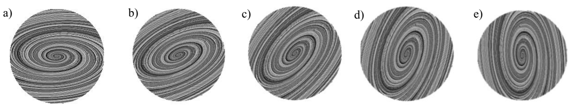

Figure 5: Domain rotated vector fields around a critical point $\mathbf{x}_0$; a) $\mathbf{v} = \mathbf{v}^{\langle 0,\mathbf{x}_0\rangle}$; b) $\mathbf{v}^{\langle \frac{\pi}{8},\mathbf{x}_0\rangle}$; c) $\mathbf{v}^{\langle \frac{\pi}{4},\mathbf{x}_0\rangle}$; d) $\mathbf{v}^{\langle \frac{3\pi}{8},\mathbf{x}_0\rangle}$; e) $\mathbf{v}^{\langle \frac{\pi}{2},\mathbf{x}_0\rangle}$.

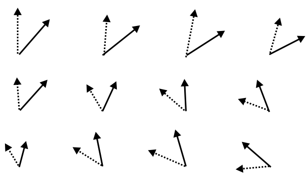

Figure 6: Rotated vector fields; if the solid arrows denote the vector field $\mathbf{v}$, the dashed arrows denote the vector field $\mathbf{v}^{[\frac{\pi}{4}]}$ (from [Theis95]).

Thus, for introducing distance measure of critical points we should distinguish only between first order critical points which cannot be transformed into each other by scaling and domain rotation. To do so, we introduce the following definitions.

Given are two vector fields $\mathbf{v}$ and $\mathbf{w}$ in the form of (4). $\mathbf{v}$ and $\mathbf{w}$ are *domain rotation equivalent* (written $\mathbf{v} \sim_{dre} \mathbf{w}$) iff they can be transformed to each other by scaling and domain rotation:

$$\begin{aligned} &\mathbf{v} \sim_{dre} \mathbf{w} \iff \\ &\exists\,\delta \in [0, 2\pi]\ \exists\lambda > 0\ :\quad \mathbf{v} = \lambda\,\mathbf{w}^{\langle\delta,\mathbf{0}\rangle}. \end{aligned} \quad (9)$$

The normalized Jacobian and the normalized divergence are constant for domain rotation equivalent vector fields, i.e. we get

$$\mathbf{v} \sim_{dre} \mathbf{w} \implies d_{norm}(\mathbf{v}) = d_{norm}(\mathbf{w}) \quad (10)$$

$$\mathbf{v} \sim_{dre} \mathbf{w} \implies \mathrm{div}_{norm}(\mathbf{v}) = \mathrm{div}_{norm}(\mathbf{w}). \quad (11)$$

4.3 Rotated vector fields

The concept of *rotated vector fields* was introduced in [Theis95]. Given a vector field $\mathbf{v}$, a new vector field $\mathbf{v}^{[\gamma]}$ can be obtained in the following way: for every point (x, y) in the domain, the direction of $\mathbf{v}(x, y)$ is rotated counterclockwise by the angle γ while the magnitude remains unchanged. Figure 6 gives an illustration of $\mathbf{v}^{[\frac{\pi}{4}]}$. The rotated vector field $\mathbf{v}^{[\gamma]}$ can be computed from $\mathbf{v}$ by

$$\mathbf{v}^{[\gamma]}(x,y) = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \cdot \mathbf{v}(x,y). \quad (12)$$

A special rotated vector field is the perpendicular vector field $\mathbf{v}^{\perp}$ of $\mathbf{v}$ which is defined as

$$\mathbf{v}^{\perp} = \mathbf{v}^{[\frac{\pi}{2}]} = \begin{pmatrix} -v(x,y) \\ u(x,y) \end{pmatrix}. \quad (13)$$

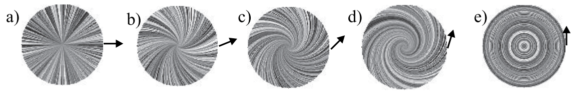

Figure 7: Rotated vector fields; a) $\mathbf{v}^{[0]}$; b) $\mathbf{v}^{[\frac{\pi}{8}]}$; c) $\mathbf{v}^{[\frac{\pi}{4}]}$; d) $\mathbf{v}^{[\frac{3\pi}{8}]}$; e) $\mathbf{v}^{[\frac{\pi}{2}]}$.

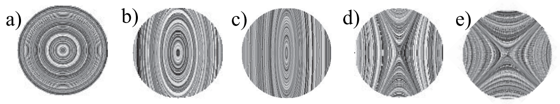

Figure 8: Domain scaling in y-direction; a) $\mathbf{v}(x,y) = (-y,x)^T$; b) $\mathbf{v}(x, 0.5\,y)$; c) $\mathbf{v}(x, 0.2\,y)$; d) $\mathbf{v}(x, -0.5\,y)$; e) $\mathbf{v}(x,-y)$.

Contrary to domain rotation, the rotation of a vector field changes the flow behavior around the critical point. Figure 7 shows the effect of vector field rotation for different angles γ.

4.4 Domain scaling in y-direction

From a given vector field $\mathbf{v}(x,y)$ by (4) we can obtain a new vector field $\mathbf{v}(x,\lambda y)$ by choosing a certain $\lambda \neq 0$. Obviously domain scaling in y-direction changes the flow behavior around the critical point. Note that in particular we allow λ to be negative. Figure 8 shows the effect of domain scaling in y-direction for different λ.

5 THE (γ, r) PHASE PLANE

In this section we introduce a new metrics of first order critical points which is based on the smooth vector field transformations introduced above. The phase plane we use here to classify first order critical points is the area inside the unit circle where (γ, r) are the polar coordinates ($\gamma \in [0, 2\pi], r \in [0,1]$). To characterize this (γ, r) phase plane, we define a *reference critical point* for each point of it. This is the critical point $(0,0)$ of the following vector field:

$$\mathbf{v}_{\gamma,r}(x,y) = \qquad (14)$$
$$\begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \cdot \begin{pmatrix} x \\ -\frac{1-2\sqrt{r(1-r)}}{1-2r}\, y \end{pmatrix}.$$

$\mathbf{v}_{\gamma,r}$ defines a vector field with a first order critical point in $(0,0)$ for each point (γ, r) of the phase plane ($\gamma \in [0,2\pi], r \in [0,1]$). Figure 9 gives an illustration of the reference critical points in the (γ, r) phase plane.

The system of reference critical points in the (γ, r) phase plane has the following properties:

- Critical points which lie on a circle $r = $ const in the (γ, r) plane can be transformed into each other by rotation:
$$\mathbf{v}_{\gamma+\alpha,r} = \mathbf{v}_{\gamma,r}{}^{[\alpha]}. \qquad (15)$$
This follows directly from (14) and the definition (12) of rotated vector fields.

- Critical points which lie on a ray through the origin $r = 0$ in the (γ, r) plane can be transformed into each other by domain scaling in y-direction:
$$\mathbf{v}_{\gamma,r_2}(x,y) = \mathbf{v}_{\gamma,r_1}(x, \lambda\, y) \qquad (16)$$
with
$$\lambda = \frac{(1-2\,r_2)\left(1-2\sqrt{r_1\,(1-r_1)}\right)}{(1-2\,r_1)\left(1-2\sqrt{r_2\,(1-r_2)}\right)}. \qquad (17)$$
This is a straightforward deduction from (14).

- The critical point in the center $r = 0$ of the (γ, r) plane deserves special attention. For this point, a rotation of the corresponding vector field gives only a domain rotated version of itself:
$$\mathbf{v}_{\gamma,0} = \mathbf{v}_{0,0}{}^{\langle \frac{\gamma}{2}, \mathbf{0}\rangle}. \qquad (18)$$
This follows as a straightforward exercise in algebra from (14) and (8). This property can also be written as $\mathbf{v}_{\gamma_1,0} \sim_{dre} \mathbf{v}_{\gamma_2,0}$ for any $\gamma_1, \gamma_2 \in [0, 2\pi]$.

- Considering the normalized Jacobian d_{norm} of the reference critical points in the (γ, r) plane , we obtain
$$d_{norm}(\mathbf{v}_{\gamma,r}) = 2\,r - 1. \qquad (19)$$
This follows from (6) and (14). Figure 10a illustrates the normalized Jacobian of the reference critical point as a height field over the (γ, r) plane.

- Considering the normalized divergence div_{norm} of the reference critical points in the (γ, r) plane, we obtain
$$\mathrm{div}_{norm}(\mathbf{v}_{\gamma,r}) = \sqrt{r} \cdot \cos\gamma. \qquad (20)$$
This follows from (7) and (14). Figure 10b illustrates the normalized divergence of the reference critical point as a height field over the (γ, r) plane.

- The normalized divergence of the perpendicular vector field of $\mathbf{v}_{\gamma,r}$ is
$$\mathrm{div}_{norm}(\mathbf{v}_{\gamma,r}{}^{\perp}) = -\sqrt{r} \cdot \sin\gamma. \qquad (21)$$
This follows from (7), (14) and (13).

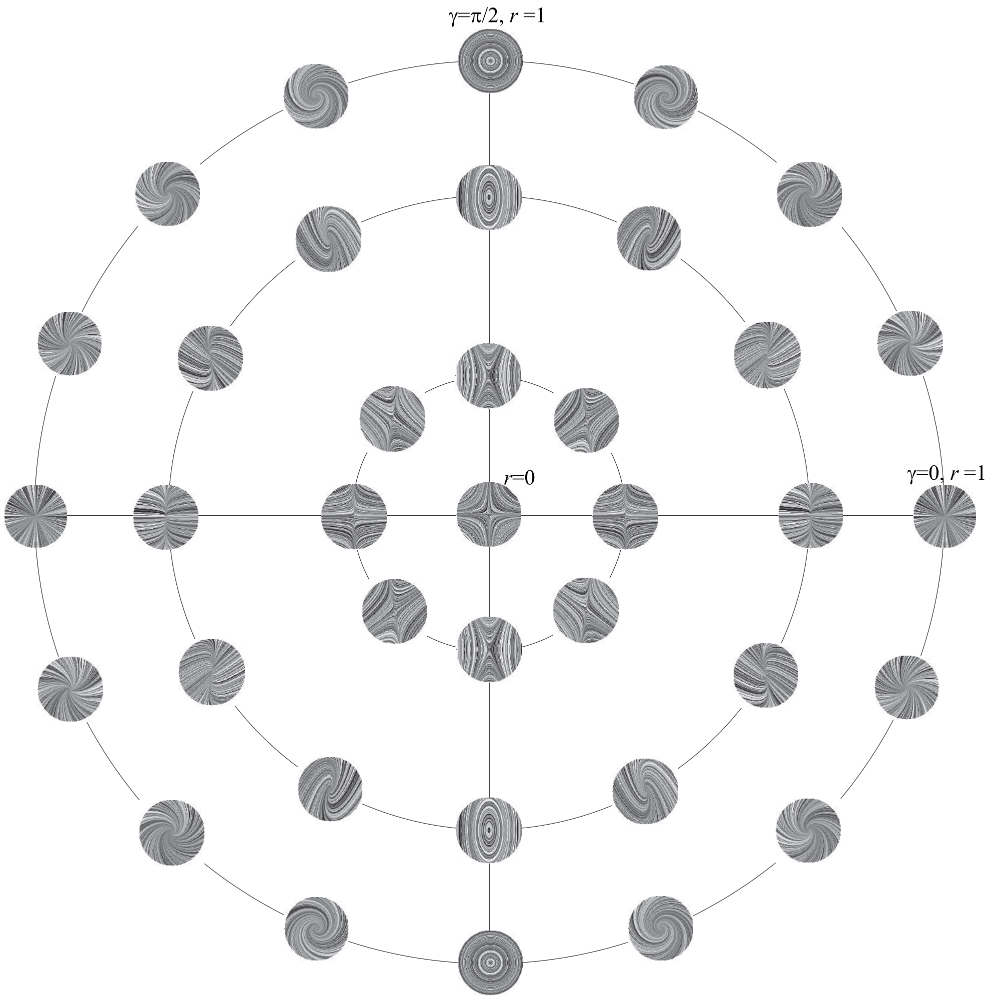

Figure 9: Reference critical points in the (γ, r) phase plane.

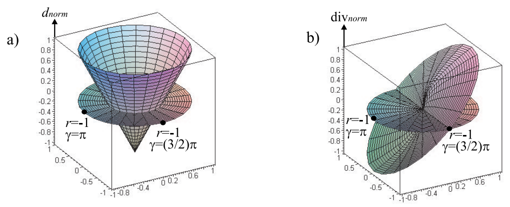

Figure 10: a) normalized Jacobian d_{norm} of the reference critical points as height field over the (γ, r) plane; b) normalized divergence div_{norm} of the reference critical point as height field over the (γ, r) plane.

- The reference critical points in the (γ, r) plane yield the following classification of first order critical points following [Helma89] and [Lavin98]: the critical point $(0,0)$ of the reference vector field $\mathbf{v}_{\gamma,r}$ is a
  - saddle point (Sa) iff ($\gamma = \frac{\pi}{2}$ and $r < \frac{1}{2}$) or ($\gamma = -\frac{\pi}{2}$ and $r < \frac{1}{2}$) or $r = 0$,
  - repelling saddle (RSa) iff $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$ and $0 < r < \frac{1}{2}$,
  - attracting saddle (ASa) iff $\frac{\pi}{2} < \gamma < \frac{3}{2}\pi$ and $0 < r < \frac{1}{2}$,
  - degenerate (D) - not a critical point - iff $r = \frac{1}{2}$,
  - center 1 (C1) iff $\gamma = \frac{\pi}{2}$ and $\frac{1}{2} < r \leq 1$,
  - center 2 (C2) iff $\gamma = -\frac{\pi}{2}$ and $\frac{1}{2} < r \leq 1$,
  - repelling focus 1 (RF1) iff $0 < \gamma < \frac{\pi}{2}$ and $\frac{1}{2} < r < \frac{1}{1+\sin^2\gamma}$,
  - repelling focus 2 (RF2) iff $-\frac{\pi}{2} < \gamma < 0$ and $\frac{1}{2} < r < \frac{1}{1+\sin^2\gamma}$,
  - attracting focus 1 (AF1) iff $\frac{\pi}{2} < \gamma < \pi$ and $\frac{1}{2} < r < \frac{1}{1+\sin^2\gamma}$,
  - attracting focus 2 (AF2) iff $\pi < \gamma < \frac{3}{2}\pi$ and $\frac{1}{2} < r < \frac{1}{1+\sin^2\gamma}$,
  - repelling star 1 (RS1) iff $0 < \gamma < \frac{\pi}{2}$ and $r = \frac{1}{1+\sin^2\gamma}$,
  - repelling star 2 (RS2) iff $-\frac{\pi}{2} < \gamma < 0$ and $r = \frac{1}{1+\sin^2\gamma}$,

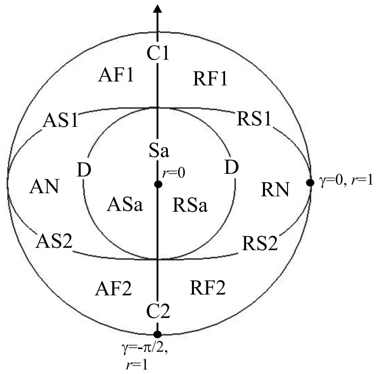

Figure 11: Classification of critical points in the (γ, r) phase plane.

- attracting star 1 (AS1) iff $\frac{\pi}{2} < \gamma < \pi$ and $r = \frac{1}{1+\sin^2\gamma}$,
- attracting star 2 (AS2) iff $\pi < \gamma < \frac{3}{2}\pi$ and $r = \frac{1}{1+\sin^2\gamma}$,
- repelling node (RN) iff $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$ and $\frac{1}{1+\sin^2\gamma} < r \leq 1$,
- attracting node (AN) iff $\frac{\pi}{2} < \gamma < \frac{3}{2}\pi$ and $\frac{1}{1+\sin^2\gamma} < r \leq 1$.

This classification of critical points has extensions to the classifications of [Helma89] and [Lavin98] in the following way:

- We distinguish between three kinds of saddle points. A saddle point (in the sense of [Helma89] and [Lavin98]) is a first order critical point which has both inflow and outflow. A repelling saddle (RSa) has more outflow than inflow, i.e. a positive divergence. An attracting saddle (ASa) has more inflow than outflow and therefore a negative divergence. A saddle point (Sa) has a zero divergence.
- The classes of points RF, RS, C, AF, AF are each subdivided into two subclasses 1 and 2. Subclass 1 means that in a neighborhood of the critical point all tangent curves turn to the left, i.e. they have non-negative curvature (see [Theis95]). In critical points of subclass 2, all tangent curves in a neighborhood turn to the right, i.e. have non-positive curvature.

Figure 11 illustrates the location of the different classes of critical points in the (γ, r) phase plane. Note that the curve $r = \frac{1}{1+\sin^2\gamma}$ defining attracting and repelling stars is not an ellipse.

After showing that the system of reference critical points in the (γ, r) phase plane has a number of useful properties, we still have to show that it describes all first order critical points uniquely (except for domain rotation and scaling). This means that we have to show that every vector field $\mathbf{v}$ defined by (4) is domain rotation equivalent to one and only one reference vector field $\mathbf{v}_{\gamma,r}$ defined by (14). The key idea is the observation that the location of any first order critical point in the $\mathbf{v}_{\gamma,r}$ plane is uniquely determined by its normalized Jacobian, its normalized divergence, and the normalized divergence of its perpendicular vector field. We formulate

Theorem 1. *Given is a vector field $\mathbf{v}(x, y)$ defined by (4) with a first order critical point at $\mathbf{x}_0 = \mathbf{0}$. Then there exists one and only one reference critical point in the (γ, r) phase plane which is domain rotation equivalent to $\mathbf{v}$. This reference critical point is the critical point of $\mathbf{v}_{\gamma,r}$ with*

$$\cos\gamma = \frac{u_x + v_y}{\sqrt{(u_x+v_y)^2 + (v_x-u_y)^2}} \quad (22)$$

$$\sin\gamma = \frac{v_x - u_y}{\sqrt{(u_x+v_y)^2 + (v_x-u_y)^2}} \quad (23)$$

$$r = \frac{1}{2} + \frac{u_x\,v_y - v_x\,u_y}{u_x^2 + v_x^2 + u_y^2 + v_y^2}. \quad (24)$$

(22) and (23) determine γ uniquely except for the case $(u_x + v_y) = (v_x - u_y) = 0$. Since in this case we obtain $r = 0$ from (24), γ is of no importance there.

Proof: Since $\mathbf{v}_{\gamma,r}$ has to fulfill $\mathbf{v} \sim_{dre} \mathbf{v}_{\gamma,r}$, we obtain (24) from (6), (10) and (19). Similarly, (22) is obtained from (7), (11) and (20). (23) follows from (7), (11), (21) and (13). Thus the only reference critical point which is a candidate for being domain rotation equivalent to $\mathbf{v}$ is $\mathbf{v}_{\gamma,r}$ with γ, r described by (22)-(24). To show that this reference critical point is indeed domain rotation equivalent to $\mathbf{v}$, we have to find a domain rotation angle δ and a scaling factor $\lambda > 0$ in such a way that

$$\lambda\,\mathbf{v}^{\langle\delta,\mathbf{0}\rangle} = \mathbf{v}_{\gamma,r}. \quad (25)$$

Choosing δ and λ as

$$\cos(2\delta) = \quad (26)$$
$$\frac{u_x^2 + v_x^2 - u_y^2 - v_y^2}{\sqrt{(u_x^2 + v_x^2 - u_y^2 - v_y^2)^2 + 4\,(u_x u_y + v_x v_y)^2}}$$

$$\sin(2\delta) = \quad (27)$$
$$\frac{-2\,(u_x u_y + v_x v_y)}{\sqrt{(u_x^2 + v_x^2 - u_y^2 - v_y^2)^2 + 4\,(u_x u_y + v_x v_y)^2}}$$

$$\lambda = \frac{\operatorname{div}(\mathbf{v}_{\gamma,r})}{\operatorname{div}(\mathbf{v})}, \quad (28)$$

(25) follows from (8), (22), (23), (24), (26), (27) and (28) [2]. If $\operatorname{div}(\mathbf{v}) = 0$, (28) has to be replaced by $\lambda = \frac{\operatorname{div}(\mathbf{v}_{\gamma,r}{}^{\perp})}{\operatorname{div}(\mathbf{v}^{\perp})}$ which yields (25) as well. Thus theorem 1 is proven.

Theorem 1 shows that the γ, r phase plane gives a continuous one-to-one representation of all first order critical points. Thus it can be used to compute the distance of two first order critical points by mapping them into the γ, r phase plane and computing their Euclidian distance there.

6 RESULTS

We have introduced a new distance measure of first order critical points which is based on a classification in a (γ, r) phase plane. This distance measure has the following properties:

- Invariance under scaling. The vector fields $\mathbf{v}$ and $\lambda\mathbf{v}$ with $\lambda > 0$ fall into the same location of the (γ, r) phase plane, i.e. have a zero distance to each other.
- Invariance under domain rotation. $\mathbf{v}$ and $\mathbf{v}^{\langle\delta,\mathbf{0}\rangle}$ have a zero distance for all δ.
- Consistent treatment of the inverted vector field. The average of the locations of the vector fields $\mathbf{v}$ and $-\mathbf{v}$ in the (γ, r) phase plane is always the point $(0, 0)$.
- Only critical points which are domain rotation equivalent to each other have a zero distance in the (γ, r) phase plane.

As a byproduct of the distance measure, the (γ, r) phase plane yields an advanced classification of first order critical points. Since all domain rotation equivalent critical points fall to one and only one location in the (γ, r) phase plane, and since all first order critical points at a certain location of the (γ, r) phase plane are domain rotation equivalent, this classification is complete in the sense that all critical points which are not domain rotation equivalent are distinguished and have a nonzero distance. Thus further refinements of the classification are not possible.

REFERENCES

[Batra99a] R. Batra and L. Hesselink. Feature comparisons of 3-d vector fields using earth mover's distance. In D. Ebert, M. Gross, and B. Hamann, editors, *Proc. IEEE Visualization '99*, pages 105–114, Los Alamitos, 1999.

[Batra99b] R. Batra, K. Kling, and L. Hesselink. Topology based vector field comparison using graph methods. In A. Varshney, C.M. Wittenbrink, and H. Hagen, editors, *Proc. IEEE Visualization '98, Late Breaking Hot Topics*, pages 25–28, 1999.

[Chong90] M. S. Chong, A. E. Perry, and B. J. Cantwell. A general classification of three-dimensional flow fields. *Physics of Fluids A*, 2(5):765–777, 1990.

[Firby82] P.A. Firby and C.F. Gardiner. *Surface Topology*, chapter 7, pages 115–135. Ellis Horwood Ltd., 1982. Vector Fields on Surfaces.

[Hecke99] B. Heckel, G.H. Weber, B. Hamann, and K.I.Joy. Construction of vector field hierarchies. In D. Ebert, M. Gross, and B. Hamann, editors, *Proc. IEEE Visualization '99*, pages 19–26, Los Alamitos, 1999.

[Helma89] J. Helman and L. Hesselink. Representation and display of vector field topology in fluid flow data sets. *IEEE Computer*, 22(8):27–36, August 1989.

[Kenwr99] D.N. Kenwright, C. Henze, and C. Levit. Feature extraction of separation and attachment lines. *IEEE Transactions on Visualization and Computer Graphics*, 5(2):135–144, 1999.

[Lavin98] Y. Lavin, R.K. Batra, and L. Hesselink. Feature comparisons of vector fields using earth mover's distance. In D. Ebert, H. Hagen, and H. Rushmeier, editors, *Proc. IEEE Visualization '98*, pages 103–109, Los Alamitos, 1998. IEEE Computer Society Press.

[Phili97] P. A. Philippou and R. N. Strickland. Vector field analysis and synthesis using three dimensional phase portraits. *Graphical Models and Image Processing*, 59:446–462, November 1997.

[Risqu98] C. Perez Risquet. Visualizing 2D flows: Integrate and Draw. In *Proceedings 9th Eurographics Workshop on Visualization in Scientific Computing*, pages 57–67, 1998.

[Scheu97] G. Scheuermann, H. Hagen, H. Krüger, M. Menzel, and A. Rockwood. Visualization of higher order singularities in vector fields. In R. Yagel and H. Hagen, editors, *Proc. IEEE Visualization '97*, pages 67–74, 1997.

[Scheu98] G. Scheuermann, H. Krüger, M. Menzel, and A. Rockwood. Visualizing non-linear vector field topology. *IEEE Transactions on Visualization and Computer Grapics*, 4(2):109–116, 1998.

[Telea99] A. Telea and J.J. van Wijk. Simplified representation of vector fields. In D. Ebert, M. Gross, and B. Hamann, editors, *Proc. IEEE Visualization '99*, pages 35–42, Los Alamitos, 1999.

[Theis95] H. Theisel. *Vector Field Curvature and Applications.* PhD thesis, University of Rostock, November 1995.

[Trott00] I. Trotts, D. Kenwright, and R. Haimes. Critical points at infinity: a missing link in vector field topology. In *Proc. NSF/DoE Lake Tahoe Workshop on Hierarchical Approximation and Geometrical Methods for Scientific Visualization*, 2000. available at http://graphics.cs.ucdavis.edu/ hvm00/program.html.

[2] Once δ and λ are specified by (26)–(28), this is just a straightforward computation for which one can use formula manipulation programs like Mathematica or Maple.